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Overlapping Domain Decomposition for a Mixed Finite Element Method in Three Dimensions

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1 Introduction

The work presented in this talk was motivated by the following question: Is it possible to find a computational technique for solving the linear system resulting from Mixed Finite Element discretizations of certain self-adjoint second order elliptic boundary value problems at the computational cost of a standard Galerkin FEM? There are two main obstacles to achieving the above stated goal. First, Mixed FEM formulations lead to a significantly larger number of unknowns compared to a standard conforming FEM of a comparable accuracy on the same triangulation of the domain. And secondly, the Mixed FEM produces a symmetric indefinite matrix problem as opposed to a symmetric and positive definite matrix in the standard FEM. The combined effect of these difficulties can often discourage end users from using Mixed Methods even in applications where a mixed approach can be beneficial (see [MSAC94]).

A number of researchers have studied this problem over the years and have contributed to developing several efficient iterative solvers. We acknowledge all their work, but for brevity, we will only consider in this talk approaches that involve Domain Decomposition ideas.

In [EW92], Ewing and Wang considered and analyzed a domain decomposition method for solving the discrete system of equations which result from mixed finite element approximation of second-order elliptic boundary value problems in two dimensions. The approach in [EW92] is first to seek a discrete velocity satisfying the discrete continuity equation through a variation of domain decomposition (static condensation), and then to apply a domain decomposition method to the reduced elliptic problem arising from elimination of the pressure and part of the velocity unknowns in the saddle-point problem. The crucial part of the approach in [EW92] is to characterize the divergence-free velocity subspaces. This is also the essential

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difference with those in [GW87], [MR94], and [CMW95]. The Lagrange multipliers approach used in [GW87, CMW95] does produce a symmetric and positive definite matrix, but fails to address the other issue: the large number of unknowns. Recently, Chen, Ewing and Lazarov suggested in [CEL96] to reduce the number of unknowns by eliminating on a element by element basis the pressure variables. Then they applied a DD algorithm on the Lagrange multiplier variables only.

When comparing the two basic ideas (i.e. Lagrange multipliers and div-free subspace) one notes that they both re-formulate the original saddle-point problem into a symmetric and positive definite one and then apply some known DD algorithm. The difference is the number of unknowns in the discrete system. The dimension of the div-free velocity subspace is always smaller than the number of Lagrange multipliers.

In this paper, we will use the domain decomposition approach in [EW92] for the solution of the algebraic system resulting from the mixed finite element method applied to second-order elliptic boundary value problems in three dimensions. As mentioned above, the basis of the divergence-free velocity subspace plays an essential role in the approach; hence we will construct a basis of this subspace for the lowest-order rectangular Raviart-Thomas-Nedelec [RT77, Ned80] velocity space. The construction in two dimensions is rather easier than in three dimensions due to the fact that any divergence-free vector in 2-D can be expressed as the curl of a scalar stream function. Extension of this work to triangular or irregular meshes and to multilevel domain decomposition will be discussed in a forthcoming paper.

This approach has several practical advantages. For an $n \times n \times n$ grid in 3-D, the number of discrete unknowns is approximately $4n^3$, essentially one pressure and three velocity components per cell. Using the divergence-free subspace, we decouple the system in such a manner that the velocity can be obtained directly by solving a symmetric positive definite system of order roughly $2n^3$ thus coming closer than any other approach so far to the number of degrees of freedom in a standard Galerkin FEM. In contrast to some other proposed procedures, this does not require the introduction of Lagrange multipliers corresponding to pressures at cell interfaces, and it permits direct computation of the velocity, which is often the principal variable of interest, alone. If the pressure is also needed, it can be calculated inexpensively in an additional step. Furthermore, the approach deals readily with the case of full-tensor conductivity (cross-derivatives), where the mass matrix is fuller than tridiagonal and methods based on reduced integration (mass lumping) are difficult to apply. This case results, for example, from anisotropic permeabilities in flows in porous media, where highly discontinuous conductivity coefficients are also common. For such problems, mixed methods are known to produce more realistic velocities than standard techniques [MSAC94].

2 Mixed Finite Element Method

In this section, we begin with a brief review of the mixed finite element method with lowest-order Raviart-Thomas-Nedelec [RT77, Ned80] approximation space for second-order elliptic boundary value problems in three dimensions. For simplicity, we consider a homogeneous Neumann problem: find p such that

$$\left\{\begin{array}{rcll} -\nabla\cdot(k\nabla p) & = & f, & \text{in} \quad \Omega=(0,\,1)^3, \\ (k\nabla p)\cdot\mathbf{n} & = & 0, & \text{on} \quad \partial\Omega, \end{array}\right. \tag{1}$$

where $f \in L^2(\Omega)$ satisfies the relation $\int_\Omega f = 0$, and $\mathbf{n}$ denotes the unit outward normal vector to $\partial\Omega$. The symbols $\nabla\cdot$ and ∇ stand for the divergence and gradient operators, respectively. Assume that $k = (k_{ij})_{3\times 3}$ is a given real-valued symmetric matrix function with bounded and measurable entries k_{ij} $(i,\, j = 1,\, 2,\, 3)$ and satisfies an ellipticity condition a.e. in Ω.

We shall use the following space to define the mixed variational problem. Let

$$H(div;\Omega) \equiv \{\mathbf{w} \in L^2(\Omega)^3 \mid \nabla\cdot\mathbf{w} \in L^2(\Omega)\},$$

which is a Hilbert space when equipped with the standard norm and the associated inner product. By introducing the flux variable

$$\mathbf{v} = -k\nabla p,$$

which is often of practical interest for many physical problems, we can rewrite the PDE of (1) as a first-order system

$$\left\{\begin{array}{rcl} k^{-1}\mathbf{v} + \nabla p & = & 0, \\ \nabla\cdot\mathbf{v} & = & f, \end{array}\right.$$

and obtain the mixed formulation of (1): find $(\mathbf{v},\, p) \in \mathbf{V}\times\Lambda$ such that

$$\left\{\begin{array}{lcll} a(\mathbf{v},\,\mathbf{w}) - b(\mathbf{w},\,p) & = & 0, & \forall\,\mathbf{w}\in\mathbf{V}, \\ b(\mathbf{v},\,\lambda) & = & (f,\,\lambda), & \forall\,\lambda\in\Lambda. \end{array}\right. \tag{2}$$

Here $\mathbf{V} = H_0(div;\Omega) \equiv \{\mathbf{w} \in H(div;\Omega) \mid \mathbf{w}\cdot\mathbf{n} = 0 \text{ on}\,\partial\Omega\}$, Λ is the quotient space $L_0^2(\Omega) = L^2(\Omega)/\{\text{constants}\}$, the bilinear forms $a(\cdot\,,\cdot) : \mathbf{V}\times\mathbf{V} \to I\!R$ and $b(\cdot\,,\cdot) : \mathbf{V}\times\Lambda \to I\!R$ are defined by

$$a(\mathbf{w},\,\mathbf{u}) = \int_\Omega (k^{-1}\mathbf{w})\cdot\mathbf{u}\,dx\,dy\,dz \quad \text{and} \quad b(\mathbf{w},\,\lambda) = \int_\Omega (\nabla\cdot\mathbf{w})\lambda\,dx\,dy\,dz$$

for any $\mathbf{w},\,\mathbf{u}\in\mathbf{V}$ and $\lambda\in\Lambda$, respectively, and $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product.

To discretize the mixed formulation (2), we assume that we are given two finite element subspaces

$$\mathbf{V}^h \subset \mathbf{V} \quad \text{and} \quad \Lambda^h \subset \Lambda$$

defined on a uniform rectangular mesh with elements of size $O(h)$. The mixed approximation of $(\mathbf{v}\,,p)$ is defined to be the pair, $(\mathbf{v}^h,\,p^h) \in \mathbf{V}^h\times\Lambda^h$, satisfying

$$\left\{\begin{array}{lcll} a(\mathbf{v}^h,\,\mathbf{w}) - b(\mathbf{w}\,,p^h) & = & 0, & \forall\,\mathbf{w}\in\mathbf{V}^h, \\ b(\mathbf{v}^h,\,\lambda) & = & (f,\,\lambda), & \forall\,\lambda\in\Lambda^h. \end{array}\right. \tag{3}$$

We refer to [RT77] for the definition of a class of approximation subspaces $\mathbf{V}^h$ and Λ^h. In this paper, we shall only consider the lowest-order R-T-N space defined on a rectangular triangulation of Ω. However, as shown in [CPRY95], this construction can readily be generalized to higher-order elements on non-orthogonal meshes and more general boundary conditions.

3 Domain Decomposition

Problem (3) is clearly symmetric and indefinite. To reduce it to a symmetric positive definite problem, we need a discrete velocity $\mathbf{v}_I^h \in \mathbf{V}^h$ satisfying

$$b(\mathbf{v}_I^h, \lambda) = (f, \lambda), \quad \forall \lambda \in \Lambda^h. \tag{4}$$

Define the discretely (as opposed to pointwise) divergence-free subspace $\mathbf{D}^h$ of $\mathbf{V}^h$:

$$\mathbf{D}^h = \{\mathbf{w} \in \mathbf{V}^h \,|\, b(\mathbf{w}, \lambda) = 0, \quad \forall \lambda \in \Lambda^h\}, \tag{5}$$

and let

$$\mathbf{v}_D^h = \mathbf{v}^h - \mathbf{v}_I^h,$$

which is obviously in $\mathbf{D}^h$ by the second equation of (3) and satisfies

$$a(\mathbf{v}_D^h, \mathbf{w}) = -a(\mathbf{v}_I^h, \mathbf{w}), \quad \forall \mathbf{w} \in \mathbf{D}^h, \tag{6}$$

by the first equation. This problem is symmetric and positive definite.

This suggests the following procedure to obtain $\mathbf{v}^h$, the solution of (3): find $\mathbf{v}_I^h \in \mathbf{V}^h$ satisfying (4), compute the projection $\mathbf{v}_D^h \in \mathbf{D}^h$ satisfying (6), then set $\mathbf{v}^h = \mathbf{v}_I^h + \mathbf{v}_D^h$. This procedure will be the basis for Algorithms 3.1 and 3.2 below. Given $\mathbf{v}_I^h$, (6) leads to a unique $\mathbf{v}^h$, which is independent of the choice of $\mathbf{v}_I^h$. For an $n \times n \times n$ grid, computing the projection $\mathbf{v}_D^h$ involves solving a SPD system of order approximately $2n^3$. Solving for p^h is optional; if it is desired, it can be obtained from the first equation in (3) once $\mathbf{v}^h$ is known.

There are many discrete velocities in $\mathbf{V}^h$ satisfying (4), and several approaches have been discussed in the literature for seeking such a discrete velocity (e.g., [EW92], [GW87], and [MR94]). All of these approaches are based on a type of domain decomposition (static condensation) method applied to problem (3). In a recent paper [CPRY95], we suggested a different approach which only requires solving a number of independent one-dimensional problems.

We shall use additive and multiplicative domain decomposition methods for approximate computation of the solution of problem (6). As usual, we first decompose the original domain Ω into non-overlapping subdomains $\Omega = \cup\tilde{\Omega}_j$, where each subdomain $\tilde{\Omega}_j$ has a diameter of size H and then extend generously (i.e. with overlap of order H) each $\tilde{\Omega}_j$. The restriction of any FE space to the coarse grid defined by the non-overlapping subdomains will be denoted by the index H, and the restriction to Ω_j by the index j. Next, we define the family of discretely divergence-free velocity subspaces $\{\mathbf{D}_j\}_{j=0}^J$ by $\mathbf{D}_0 = \mathbf{D}^H$, and for $j \in \{1, 2, ..., J\}$,

$$\mathbf{D}_j = \{\mathbf{u} \in \mathbf{V}_j \,|\, b(\mathbf{u}, \lambda) = 0, \quad \forall \lambda \in \Lambda_j\}.$$

For any $\mathbf{u} \in \mathbf{D}^h$, we define the projection operators $\mathbf{P}_j : \mathbf{D}^h \longrightarrow \mathbf{D}_j$ associated with the bilinear form $a(\cdot\,,\cdot)$ by

$$a(\mathbf{P}_j\mathbf{u}, \mathbf{w}) = a(\mathbf{u}, \mathbf{w}), \quad \forall \mathbf{w} \in \mathbf{D}_j,$$

for $j \in \{0, 1, ..., J\}$.

Algorithm 3.1 (Additive Domain Decomposition) *1. Compute* $\mathbf{v}_I^h \in \mathbf{V}^h$ *as in [CPRY95].*

2. Compute an approximation, $\mathbf{v}_D$*, of* $\mathbf{v}_D^h \in \mathbf{D}^h$ *by applying conjugate gradient iteration to*

$$\mathbf{P}\mathbf{v}_D = \mathbf{F} \tag{7}$$

where $\mathbf{P} = \mathbf{P}_0 + \mathbf{P}_1 + \cdots + \mathbf{P}_J$*,* $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1 + \cdots + \mathbf{F}_J$*, and* $\mathbf{F}_j = \mathbf{P}_j \mathbf{v}_D^h$*.*

3. Set

$$\mathbf{v}^h = \mathbf{v}_D + \mathbf{v}_I^h.$$

Remark 3.1 *The right-hand side* $\mathbf{F}$ *in* (7) *can be computed by solving the coarse-grid problem and local subproblems. Specifically, for each* $j \in \{0, 1, ..., J\}$*,* $\mathbf{F}_j$ *is the solution of the following problem:*

$$a(\mathbf{F}_j, \mathbf{w}) = a(\mathbf{P}_j \mathbf{v}_D^h, \mathbf{w}) = -a(\mathbf{v}_I^h, \mathbf{w}), \quad \forall\, \mathbf{w} \in \mathbf{D}_j. \tag{8}$$

Algorithm 3.2 (Multiplicative Domain Decomposition) *1. Compute* $\mathbf{v}_I^h$ *as in the first step of* Algorithm 4.

2. Given an approximation $\mathbf{v}_D^l \in \mathbf{D}^h$ *to the solution* $\mathbf{v}_D^h$ *of* (6)*, define the next approximation* $\mathbf{v}_D^{l+1} \in \mathbf{D}^h$ *as follows:*

a) Set $W_{-1} = \mathbf{v}_D^l$*.*

b) For $j = 0, 1, ..., J$ *in turn, define* W_j *by*

$$W_j = W_{j-1} + \omega \mathbf{P}_j (\mathbf{v}_D^h - W_{j-1})$$

where the parameter $\omega \in (0, 2)$*.*

c) Set $\mathbf{v}_D^{l+1} = W_J$*.*

3. Set

$$\mathbf{v}^h = \mathbf{v}_I^h + \mathbf{v}_D^L.$$

Remark 3.2 $\mathbf{P}_j(\mathbf{v}_D^h - W_{j-1})$ *can be computed by solving the following problem:*

$$a(\mathbf{P}_j(\mathbf{v}_D^h - W_{j-1}), \mathbf{w}) = -a(\mathbf{v}_I^h + W_{j-1}, \mathbf{w}), \quad \forall\, \mathbf{w} \in \mathbf{D}_j. \tag{9}$$

A simple computation implies that the error propagation operator of multiplicative domain decomposition at the second step of Algorithm 3.2 has the form of

$$\mathbf{E} = (\mathbf{I} - \mathbf{P}_J)(\mathbf{I} - \mathbf{P}_{J-1}) \cdots (\mathbf{I} - \mathbf{P}_0). \tag{10}$$

Define a norm associated with the bilinear form $a(\cdot, \cdot)$ by

$$\|\mathbf{u}\|_a = a(\mathbf{u}, \mathbf{u})^{1/2}, \quad \forall\, \mathbf{u} \in \mathbf{D}^h.$$

We shall show in the last section that $\|\mathbf{E}\|_a$ is bounded by a constant which is less than one and independent of the mesh size h and the number of subdomains.

4 Construction of Divergence-Free Basis

Since the technique of the mixed method leads to a saddle-point problem which causes the final system to be indefinite, many well-established efficient linear system solvers cannot be applied. As we mentioned earlier, (3) could be symmetric and positive definite if we discretize it in the discrete divergence-free subspace $\mathbf{D}^h$. The construction of a basis for $\mathbf{D}^h$ is essential.

In this section, we will construct a computationally convenient basis for $\mathbf{D}^h$—the divergence-free subspace of $\mathbf{V}^h$. We will do this by first constructing a vector potential space $\mathbf{U}^h$ such that

$$\mathbf{D}^h = \mathbf{curl}\,\mathbf{U}^h. \tag{11}$$

Next, we will find a basis for $\mathbf{U}^h$ and we will define a basis for $\mathbf{D}^h$ by simply taking the curls of the vector potential basis functions.

Denote the mesh on $\Omega = (0,1)^3$ by $0 = x_0 < \cdots < x_i < \cdots < x_n = 1$, and similarly with y_j and z_k, $0 \le j, k \le n$. The assumption of the same number n of intervals in each direction is merely for convenience and is not necessary for the construction to follow. Let $\mathbf{U}^h$ be defined as follows:

$$\mathbf{U}^h = \text{span}\left\{ \begin{pmatrix} \phi_i(y,z) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_j(x,z) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \phi_k(x,y) \end{pmatrix} \right\}, \tag{12}$$

where $1 \le i \le (n-1)^2$ (thus, only the first yz-slice is included) and $1 \le j, k \le n(n-1)^2$ (all xz- and xy-slices are included) and ϕ_i is the standard bi-linear nodal basis function associated with the edge i and is piece-wise constant in the third dimension. Note that the number of excluded ϕ_i's is $(n-1)^3$. If the number of intervals in the x-, y-, and z-directions were ℓ, m, and n, respectively, the number excluded would be $(\ell-1)(m-1)(n-1)$, and would be the same if all but one xz- or xy-slice were excluded instead of all but one yz-slice.

Next, we list some properties of $\mathbf{U}^h$ which follow directly from the definition of the potential space.

Remark 4.1 $\mathbf{U}^h \not\subset H(div;\Omega)$ *, and hence,* $\mathbf{U}^h \not\subset H^1(\Omega)^3$.

Remark 4.2 *Every* $\Phi \in \mathbf{U}^h$ *satisfies* $\Phi \times \mathbf{n} = \mathbf{0}$ *on* $\partial\Omega$.

Remark 4.3 $\mathbf{U}^h$ *is locally divergence-free, i.e.* $\nabla \cdot \Phi = 0$ *on each element* $K \in \mathcal{T}^h$ *for every* $\Phi \in \mathbf{U}^h$.

Remark 4.4 $\mathbf{U}^h \subset H(curl;\Omega)$*, and hence* $\mathbf{curl}\,\mathbf{U}^h \subset \mathbf{V}^h$.

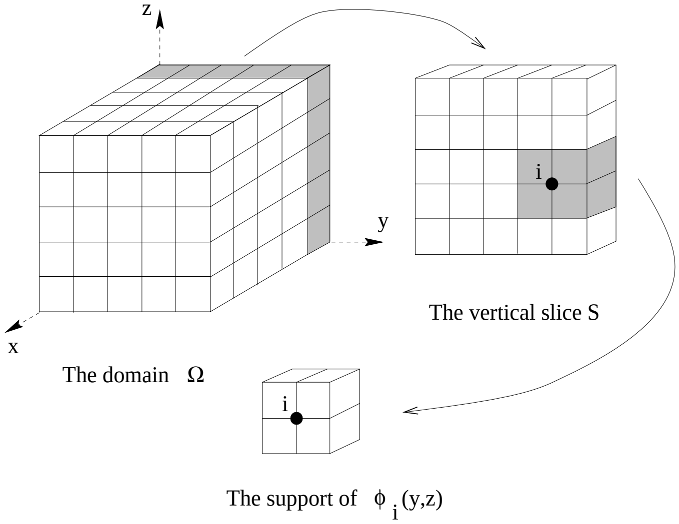
Since $\operatorname{div}\mathbf{curl} \equiv 0$, we have $\mathbf{curl}\,\mathbf{U}^h \subset \mathbf{D}^h$. Counting dimensions,

$$\dim \mathbf{U}^h = (2n+1)(n-1)^2 = 2n^3 - 3n^2 + 1.$$

Also, $\operatorname{div}\mathbf{V}^h$ consists of those piecewise constants with integral zero over Ω, hence has dimension $n^3 - 1$, and we obtain

$$\dim \mathbf{D}^h = \dim \mathbf{V}^h - \dim \operatorname{div}\mathbf{V}^h = 3(n-1)n^2 - (n^3 - 1) = 2n^3 - 3n^2 + 1.$$

Figure 1 The support of a typical potential basis function



We show in [CPRY95] that the curls of the vectors in (12) are linearly independent, so that

$$\dim \mathbf{D}^h = \dim \mathbf{curl}\, \mathbf{U}^h = \dim \mathbf{U}^h = 2n^3 - 3n^2 + 1,$$

which implies that for every divergence-free vector $\mathbf{v} \in \mathbf{D}^h$ there exists a unique potential vector $\Phi \in \mathbf{U}^h$ such that

$$\mathbf{v} = \mathbf{curl}\, \Phi.$$

The vector functions in (12) constitute only one possible choice of a basis for $\mathbf{U}^h$.

Remark 4.5 *The above-defined basis for $\mathbf{U}^h$ (and hence for $\mathbf{D}^h$) consists of vector functions with minimal possible support (4 elements).*

In [CPRY95] we prove the following Poincaré-type inequality:

Lemma 4.1 *There exists a constant $C(\Omega) > 0$ independent of the quasi-uniform mesh size h, such that for all $\Phi \in \mathbf{U}^h$ we have*

$$\|\Phi\|_{L^2(\Omega)^3} \leq C(\Omega)\, \|\mathbf{curl}\, \Phi\|_{L^2(\Omega)^3}. \tag{13}$$

(Since the vector potential space $\mathbf{U}^h \not\subset H^1(\Omega)^3$, inequality (13) does not follow from the standard Poincaré inequality.)

Corollary 4.2 *The linear system (6) to be solved in $\mathbf{D}^h$ has a symmetric and positive definite matrix with condition number of order $O(h^{-2})$.*

The result of the Lemma suggests that the curl semi-norm behaves like the H_0^1 semi-norm for scalar functions and thus allowing us to use fairly standard DD tools (as in [BPWX91, BX91, Cai93, DW87, Lio88, GR86]) to prove in [CPRY95] the following uniform convergence rate estimates:

Theorem 4.1 *For any vector* $\mathbf{v} \in \mathbf{D}^h$, *we have*

$$C_1 a(\mathbf{v}, \mathbf{v}) \leq a(\mathbf{P}\mathbf{v}, \mathbf{v}) \leq C_2 a(\mathbf{v}, \mathbf{v}) \tag{14}$$

where the positive constants C_1 *and* C_2 *are independent of* h *and* J.

Theorem 4.2 *The iterative method defined at the second step in* Algorithm 3.2 *is uniformly convergent, i.e.,*

$$\|\mathbf{E}\|_a \leq \gamma < 1 \tag{15}$$

where γ *is a constant that does not depend on the number of subdomains and the mesh size.*

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REFERENCES

[BPWX91] Bramble J. H., Pasciak J. E., Wang J., and Xu J. (1991) Convergence estimates for product iterative methods with applications to domain decomposition and multigrid. *Math. Comp.* 57: 1–21.

[BX91] Bramble J. H. and Xu J. (1991) Some estimates for a weighted L^2 projection. *Math. Comp.* 56: 463–476.

[Cai93] Cai Z. (1993) Norm estimates of product operators with application to domain decomposition. *Math. Comp.* 53: 251–276.

[CEL96] Chan Z., Ewing R. E., and Lazarov R. D. (1996) Domain decomposition algorithms for mixed methods for second-order elliptic problems. *Math. Comp.* 65: 467–490.

[CMW95] Cowsar L. C., Mandel J., and Wheeler M. F. (1995) Balancing domain decomposition for mixed finite elements. *Math. Comp.* 64: 989–1015.

[CPRY95] Cai Z., Parashkevov R. R., Russell T. F., and Ye X. (1995) Domain decomposition for a mixed finite element method in three dimensions. *SIAM J. Numer. Anal.* to appear.

[DW87] Dryja M. and Widlund O. (1987) An additive variant of the Schwarz alternating method for the case of many subregions. Tech. Rep. 339, Courant Institute.

[EW92] Ewing R. E. and Wang J. (1992) Analysis of the Schwarz algorithm for mixed finite element methods. *RAIRO Math. Modél. Anal. Numér.* 26: 739–756.

[GR86] Girault V. and Raviart P. A. (1986) *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms.* Springer-Verlag, New York.

[GW87] Glowinski R. and Wheeler M. F. (1987) Domain decomposition and mixed finite element methods for elliptic problems. In Glowinski R., Golub G. H., Meurant G. A., and Périaux J. (eds) *Proc. 1st Int. Symp. on Domain Decomposition Methods.* SIAM, Philadelphia.

[HY95] Hall C. and Ye X. (1995) Construction of null bases for the divergence operator associated with incompressible Navier-Stokes equations. *J. Linear Algebra and Applications* to appear.

[Lio87] Lions P. L. (1987) On the Schwarz alternating method I. In Glowinski R., Golub G. H., Meurant G. A., and Périaux J. (eds) *Proc. 1st Int. Symp. on Domain Decomposition Methods.* SIAM, Philadelphia.

[Mat89] Mathew T. F. (1989) *Domain decomposition and iterative refinement methods for mixed finite element discretizations of elliptic problems.* PhD dissertation, Courant Institute, New York.

[MSAC94] Mosé R., Siegel P., Ackerer P., and Chavent G. (1994) Application of the mixed hybrid finite element approximation in a groundwater flow model: Luxury or necessity? *Water Resour. Res.* 30: 3001–3012.

[Ned80] Nedelec J. C. (1980) Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.* 35: 315–341.

[RT77] Raviart P. A. and Thomas J. M. (1977) *A mixed finite element method for 2nd order elliptic problems*, volume 606 of *Mathematical Aspects of Finite Element Methods, Lecture Notes in Mathematics*, pages 292–315. Springer-Verlag, New York.