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A Hybrid Domain Decomposition Method For Convection-Dominated Problems

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1 Introduction

Domain decomposition methods have been intensively studied for partial differential equations. They are efficient parallel methods especially for the elliptic equations. However, when domain decomposition are used for convection-dominated problems, the flow directions must be carefully considered. We refer to [WY97], [TJDE97], [RZ94], [RT97], [KL95] for some results of domain decomposition methods for convection-dominated problems.

In this work a hybrid domain decomposition method is proposed. When the flow is simple, a non-iterative domain decomposition approach can be used. The subdomains in the upwind side shall be computed first and the subdomains in downwind direction are computed one after another. For each subdomain, Dirichlet boundary condition is used on the inflow boundary and an artificial boundary condition is used on the outflow boundary. When the flow is complicated, an iterative method must be used. The proposed methods are suitable for problem (1) when the diffusion parameter ϵ is relatively small. For small ϵ, the error introduced by the domain decomposition methods is small, and one can easily use finer meshes in the subdomains that intersect with singular layers. When the proposed methods are used for time dependent problems, the convergence properties are even better. The proposed methods of this work are easy to implement and easy to do local refinement.

2 The Hybrid Domain Decomposition Method

Consider the advection diffusion problem:

$$\begin{cases} -div(\epsilon \bigtriangledown u) + div(\beta u) + \alpha u = f, & in\ \ \Omega, \\ u = 0, & on\ \ \partial\Omega, \end{cases} \tag{1}$$

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where α, f are bounded functions, $\boldsymbol{\beta}$ is a vector-valued function. For simplicity, it is assumed that ϵ is a small constant, all results can be extended to the case that ϵ is a symmetric and positive definite matrix-valued function with small entries ϵ_{ij}.

The standard Galerkin method for (1) is to seek $u \in S_0^h$ such that

$$(\epsilon \bigtriangledown u^h, \bigtriangledown v) + (div(\boldsymbol{\beta} u^h) + \alpha u^h, v) = (f, v), \quad \forall v \in S_0^h, \tag{2}$$

where $S_0^h \in H_0^1(\Omega)$ is a finite element space on Ω with zero Dirichlet boundary condition.

To describe the domain decomposition algorithms, we first divide the domain Ω into some nonoverlapping subdomains Ω_i satisfying $\bar{\Omega} = \bigcup_i \bar{\Omega}_i, \quad \Omega_i \bigcap \Omega_j = \emptyset, \ \ i \neq j$. Let $S^h(\Omega_i) \subset H^1(\Omega_i)$ be the finite element space on Ω_i, we define

$$V_i = \{v \in S^h(\Omega_i);\ v = 0 \ on \ \ \partial\Omega_i \bigcap \partial\Omega\},$$

$$\hat{S}_0^h = \sum_i V_i = \{v \in S^h(\Omega_i),\ \forall i, \ \ v = 0 \ on \ \partial\Omega\}.$$

Notice that functions from $\hat{S}_0^h$ can have jumps along the interfaces. Bilinear form $A_i(\cdot,\cdot)$ is defined as:

$$A_i(w, v) = (\epsilon \bigtriangledown w, \bigtriangledown v)_{\Omega_i} + (div(\boldsymbol{\beta} w) + \alpha w, v)_{\Omega_i} - \int_{\partial\Omega_i^-} w_+ v_+ \boldsymbol{n\beta} ds,$$

where $\boldsymbol{n}$ is the unit outer normal vector on $\partial\Omega_i$ and

$$w_\pm(x) = \lim_{s\to 0^\pm} w(x + s\boldsymbol{\beta}), \qquad (w, v)_{\Omega_i} = \int_{\Omega_i} wv dx,$$

$$\partial\Omega_i^- = \{x \in \partial\Omega_i, \ \ \boldsymbol{\beta}(x) \cdot \boldsymbol{n}(x) \leq 0\}.$$

Our hybrid domain decomposition finite element solution is to find $\hat{u}^h = \sum \hat{u}_i^h$ such that $\hat{u}_i^h = 0$ in $\Omega \setminus \Omega_i$, and in Ω_i, $\hat{u}_i^h \in V_i$ satisfies

$$A_i(\hat{u}_i^h, v) = (f, v)_{\Omega_i} - \int_{\partial\Omega_i^-} (\hat{u}^h)_- v_+ \boldsymbol{n\beta} ds, \quad \forall v \in V_i, \tag{3}$$

where $(\hat{u}^h)_-$ is the boundary value of the solution of the adjacent subdomains in the upwind direction.

In order to solve the subdomain problem (3) to get $\hat{u}_i^h$, the inflow boundary condition $\hat{u}^h|_{\partial\Omega_i^-}$ must be known. Therefore, we need to assume that the flow is simple so that the domain Ω can be divided into subdomains and when the subdomain problems are solved one after another in the flow direction, the inflow boundary condition is always known from the neighbouring subdomains. If the flow does not have closed streamlines, this kind of division is always possible. By suitably organising the subdomains, the computation of the subdomains in the cross-wind direction can be done in parallel.

In the domain decomposition scheme (3), an artificial boundary condition on the outflow boundary is introduced, i.e. we are in fact using

$$\frac{\partial u}{\partial n} = 0, \ \text{on} \ \partial\Omega_i^+, \ \forall i. \tag{4}$$

An error will be produced by this artificial boundary condition. It can be proved, see [ETY96], that when ϵ is small, the effect from the artificial boundary condition is small. This is also confirmed in our numerical experiments.

Theorem 2.1 *Suppose u is the solution of (1), $\hat{u}^h$ is the hybrid domain decomposition finite element solution of (3), $\alpha + \frac{1}{2}div\boldsymbol{\beta} \geq \gamma > 0$ in Ω, and $|\boldsymbol{n\beta}| \geq \gamma_1 > 0$ on inner boundaries $\partial\Omega_i^- \backslash \partial\Omega$, $\forall i$, then*

$$\|u - \hat{u}^h\|_0 \leq C(\|u - u^I\|_1 + \epsilon \sum_i \|\frac{\partial u}{\partial n}\|_{0,\partial\Omega_i^- \backslash \partial\Omega}) , \tag{5}$$

where $u^I \in S_0^h$ is the interpolation of the solution u, C is a positive constant independent of h, ϵ and u.

Remark 2.2 *Compare (5) with the standard error estimate, one sees that the error resulted from the artificial boundary condition is only*

$$O(\epsilon)(\sum_i \|\frac{\partial u}{\partial n}\|_{0,\partial\Omega_i^- \backslash \partial\Omega}). \tag{6}$$

For convection-dominated problems, boundary layers and transient layers can appear inside the domain Ω. In getting the subdomains, we shall avoid the situation that the subdomain boundaries are parallel to the streamlines in the singular layers. Outside the singular layers, there is no problem. Due to the reason that the boundary layers are always narrow, i.e. of width less or equal $O(\epsilon)$, we can construct the subdomains in such a way that the part of $\partial\Omega_i^-$ contained in the singular layers is only of size $O(\epsilon)$. Then, in the worst case, the summation of the error from all the subdomains is

$$\epsilon \sum_i \|\frac{\partial u}{\partial n}\|_{\partial\Omega_i^- \backslash \partial\Omega} = O(\epsilon^{\frac{1}{2}}). \tag{7}$$

If linear finite elements are used for the approximation and the mesh size is h in the part of the domain where the solution is smooth, then the error caused by the artificial boundary condition is negligible when $\epsilon \leq O(h^2)$.

Remark 2.3 *If $\alpha = 0$, β = constant, then condition $\alpha + \frac{1}{2}div\boldsymbol{\beta} \geq \gamma > 0$ is not satisfied. Theorem 2.1 is still correct if we just replace $\|u - \hat{u}^h\|_0$ by $\|u - \hat{u}^h\|_A$. Here*

$$\|v\|_A^2 = \sum_i (\epsilon \nabla v, \nabla v)_{\Omega_i} + \frac{1}{2} \sum_i \int_{\partial\Omega_i^-} [v]^2 |\boldsymbol{n\beta}| ds \ ,$$

and $[v]$ denotes the jump of v on the inflow boundaries. So, we cannot control the errors in the L^2 norm, instead we can only control the errors on the subdomain boundaries, see [ETY96].

Remark 2.4 *The streamline diffusion finite element method (SDFEM) is stable and shall be used to compute the subdomain solutions preferably. Corresponding error estimate can also be obtained for SDFEM, see [ETY96].*

3 Some Discussions and Extensions

An Iterative Domain Decomposition Method

When the flow is complicated and there are closed streamlines, it could be difficult to construct the subdomains in such a way that the subdomain solutions can be computed in the flow direction and inflow boundary condition is always available when we come to compute the solution of a new subdomain. In this case, we only need to construct the subdomains to guarantee $\boldsymbol{n\beta} \geq \gamma_1 > 0$ on $\partial\Omega_i^-$, $\forall i$. Now, the subdomain solutions are all coupled to each other. An iterative scheme is needed. During the iteration, the inflow boundary condition is taken from the previous iterative step, and the algorithm can be written as:

Algorithm 1
Step 1. Choose initial value $\hat{u}_h^0$*;*
Step 2. For $n \geq 1$*, in every subdomain* Ω_i*, find* $\hat{u}_h^{n+1}|_{\Omega_i} = \hat{u}_i^{n+1} \in V_i$ *such that*

$$A_i(\hat{u}_i^{n+1}, v) = (f, v) - \int_{\partial\Omega_i^-} (\hat{u}_h^n)_- v_+ \boldsymbol{n\beta} ds, \quad \forall v \in V_i \ ; \tag{8}$$

Step 3. Go to the next iteration.

For the above scheme, it can be proved, see [ETY96], that when $\hat{u}^h$, $\hat{u}_h^n$ are the solutions of (3) and (8), $\alpha + \frac{1}{2} div\boldsymbol{\beta} \geq 0$, then the iterative scheme (8) is convergent, i.e. $\|\hat{u}^h - \hat{u}_h^n\|_0 \to 0$ as $n \to \infty$, and when $\alpha + \frac{1}{2} div\boldsymbol{\beta} \geq \gamma > 0$, the spectral radius of the iteration operator T_0 satisfies $\rho(T_0) \leq \left(\dfrac{1}{1 + C\epsilon + C\gamma h}\right)^{\frac{1}{2}}$.

Remark 3.1 *When ϵ is not small, different kinds of boundary condition on the outflow boundary should be used to improve the accuracy. For example, Lagrange multiplier can be used on the inner boundaries, see [WY97] for the details.*

Time-dependent Problems

Consider the time dependent convection-diffusion problem:

$$\begin{cases} u_t - div(\epsilon \bigtriangledown u) + div(\boldsymbol{\beta} u) = f, \text{ in } \Omega \times [0, T], \\ u(x, t) = 0, \text{ on } \partial\Omega \times [0, T], \quad u(x, 0) = u_0(x), \text{ in } \Omega. \end{cases}$$

We can use the backward difference scheme for t. In every time step, we just need to solve

$$-div(\epsilon \bigtriangledown \bar{u}^{k+1}) + div(\boldsymbol{\beta}\bar{u}^{k+1}) + \frac{\bar{u}^{k+1}}{\triangle t} = f + \frac{\bar{u}^k}{\triangle t}, \text{ in } \Omega, \tag{9}$$

with $\bar{u}^{k+1} = 0$ on $\partial\Omega$. Problem (9) is the same kind of problem as (1) with $\alpha = \frac{1}{\triangle t}$. So, the domain decomposition schemes can be used to solve (9). Similar to theorem 2.1, it can be proved that

$$\|\bar{u}^k - \hat{u}_h^k\|_0 \leq C(\epsilon\sqrt{\triangle t} \sum_i (\|\frac{\partial u}{\partial n}\|_{0,\partial\Omega_i^- \backslash \partial\Omega}) + \triangle t \, \|u - u^I\|_1 + \|u - u^I\|_0), \ \ \forall k,$$

where $\hat{u}_h^k$ is the domain decomposition solution of (9) by the non-iterative scheme (3). If the iterative domain decomposition is used to solve (9), because $\alpha = \frac{1}{\triangle t}$ is large, $\alpha + \frac{1}{2}div\boldsymbol{\beta} \geq 0$, so the iteration is convergent, and the spectral radius of the iteration operator is $\rho(T_0) \leq \left(\frac{1}{1 + C\epsilon + Ch(\triangle t)^{-1}}\right)^{\frac{1}{2}}$. Hence, when $\triangle t = O(h)$, $\rho(T_0) \leq C < 1$, i.e. the error reduction of the iteration is uniform. Especially, when $\triangle t = O(h^2)$, $\rho(T_0) \leq C\sqrt{h}$, therefore only a few iteration steps are required at every time level, see [ETY96] for the analyses.

4 Numerical Experiments

As a test example, we solve the model problem

$$-\epsilon \triangle u + divu + 2u = f, \text{ in } \Omega, \tag{10}$$

with $\Omega = [0,1] \times [0,1]$ and $u = 0$ on $\partial\Omega$. We choose

$$f = C_1(e^{a(1-x)} + e^{a(1-y)}) + C_2(e^{b(1-x)} + e^{b(1-y)}) + 2$$

with

$$C_1 = \frac{1-e^b}{e^b - e^a}, \quad C_2 = \frac{e^a - 1}{e^b - e^a}, \quad a = \frac{-1+\sqrt{1+4\epsilon}}{2\epsilon}, \quad b = \frac{-1-\sqrt{1+4\epsilon}}{2\epsilon}.$$

Then the analytical solution is

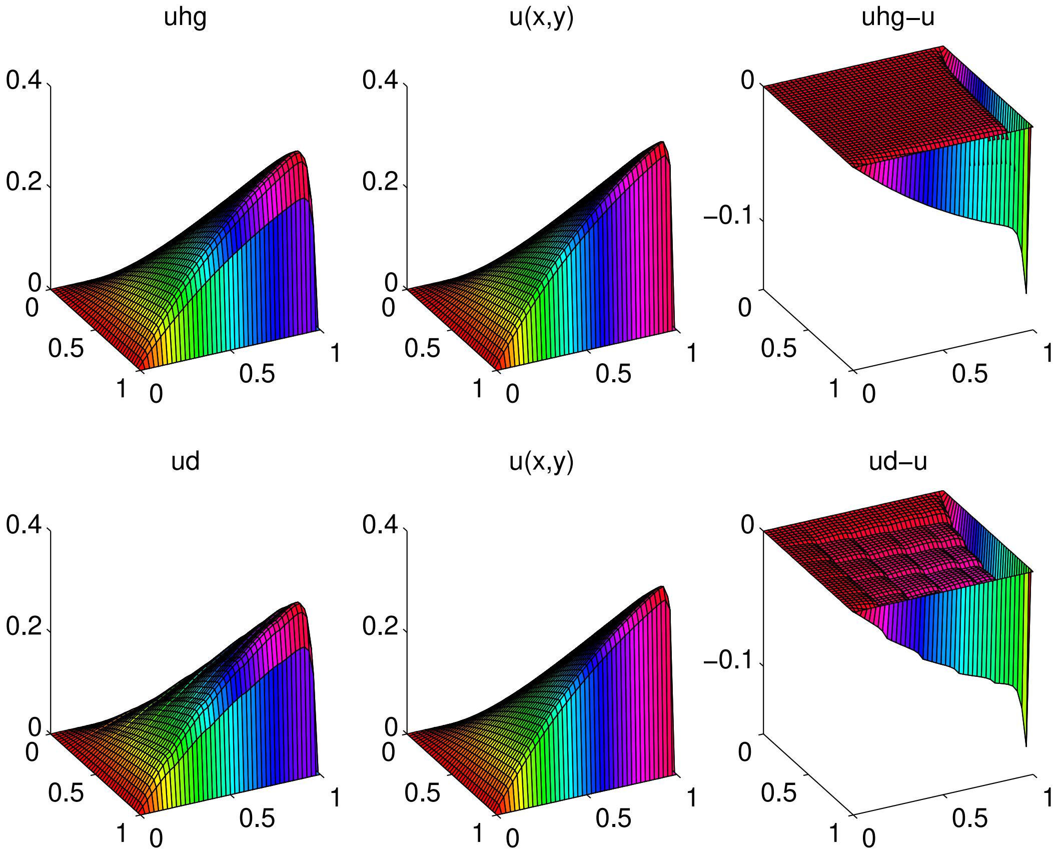
$$u = (C_1 e^{a(1-x)} + C_2 e^{b(1-x)} + 1)(C_1 e^{a(1-y)} + C_2 e^{b(1-y)} + 1).$$

In the computations, the domain Ω is divided into 5×5 subdomains. Piecewise linear finite element functions on uniform triangular meshes is used. In each subdomain, a first order upwind approximation is used for the convection term and the inflow boundary condition is realised exactly which is taken from the subdomains in the upwind direction. Let i=1,2,3,4,5, and j=1,2,3,4,5 be the numbers associated with the subdomains in the x- and y-directions. We solve the subdomain problems by first sweeping over i=1,2,3,4,5 and then sweeping over j=1,2,3,4,5. By solving the subdomain problems in this order, the inflow boundary condition is always available when we come to compute a subdomain solution.

In table 1, some numerical results for different ϵ and different mesh sizes h are shown, where $\|e_g\|_0$ and $\|e_d\|_0$ represent the error of the global finite element solution and the error of the domain decomposition solution for problem (10) in L^2-norm, respectively.

Figure 1 shows the computed solutions and their errors for $\epsilon = 0.01$ and $h = 0.025$, where u, uhg and ud represent the exact solution, the global finite element solution and the domain decomposition solution of (10), respectively. From table 1 and figure 1, one observes that when ϵ is small, the error of the domain decomposition solution is of the same order as the global finite element solution (see Table 1 for ϵ =0.01, 0.001, 0.00001). From figure 1, one finds that the large errors both for the global FEM solution and the domain decomposition solution are concentrated in the neighbourhood of

Figure 1 The global FEM solution and the domain decomposition solution for $\epsilon = 0.01,\ h = 0.025$ and the corresponding errors.



the outflow boundary. Due to the relative large mesh size used near the outflow boundary, the boundary layer is not properly resolved. Here comes the advantage of the proposed domain decomposition methods. Once we know that which subdomain contains the singular layers, we can use finer mesh in this subdomain. By doing so, the error introduced by the artificial boundary condition does not increase, but the singular layers can be efficient resolved by using the known boundary conditions from the neighbouring subdomains and a sufficient fine mesh in this subdomain. Different examples have been tested by the proposed algorithms. The numerical results always show that when ϵ is small, the domain decomposition solution and the global finite element solution have errors of the same order and the large errors are in the singular layers. To do grid refinement for the global problem is not easy, but it is very easy to use fine meshes for the subdomains that contains the singular layers.

Table 1. L^2-error of the global solution and the domain decomposition solution.

	$\epsilon = 0.1$		$\epsilon = 0.01$		$\epsilon = 0.001$		$\epsilon = 0.00001$	
	$\|e_g\|_0$	$\|e_d\|_0$	$\|e_g\|_0$	$\|e_d\|_0$	$\|e_g\|_0$	$\|e_d\|_0$	$\|e_g\|_0$	$\|e_d\|_0$
h=0.1	0.0155	0.0354	0.0126	0.0200	0.0066	0.0095	0.0061	0.0084
h=0.05	0.0085	0.0323	0.0131	0.0186	0.0038	0.0051	0.0029	0.0035
h=0.025	0.0045	0.0297	0.0122	0.0165	0.0029	0.0037	0.0014	0.0016
h=0.0125	0.0023	0.0280	0.0082	0.0122	0.0032	0.0036	0.0007	0.0007

5 Conclusion

Both theoretical analysis and numerical tests reveal that the proposed algorithms are suitable for problems with small ϵ. When the diffusion parameter is small, the singular layers are very narrow. In order to resolve the singular layers, the ratio between the mesh size in the singular layers and the mesh size in the part of the domain where the solution is smooth shall be very large. In this case, the error introduced by the domain decomposition algorithms are negligible in comparison with the errors in the singular layers. However, the domain decomposition algorithms allow easy and efficient grid refinement in the subdomains that contain the singular layers.

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