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A Characteristic Domain Decomposition Method for Modeling Flow in a Coastal Aquifer

Helge K. Dahle, Torbjørn O. Widnes Johansen, Tone Botnen, and Xue-Cheng Tai

1 Introduction

A characteristic domain splitting method is implemented for the concentration equation of a coastal aquifer with intrusion and discharge. In each timestep the concentration is advected along streamlines. The diffusion part is then solved using an overlapping domain decomposition technique. If the overlapping size is suitably chosen, no iterations are needed between the subdomain problems at each time level. If the diffusion parameter or the timestep is sufficiently small, only one or two elements of overlap is needed. For problems with large diffusion, or if we use large timesteps, a few iterations between the subproblems are needed to further reduce the domain decomposition error. Numerical results show the potential of this method for the ground water flow problem.

Salt water consists of one liquid phase composed of salt and water components. The mixing of salt and fresh water in coastal aquifers may be described by Darcy's law and conservation of mass. Here, a two-dimensional model of a coastal aquifer is considered [Bot93, SRMS92]. The x-direction is aligned with the main horizontal flow direction and z denotes the vertical direction pointing upwards from the bottom of the aquifer; see Figure 1. We shall assume that the aquifer is completely saturated by water, that the density of water depends linearly on salt concentration, i.e., $\rho = \rho_0(1 + \beta c)$, that the hydraulic conductivity tensor is isotropic and diagonal, $\mathbf{K} = K(\mathbf{x})\mathbf{I}$, and porosity $n = 0.3$. These assumptions leads to a somewhat simplified set of governing equations for $(\mathbf{x}, t) \in \Omega \times (0, T]$:

$$\mathbf{q} = -K(\nabla\phi + \beta c \nabla z), \tag{1.1}$$

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$$-K\nabla^2\phi = \beta(K\frac{\partial c}{\partial z} - n\frac{Dc}{Dt}), \tag{1.2}$$

$$\frac{Dc}{Dt} - \nabla \cdot D_h \nabla c = 0. \tag{1.3}$$

Here $\mathbf{q} = n\mathbf{v}$ is the volumetric flow, $\mathbf{v}$ is the particle velocity, c is the concentration, $\phi = p/\rho_0 g + z$ is the fresh water head, D_h represents hydromechanical dispersion and $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ denotes the convective derivative. According to site investigations, a good estimate of the dispersion term is $D_h \sim 10^{-5} m^2/s$, see [SRMS92]. However, we note that this term generally is given by a tensor which is difficult to model and may be impossible to measure by direct means, e.g., [BB90]. Hence, a reliable and efficient numerical model may be needed to estimate this term. A similar statement may be made about the conductivity. However, this term is somewhat easier to measure by direct means, and in Figure 1 data for the conductivity are given. The flow takes place in a two-dimensional region Ω bounded by a river, a channel, the ocean (fjord) and impervious rock. For simplicity, we assume stationary boundary conditions. The boundary conditions, being a mixture of Neumann and Dirichlet conditions, are shown in Figures 2 and 3. Initially, the aquifer is completely filled with salt water. Since the boundary conditions are stationary this leads to a stationary solution after some time ($\sim$ 200 days), describing a mixing zone between fresh and salt water. We do not intend to describe the effect of tidal water, seasonal changes or wells on this mixing zone. Typical length and time scales for this problem are $L = 100m$ and $T = 100$ days respectively. This gives a typical diffusion in the range $0.005 < D_h < 0.5$. For simplicity we keep the parameters in dimensional form, although concentration is scaled to vary between 0 and 1 in the figures. The main aim of the present paper is to report numerical experiments when the solution method for the advection-dispersion equation is a part of a complicated, coupled, nonlinear system of partial differential equations. This is done in section 1.3. The rest of the paper describes how equation (1.3) is solved, by extending and modifying the algorithm given in [TJDE96].

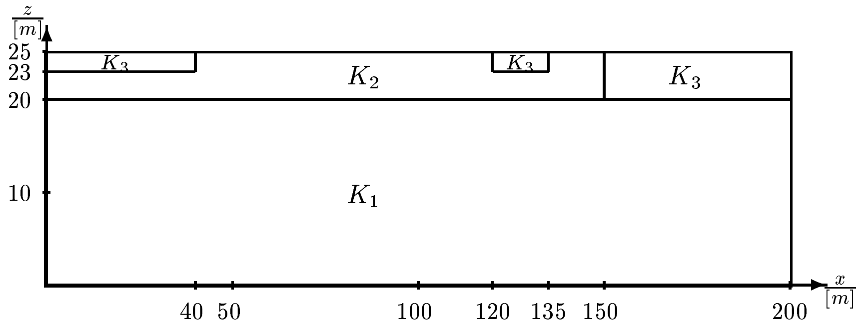
2 Algorithms

Equations (1.1)-(1.3) are solved using a sequential time-marching procedure: At each time level t^n the pressure/velocity equations (1.1), (1.2) are solved using the previous known concentration values. The concentration distribution is then updated using the new velocity field. In this way the issue of determining pressure/velocity is decoupled from the problem of finding a concentration distribution. In particular, the pressure velocity equations are solved by a control volume technique, see [DEEa90]. Here, we focus on the transport equation (1.3), describing the mixing process. Thus, from now on the velocity field is assumed to be a known function of space and time.

Discretization

The concentration equation will be solved by the Modified Method of Characteristics (MMOC), see [DES92, DR82]. This choice is important in two ways. First, it gives

Figure 1 Hydraulic conductivity of the modeled profile.
$K_1 = 6.5 \cdot 10^{-4}\, m/s$, $K_2 = 6.5 \cdot 10^{-3}\, m/s$, and $K_3 = 0.1\, m/s$.



an accurate solution even for large timesteps since the coefficients of the time-truncation error only depends on higher order derivatives along the (approximate) characteristics, see [DR82]. Secondly, it gives very accurate internal boundary values for the subdomains, to be used in the domain decomposition method. To be more precise, let $S_h(\Omega) \subset H^1(\Omega)$ be the space of piecwise bilinear functions on a rectangular discretization of Ω. This defines a finite element discretization $\Omega = \cup_{e \in \mathcal{T}_h} e$. Let V^h be the subset of $S_h(\Omega)$ satisfying the given Dirichlet conditions and $S_h^0(\Omega)$ be the subspace of functions which are zero at the Dirichlet part of the boundary. The MMOC approximation may then be written: For $n = 1, 2, \ldots$, find $c^n \in V_h$ such that

$$(c^n, v) + (\Delta t D_h \nabla c^n, \nabla v) = (\bar{c}^n, v) + \mathrm{B.}T., \quad \forall v \in S_h^0(\Omega). \tag{2.4}$$

Here $\Delta t = t^n - t^{n-1}$ is the timestep, $(\cdot, \cdot)$ is the usual L_2-inner product on Ω and B.T. denotes boundary terms. The characteristic solution $\bar{c}^n \in V_h$ is obtained by tracking particle trajectories backwards in time from each node $\mathbf{x}_i$, i.e., $\bar{c}^n(\mathbf{x}_i) = c^{n-1}(\bar{\mathbf{x}}(\mathbf{x}_i, t^{n-1}))$, where

$$\frac{d\bar{\mathbf{x}}}{d\tau} = \mathbf{v}(\bar{\mathbf{x}}(\mathbf{x}_i, \tau), \tau) \;\; and \;\; \bar{\mathbf{x}}(\mathbf{x}_i, t^n) = \mathbf{x}_i, \quad \tau \in [t^{n-1}, t^n]. \tag{2.5}$$

In the present work $\mathbf{v}(\mathbf{x}, t)$ is approximated by $\mathbf{v}(\mathbf{x}, t^n)$. If tidal effects, etc., are important linear interpolation between successive time levels may be necessary. In the numerical experiments presented here, Equation (2.5) is solved by a one-point approximation using the tangent of the velocity field at the node in consideration. Note that this can be done in parallel and gives a fast and robust method. More accurate methods are, e.g., analytical integration or Runge-Kutta methods. The main difficulty with the MMOC is when a characteristic traced backward in time crosses the physical boundary; see [WDE+96] and references therin. This may happen at an inflow or a noflow boundary. In the present case, inflow boundaries are easy to treat since concentration values are specified to be either 0 or 1. Crossing a noflow boundary, due to inexact tracing, is treated by projecting the characteristic back onto the boundary.

Figure 2 Boundary conditions for the fresh water head equation.

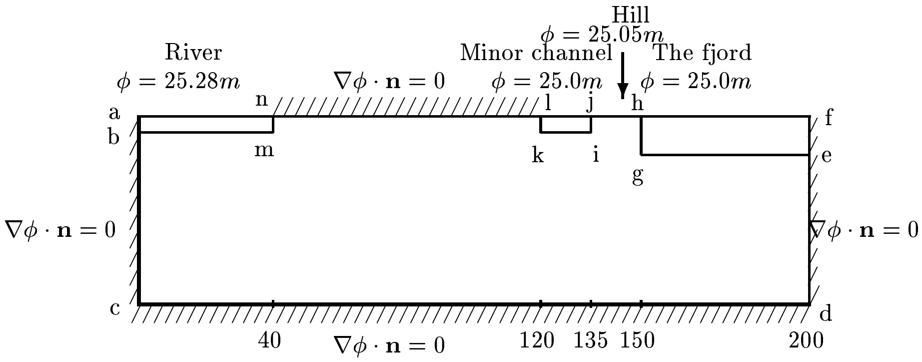
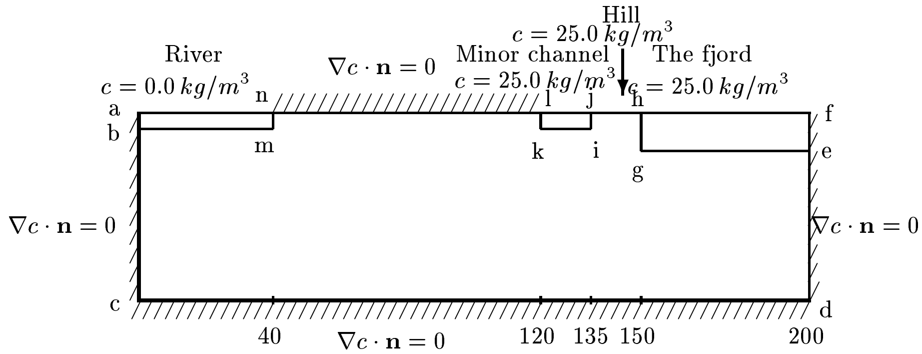


Figure 3 Boundary conditions for the concentration equation.



Characteristic Domain Decomposition

Let Ω be decomposed into M nonoverlapping subdomains Ω_i. For simplicity Ω is only subdivided in the horizontal direction. To each Ω_i we associate an enlarged subdomain

$$\Omega_i^\delta = \{e \in \mathcal{T}_h |\ dist(e, \Omega_i) \le \delta\},$$

which forms an overlapping domain decomposition of Ω with overlapping size δ. In practice δ is measured in terms of the number of elements that Ω_i extends into its neighbors. On each subdomain, solve the following problem: Find $c_i^n \in V_i^h$ such that $c_i^n = \bar{c}^n$ on $\partial\Omega_i^\delta \backslash \partial\Omega$ and

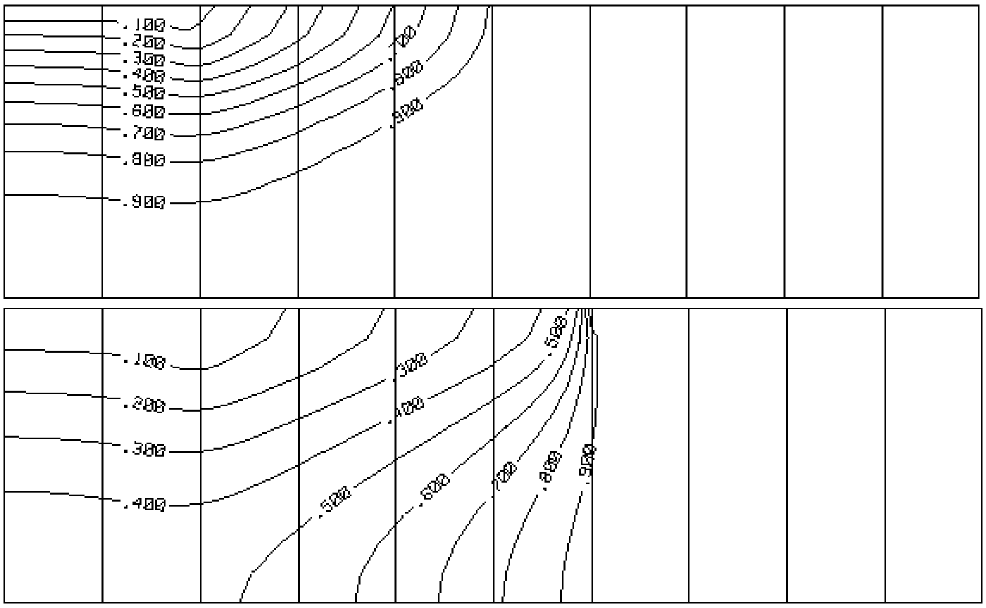
$$(c_i^n, v) + (\Delta t D_h \nabla c_i^n, \nabla v) = (\bar{c_i}^n, v) + \text{B}.T., \quad \forall v \in S_h^0(\Omega_i^\delta). \tag{2.6}$$

Here, $S_h^0(\Omega_i^\delta)$ is the finite element subspace of functions which are zero at the Dirichlet part of $\partial\Omega_i^\delta$ and V_i^h is the restriction of V^h to Ω_i^δ.

Note that the subproblems can be solved in parallel. After solving the subproblems, we assemble a global solution from the subdomain solutions. The following algorithm may now be stated, see also [TJDE96]:

Algorithm *For each time level t^n:*

Figure 4 10 (top) and 50 (bottom) timesteps



1. Solve (2.5) to obtain $\bar{c}^n \in V^h$*;*
2. Solve (2.6) on each subdomain Ω_i^δ *to get* c_i^n*;*
3. From the patchwise solution c_i^n*, a global solution*

$$c^n = \mathcal{C}\left(\{c_i^n\}_{i=1}^M\right) \in V^h$$

is constructed such that

$$\| c^n \|_{L^2(\Omega)} \leq \sum \| c_i^n \|_{L^2(\Omega_i)}; \tag{2.7}$$

4. If $t^n < T$*, got to next time level;*

We may iterate between Step 2 and 3 to further improve the solution. Step 3 is achieved in practice by a cutting and averaging technique [BLR92, TJDE96], i.e. we set the value of c^n by

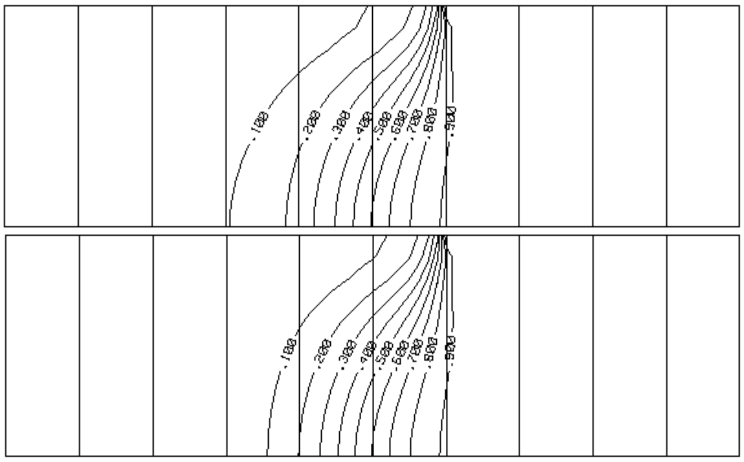
$$c^n(x_k) = \begin{cases} c_i^n(x_k), & \text{if } x_k \text{ is an inner node of } \Omega_i, \\ \frac{1}{2}(c_i^n(x_k) + c_j^n(x_k)), & \text{if } x_k \text{ is a node on the interface between of } \Omega_i \text{ and } \Omega_j. \end{cases}$$

It was proved in [TDE] that if

$$\delta > c_0 \max(\sqrt{\epsilon \Delta t}, h)|ln \Delta t|,$$

where c_0 is a constant associated with the finite element mesh and h is the mesh size, then the computed solution c^n is of first order of convergence with respect to Δt and second order of convergence with respect to h.

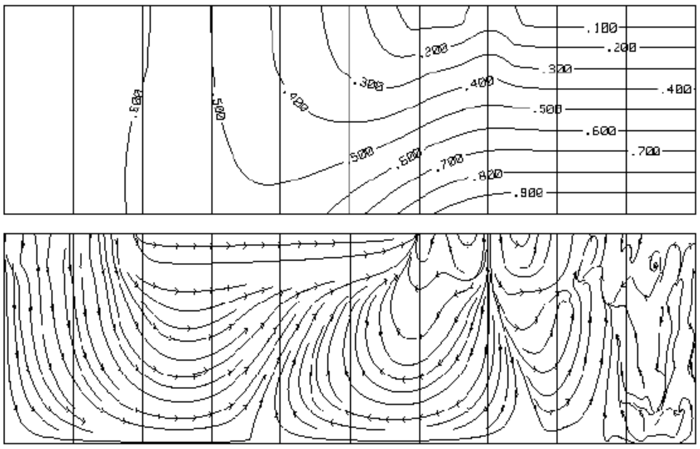
Figure 5 Concentration distribution. $D_h = 5 \cdot 10^{-5} m^2/s$.



3 Numerical Experiments

In the experiments performed, the problem is solved on a uniform grid with 81×11 grid lines, and the domain is divided into 8 equally sized subdomains. The timestep is fixed to be 24 hours. Initially the aquifer is filled with salt water ($c = 1$) and the boundary conditions are prescribed as shown in Figure 3. Fresh water ($c = 0$), then infiltrates from the river (upper right corner). In Figure 5 contour plots of the concentration distribution is shown after 10, 50, 150 and 250 timesteps for $D_h = 5 \cdot 10^{-5} m^2/s$. Figure 6 show the related pressure distribution and streamlines after 250 timesteps. At this point the problem has reached a stationary solution which is independent of the initial conditions. A reference (global) solution is computed on the same grid without domain decomposition. Figures 7 - 8 compare the global solution with the domain decomposition solution in a discrete L_2-norm, i.e. the L_2-error. Figure 7 show the error as a function of number of overlapping elements (no iterations) for a fairly small diffusion ($D_h = 5 \cdot 10^{-5} m^2/s$). The error decays as expected and only 2-3 elements were necessary to obtain an accurate solution. This is due to the fact that the characteristic solution is nearly exact for small diffusion. On the other hand, the discontinuity at the inflow boundary produced a transient phase with big errors. After two timesteps the error could not be forced to zero by a reasonable increase of the number of overlapping elements, as shown in Figure 7. This problem is explained by the fact that a small change in the velocity field, caused by small differences in concentration values, produces big differences in the concentration values in vicinity of a discontinuous infiltration front. The problem disappeared after $\sim$ 50 timesteps. Figure 8 shows the error for a large diffusion ($D_h = 3 \cdot 10^{-4} m^2/s$). For this problem a combination of 2-3 elements of overlap and 1-2 iterations between the subdomains at each time level reduced the error to an accepted level. We did not observe any difficulties in this case, since the infiltration front was immediately smeared by the

Figure 6 Fresh water head and streamlines after 250 timesteps.



diffusion term.

4 Conclusions

The combination of sequential timestepping and characteristic domain decomposition as described above, is easy to implement and gives fast and robust methods. The numerical experiments performed here show results that are expected from analysis for a single linear advection diffusion equation. In fact, for small diffusion 2-3 elements of overlap without iterations seems to be sufficient, for a larger diffusion 2-3 elements of overlap combined with 1-2 iterations between the subdomains at each time level are needed. However, more experiments have to be done and the algorithms should be implemented on a parallel machine for measuring speed up times.

Acknowledgement

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Figure 7 Error after 250 timesteps (top figure) and after 2 timesteps (bottom figure) as a function of the number of overlapping elements. $D_h = 5 \cdot 10^{-5} m^2/s$.

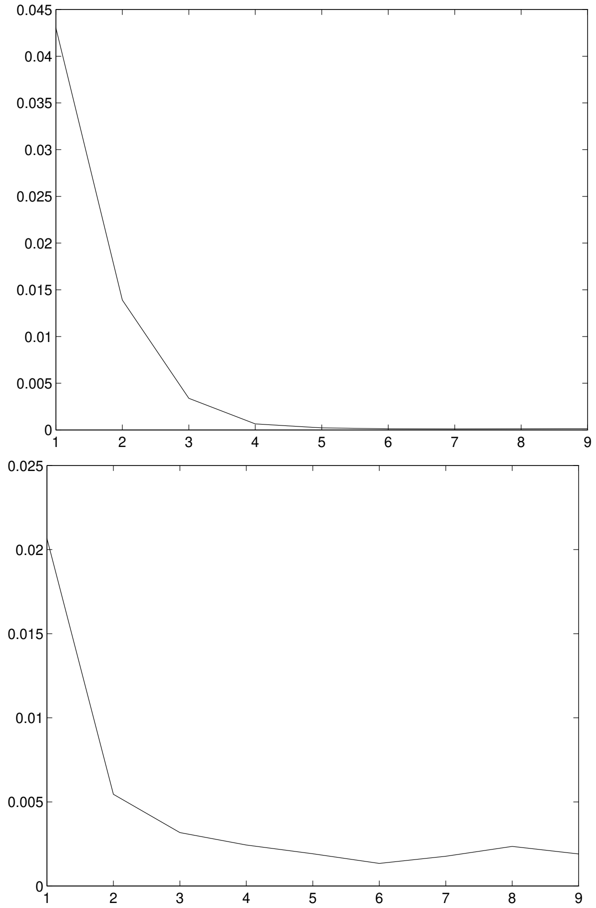

Figure 8 Error after 250 timesteps as a function of the number of overlapping elements. $D_h = 3 \cdot 10^{-4} m^2/s$.