26

A Non–overlapping Domain Decomposition Method for Solving Elliptic Problems by Finite Element Methods

Xiaobeng Feng

1 Introduction

Non-overlapping domain decomposition methods have received a lot attention during the last few years, due to the restrictions of overlapping domain decomposition methods. Several families of non–overlapping decomposition methods for the solutions of elliptic problems have been proposed, analyzed, and successfully implemented [BW86, BPWX91, DPLRW93, Dry89, GW88, Lio90, MQ89, LTDRV91, Tan92].

In a non–overlapping domain decomposition method, the original problem is first decomposed into smaller problems defined on non–overlapping subdomains. Parallel or sequential iterative procedures are then constructed for decoupling the whole domain problem into subdomain problems. During the iterative process, information must be transmitted between subdomains in order to guarantee convergence. This "information transmission" step is the key part of a domain decomposition method; it distinguishes one domain decomposition method from another. Several methods for passing information have been proposed in the literature [BF96, DPLRW93, GLT90, Lio90, MQ89]. The common approach was to develop a transmission condition for the differential problem and adapt the same condition to the corresponding discrete problem.

The purpose of this paper is to present a parallelizable, iterative, non–overlapping domain decomposition method for solving second oreder elliptic problems discretized by finite element methods. Unlike the usual approach, we bypass the differential problems and construct the iterative procedure based on domain decomposition techniques directly for the finite element equations. To obtain the split subdomain problems, our main idea is to use a penalty method on each subdomain and to introduce a local (in the pointwise sense), non–Robin type transmission condition which not only enhances the convergence and passes the information between the

Ninth International Conference on Domain Decomposition Methods
Editor Petter E. Bjørstad, Magne S. Espedal and David E. Keyes

subdomains but also traces the jumps of the discrete Neumann data of the finite element solutions across the interfaces.

The rest of the paper is organized as follows. In Section 2, a model second order elliptic problem and its finite element discretization are introduced. In Section 3, a parallelizable iterative procedure based on domain decomposition techniques for solving the finite element equations of the elliptic problem is presented. Finally, in section 4, the convergence analysis and the rate of convergence are demonstrated in the case when subdomains are chosen as small as the individual finite elements to show the effectiveness of the procedure. It is shown that the domain decomposition procedure converges at a rate which is independent of the mesh size h if the relaxation parameters are chosen properly. All the theorems and lemmas are stated either without proofs or with schematic proofs. For the details, we refer to [Fen96], where numerical experiments are also presented, and closely related domain decomposition procedures are developed for the biharmonic equation and the Helmholtz equation.

2 The Model Problem

Let $\Omega \subset \mathbf{R}^2$ be a bounded polygonal domain. Consider the model Dirichlet problem:

$$
\begin{aligned}
-\Delta u(x) + c(x)u(x) &= \quad f(x), \qquad \text{in } \Omega, & (1)\\
u(x) &= \quad 0, \qquad \text{on } \partial\Omega, & (2)
\end{aligned}
$$

where the coefficient function $c(x) \geq 0$. The weak formulation of (1)–(2) is to find $u \in H_0^1(\Omega)$ such that

$$
a(u,v)_\Omega = (f,v)_\Omega, \qquad \forall v \in H_0^1(\Omega), \tag{3}
$$

where

$$
a(w,v)_\Omega = \int_\Omega \nabla w \cdot \nabla v \; dx, \qquad (w,v)_\Omega = \int_\Omega wv dx.
$$

Let $\mathcal{T}_h$ be a quasiuniform triangular or rectangular partition of Ω and $V^h \subset H_0^1(\Omega)$ denote a finite element space of piecewise polynomials of degree r (≥ 1). Then the finite element method for problem (1)–(2) is to find $u^h \in V^h$ such that

$$
a(u^h,v)_\Omega = (f,v)_\Omega, \qquad \forall v \in V^h. \tag{4}
$$

Let $\{\phi_j^h\}_{j=1}^n$ denote the nodal basis of the finite element space V^h and $\{p_j^h\}_{j=1}^n$ denote the nodal set corresponding to the nodal parameters (a node is counted k times if there are k nodal parameters attached to it). Then (4) gives the following linear system:

$$
A\xi = b, \tag{5}
$$

where

$$A = [a(\phi_i^h, \phi_j^h)]_{n\times n}, \quad \xi = (u^h(p_1^h), \cdots, u^h(p_n^h))^t, \quad b = ((f, \phi_1^h)_\Omega, \cdots, (f, \phi_n^h)_\Omega)^t.$$

It is well–known that the condition number of the system is of order h^{-2}, which implies that the system is ill–conditioned. Therefore, to solve problem (4), in particular for small h, a fast solver other than a classical iterative method is necessary. To find a required fast solver using domain decomposition techniques is the goal of this paper.

The following notations are adopted throughout the rest of this paper.

Let $\{\Omega_j\}_1^J$ be a non–overlapping partition of Ω, that is,

$$\overline{\Omega} = \cup_{j=1}^J \overline{\Omega}_j; \qquad \Omega_j \cap \Omega_k = \emptyset, \quad \text{if } j \neq k.$$

Assume that $\partial\Omega_j$, $j = 1, 2, \cdots, J$ is Lipschitz and Ω_j is a star–shaped domain. We also assume that the non-overlapping partition aligns with the triangulation $\mathcal{T}_h$. In practice, with the exception of perhaps a few Ω_j's along $\partial\Omega$, each Ω_j will be convex with a piecewise–smooth boundary. For example, an interesting choice for the domain decomposition of a finite element discretization is to let each finite element be a subdomain.

Finally, define

$$\Gamma = \partial\Omega, \qquad \Gamma_j = \Gamma \cap \partial\Omega_j, \qquad \Gamma_{jk} = \Gamma_{kj} = \partial\Omega_j \cap \partial\Omega_k. \tag{6}$$

$$H^1_{\Gamma_j}(\Omega_j) = \{v \in H^1(\Omega_j);\ v = 0, \text{ on } \Gamma_j\}. \tag{7}$$

3 The Domain Decomposition Iterative Method

The objective of this section is to construct a domain decomposition iterative method to solve the finite element equations (4). The key step is to construct the split subdomain problems and the local (in the pointwise sense) transmission conditions on the interfaces of the subdomains. We notice that the pointwise continuity across the element interfaces does not hold for the flux since, in general, $\frac{\partial u_i^h}{\partial n_i}$ is different from $\frac{\partial u_j^h}{\partial n_j}$ on Γ_{ij}. Therefore, any attempt to enforce the pointwise continuity of the flux will not succeed. The above observation leads us to take the following approach: find transmission conditions that preserve the continuity of u^h and trace the *discontinuity* of $\frac{\partial u^h}{\partial n}$ on the interfaces.

To construct the subdomain problems, first, we rewrite (4) as

$$\sum_j a(u^h, v)_{\Omega_j} = \sum_j (f, v)_{\Omega_j}, \qquad \forall v \in V^h. \tag{8}$$

Next, we observe the following fact:

$$\sum_{i,j} \langle \beta(\frac{\partial u_i^h}{\partial n_i} - \frac{\partial u_j^h}{\partial n_j}), v\rangle_{\Gamma_{ij}} = 0, \qquad \forall v \in V^h,$$

for any nonzero constant β.

Now for $V_j^h = V^h|_{\Omega_j}$ and $u_j^h = u^h|_{\Omega_j}$, it is easy to see that $\{u_j^h\}$ satisfies

$$
\begin{aligned}
a_i(u_i^h, v) - \sum_j \langle \beta(\frac{\partial u_i^h}{\partial n_i} - \frac{\partial u_j^h}{\partial n_j}), v\rangle_{\Gamma_{ij}} &= (f, v)_{\Omega_i},\ \forall v \in V_i^h, && (9)\\
u_i^h &= u_j^h. \quad \text{on } \Gamma_{ij}. && (10)
\end{aligned}
$$

We remark that the second term on the right hand side of (9), which measures the total *jumps* of the Neumann data being transmitted into the subdomain Ω_i, can be viewed as a penalty term for the subdomain problem on Ω_i, and the size of the free parameter β strongly influences the size of the penalty term.

On the other hand, it is not convenient to decouple the whole domain problem (4) based on (9)–(10), since the interface condition (10) is a Dirichlet condition on the interfaces for each subdomain problem. To overcome this difficulty, we replace equation (10) by the following equivalent one:

$$
-\beta\frac{\partial u_j^h}{\partial n_j} + \alpha u_i^h = -\beta\frac{\partial u_j^h}{\partial n_j} + \alpha u_j^h. \quad \text{on } \Gamma_{ij}, \tag{11}
$$

which is obtained by adding $-\beta\frac{\partial u_j^h}{\partial n_j}$ to both sides of (10) after multiplying it by another nonzero constant α.

Remark 3.1 The "new" interface condition (11) still holds in the pointwise sense. This condition is *not* a Robin type transmission condition since the partial derivative on the left hand side is $\frac{\partial u_j^h}{\partial n_j}$ not $\frac{\partial u_i^h}{\partial n_i}$.

Now based on (9)–(10), we propose the following domain decomposition iterative algorithm:

Algorithm 1

Step 1. $\forall u_i^0 \in V_i^h$, $i = 1, 2, \cdots, J$.

Step 2. Compute $\{u_i^n\}$ for $i = 1, 2, \cdots, J$ and $n \geq 1$ by solving

$$
a_i(u_i^n, v) - \sum_j \langle \beta\frac{\partial u_i^n}{\partial n_i} + \lambda_{ji}^n, v\rangle_{\Gamma_{ij}} = (f, v)_{\Omega_i}, \quad \forall v \in V_i^h, \tag{12}
$$

$$
\lambda_{ji}^n + \alpha u_i^n = -\beta\frac{\partial u_j^{n-1}}{\partial n_j} + \alpha u_j^{n-1}, \quad \text{on } \Gamma_{ij}, \tag{13}
$$

Note that we have omitted all super indices h in the algorithm.

4 Convergence Analysis

In this section we will establish the convergence of Algorithm 1 and derive an upper bound for its rate of convergence. Our analysis based on the discrete version of an energy method (cf. [DPLRW93]), which was first proposed by Després in [Des91] for analyzing convergence of a Lions' type domain decomposition method for the Helmholtz equation at the differential level.

Let

$$e_i^n = u_i^h - u_i^n, \qquad \mu_{ji}^n = -\beta \frac{\partial u_j^h}{\partial n_j} - \lambda_{ji}^n.$$

Then from (9)–(13) we get

$$a_i(e_i^n, v) - \sum_j \langle \beta \frac{\partial e_i^n}{\partial n_i} + \mu_{ji}^n, v_i \rangle_{\Gamma_{ij}} = 0, \quad \forall v \in V_i^h, \tag{14}$$

$$\mu_{ji}^n + \alpha e_i^n = -\beta \frac{\partial e_j^{n-1}}{\partial n_j} + \alpha e_j^{n-1}, \quad \text{on } \Gamma_{ij}. \tag{15}$$

Clearly,

$$a_i(e_i^n, e_i^n) = \sum_j \langle \beta \frac{\partial e_i^n}{\partial n_i} + \mu_{ji}^n, e_i^n \rangle_{\Gamma_{ij}}. \tag{16}$$

Define the "pseudo–energy"

$$E_n = E(\{e_i^n\}) = \sum_{i,j} |\mu_{ji}^n + \alpha e_i^n|_{0,\Gamma_{ij}}^2. \tag{17}$$

By (13)–(16) we can show the following lemmas (cf. [Fen96]).

Lemma 4.1

$$E_n = E_{n+1} - R_{n-1} = E_0 - \sum_{\ell=0}^{n-1} R_\ell, \tag{18}$$

where

$$R_{n-1} = R(\{e_j^{n-1}\}) = \sum_{i,j} [|\mu_{ij}^{n-1}|_{0,\Gamma_{ij}}^2 - \beta^2 \left| \frac{\partial e_j^{n-1}}{\partial n_j} \right|_{0,\Gamma_{ij}}^2] + 2\alpha \sum_j a_j(e_j^{n-1}, e_j^{n-1}). \tag{19}$$

Lemma 4.2 *If the parameters α and β are chosen to satisfy $\frac{\alpha}{\beta^2} = O(h^{-1})$, then $R_n \geq 0$ for $n \geq 1$.*

Remark 4.1 The following are sample choices of α and β which satisfy the assumption of Lemma 4.2

1. $\alpha = O(1)$ and $\beta = O(\sqrt{h})$.
2. $\alpha = O(h^{-1})$ and $\beta = O(1)$.
3. $\alpha = O(h)$ and $\beta = O(h)$.

Theorem 4.1 *Choose the parameters α and β such that $\frac{\alpha}{\beta^2} = O(h^{-1})$, then*

1. $\lambda_{ij}^{\ell} \to -\beta \frac{\partial u_i^h}{\partial n_i}$ *in* $L^2(\Gamma_{ij})$ *as* $\ell \to \infty$.
2. $u_j^{\ell} \to u_j^h$ *in* $H^1(\Omega_j)$ *as* $\ell \to \infty$.

Proof. Notice that if $\frac{\alpha}{\beta^2} = O(h^{-1})$, then $\{E_n\}$ is a decreasing sequence. Therefore, if $c(x) \geq C_0 > 0$, the theorem immediately follows from Lemma 4.1 and Lemma 4.2. If $c(x) = 0$ or $c(x) \geq 0$, Lemma 4.1 and Lemma 4.2 imply the convergence of ∇e_i^{ℓ} in $L^2(\Omega_i)$ for each Ω_i. To show the convergence of e_i^{ℓ} in $L^2(\Omega_i)$, we first consider all boundary subdomains Ω_j. Since $e_j^{\ell} = 0$ on Γ_j, by Poincaré inequality we have $e_j^{\ell} \to 0$ in $L^2(\Omega_j)$ for each boundary subdomain Ω_j. Suppose Ω_i is a subdomain which has a common interface Γ_{ij}^* with one of the boundary subdomains, say, Ω_j. From (15) we have

$$\alpha e_i^{\ell} = \left(-\frac{\partial e_j^{\ell-1}}{\partial n_j} - \mu_{ji}^{\ell} \right) + \alpha e_j^{\ell-1}, \quad \text{on } \Gamma_{ij}^*.$$

And

$$\|e_i^{\ell}\|_{0,\Omega_j} \leq C \left[\|\nabla e_i^{\ell}\|_{0,\Omega_i} + \int_{\Gamma_{ij}^*} |e_i^{\ell}|^2 ds \right] \to 0 \qquad \text{as } \ell \to \infty.$$

Hence

$$\|e_i^{\ell}\|_{H^1(\Omega_j)} \to 0 \qquad \text{as } \ell \to \infty.$$

So the convergence takes place on the subdomain Ω_i. The argument can be repeated until the domain is exhausted. The proof is completed.

The above convergence theorem says that, for appropriately chosen parameters α and β, Algorithm 1 produces a strongly convergent sequence. In the rest of this section we will address the issue of the algorithm's speed of convergence by giving an upper bound estimate for the rate of convergence.

Define

$$T_f(\{u_i^{n-1}\}) = \{u_i^n\}. \tag{20}$$

Then

$$T_f(u) = T_0(u) + T_f(0).$$

If u^* is a fixed point of T_f, then

$$(I - T_0)(u^*) = T_f(0). \tag{21}$$

Lemma 4.3 *Suppose $c(x) \geq C_0 > 0$, and let u be an eigenfunction of T_0. Under the assumption of Theorem 4.1 there exists a constant $Q(h) > 0$ such that*

$$E(u) \leq Q(h) R(u), \qquad \textit{with} \quad Q(h) = 2 + \frac{\alpha h^{-1} C_1}{C_0}, \tag{22}$$

where C_1 is some positive constant which is independent of h.

Remark 4.2 The conclusion of Lemma 4.3 still holds in the case $c(x) \geq 0$. For a detailed proof, see [Fen96].

Theorem 4.2 *Let $\rho(T_0)$ denote the spectral radius of T_0. Then under the assumptions of Lemma 4.3 the following estimate holds:*

$$\rho(T_0) \leq 1 - \frac{1}{Q(h)}, \tag{23}$$

where $Q(h)$ is given in (22).

Proof. Suppose

$$T_0(u) = \gamma u,$$

then from (18) we get

$$|\gamma|^2 E(u) = E(u) - R(u). \tag{24}$$

Hence, the theorem follows from combing Lemma 4.3 and (24)

Remark 4.3 From Theorem 4.2, we immediately conclude that the spectral radius of the iteration matrix of Algorithm 1 has an upper bound of the form $O(h^{-1})$ if $\alpha = O(1)$ and $\beta = O(\sqrt{h})$, moreover, it is bounded by an absolute constant which is less than one if $\alpha = O(h)$ and $\beta = O(h)$, that is, the algorithm converges optimally when $\alpha = O(h)$ and $\beta = O(h)$.

REFERENCES

[BF96] Bennethum L. S. and Feng X. (1996) A domain decomposition method for solving a helmholtz–like problem in elasticity based on the Wilson nonconforming finite element. *R.A.I.R.O. Anal. Numer.* (to appear).

[BPS89] Bramble J. H., Pasciak J. E., and Schatz A. H. (1989) The construction of preconditioners for elliptic problems by substructuring, IV. *Math. Comp.* 53: 1–24.

[BW86] Bjørstad P. and Widlund O. (1986) Iterative methods for the solution of elliptic problems on regions partitioned into substructures. *SIAM J. Numer. Anal.* 23: 1093–1120.

[Des91] Després B. (1991) Domain decomposition method and helmhotz problem. In G. Cohn L. H. and Joly P. (eds) *Proc. SIAM Mathematical and Numerical Aspects of Wave Propagation Phenomena*, pages 44–52. SIAM, Philadelphia.

[DPLRW93] Douglas Jr. J., Paes Leme P. J. S., Roberts J. E., and Wang J. (1993) A parallel iterative procedure applicable to the approximate solution of second order partial differential equations by mixed finite element methods. *Numer. Math.* 65: 95–108.

[DW90] Dryja M. and Widlund O. B. (1990) Towards to a unified theory of domain decomposition algorithms for elliptic problems. In *Proc. Third International Symposium on Domain Decomposition Method for Partial Differential Equations*, pages 53–61. SIAM, Philadelphia.

[Fen96] Feng X. (1996) Parallel iterative domain decomposition method for second and fourth order elliptic problems. preprint.

[GLT90] Glowinski R. and Le Tallec P. (1990) Augmented lagrangian interpretation of the nonoverlapping schwarz alternating method. In *Proc. Third International Symposium on Domain Decomposition Method for Partial Differential Equations.* SIAM, Philadelphia.

[GW88] Glowinski R. and Wheeler M. F. (1988) Domain decomposition and mixed finite element methods for elliptic problems. In *Proc. First International Symposium on Domain Decomposition Method for Partial Differential Equations*. SIAM, Philadelphia.

[Lio90] Lions P. L. (1990) On the schwartz alternating method III: a variant for nonoverlapping subdomains. In *Proc. Third International Symposium on Domain Decomposition Method for Partial Differential Equations*. SIAM, Philadelphia.

[LTDRV91] Le Tallec P., De Roeck Y., and Vidrascu M. (1991) Domain decomposition methods for large linearly elliptic three–dimensional problems. *J. Comp. and Appl. Math.* 34: 93–117.

[MQ89] Marini L. and Quarteroni A. (1989) A relaxation procedure for domain decomposition methods using finite elements. *Numer. Math.* 55: 575–598.

[Tan92] Tang W. P. (1992) Generalized schwarz splitting. *SIAM J. Sci. Stat. Comp* 13: 573–595.