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A New Approach to Domain Decomposition Methods with Non-matching Grids

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1 Introduction

Attempts at solving actual problems, e.g. heterogeneous equations, have revealed limitations of many classical domain decomposition methods. As a result, there has been renewed interest in new and alternative approaches.

In the context of non-matching grids, to our knowledge, three approaches are considered in literature: Mortar element methods, in *primal formulation* ([AMMP90],[BDM90],[BMP92]), or *mixed or equilibrium formulation* ([Ago96]); hybrid methods ([AT95],[RT97]); and primal-equilibrium coupling methods ([AL94a]). Mortar element type methods are based on the explicit construction of an approximation space. The approach we present here is a conforming one, in which no global approximation is constructed. The domain is decomposed in two block-subdomains allowing for internal subdomain decomposition. A primal variational formulation is used in one region, whereas an equilibrium one is used on the other. The flexibility of the method allows for use of different discretizations on each subdomain; of low-order type (e.g. finite element methods [Cia91]) or high-order type (e.g. spectral element methods [CHQZ88]). We will use in this paper either finite element or spectral element versions of the method. The solution is discontinuous on the interface, and the matching is implicitly contained in the equations formulation.

The main characteristics of the approach we introduce can be summarized as:

- Flexibility on the choice of discretizations on each subdomain;
- No global discrete space to contruct : The global space is a product of local ones; the solution is *"discontinuous"* on the interface;
- No Lagrange multiplier is used to take into account the constraint on the interface.

This paper describes recent advances in the development of the present approach. We give here the main results and leave the detailed analysis for related papers.

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2 Problem Formulation

The Continuous Case

The discussion here is restricted to second-order linear partial differential equations. We consider the solution of the Poisson equation on a domain Ω : Find u such that

$$\begin{cases} Lu &= -\Delta u + u = f \quad \text{in } \Omega, \\ u &= 0 \quad\quad\quad\quad\quad \text{on } \Gamma = \partial\Omega. \end{cases}$$

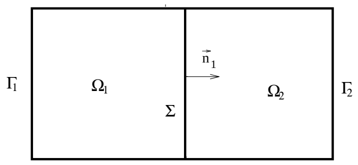
where $f \in L^2(\Omega)$.
Remark *In all what follows, L can be replaced by $Lu = -div(K$ **grad** $u) + \beta$. **grad** $u + \sigma u$, provided that all corresponding problem is satisfy standard solvability hypotheses.*
We suppose (for simplicity) that Ω is rectangularly decomposable, that is, there exist rectangular subdomains Ω_1 and Ω_2 such that

$$\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}\ ,\ \Omega_1 \cap \Omega_2 = \emptyset.$$

In the sequel, we set

$$\Sigma = \partial\Omega_1 \cap \partial\Omega_2.$$

Figure 1



First, we remark that this Problem can be factored to give the coupled system :

$$\begin{aligned} \textbf{grad}\, u_1 = p_1 \ \text{ in } \Omega_1, \\ -divp_1 + u_1 = f \ \text{ in } \Omega_1, \\ -\Delta u_2 + u_2 = f \text{ in } \Omega_2, \\ u_1 = u_2 \ \text{ on } \Sigma, \\ u_1 = 0 \ \text{ on } \Gamma_1 = \partial\Omega_1/\Sigma, \\ u_2 = 0 \ \text{ on } \Gamma_2 = \partial\Omega_2/\Sigma, \\ p_1.n_1 + \frac{\partial u_2}{\partial n_2} = 0 \ \text{ on } \Sigma \end{aligned} \tag{1}$$

where $\frac{\partial}{\partial n_2}$ is the outward normal derivative, and $p_1.n_1$ is the outward normal trace of p.

In the framework of the numerical solution of (1) by finite element type or spectral element type methods [Cia91], it is essential to work in a suitable variational context. Otherwise stated, one has to use variational forms leading to a well-posed problem equivalent to system (1) in a given sense.

The weak form of (1) is given by seeking a pai $(p_1, u_2) \in H(div, \Omega_1) \times H^1_{\Gamma_2,0}(\Omega_2)$ such that :

$$\forall q_1 \in H(div, \Omega_1), \quad \int_{\Omega_1} (p_1.q_1 + divp_1 \ divq_1)dx - < q_1.n_1, u_2 >_\Sigma = -\int_{\Omega_1} f \ divq_1 dx,$$
$$\forall v_2 \in H^1_{\Gamma_2,0}(\Omega_2), \quad \int_{\Omega_2} \{ \ \mathbf{grad} \ u_2. \ \mathbf{grad} \ v_2 + u_2 v_2\}dx + < p_1.n_1, v_2 >_\Sigma = \int_{\Omega_2} f v_2 dx \tag{2}$$

where $< .,. >_\Sigma$ is the duality pairing between the function spaces $H^{\frac{1}{2}}_{0,0}(\Sigma)$ and $H^{-\frac{1}{2}}(\Sigma)$, and

$$H^1_{0,\Gamma_2}(\Omega_2) = \{v \in H^1(\Omega_2); \text{ such that } v = 0 \text{ on } \Gamma_2\}.$$

Note that the unknowns p_1 and u_2 are coupled only through the boundary integrals appearing in (2).

One can also write the problem (2) in another useful form, namely
Find $(p_1, u_2) \in H(div, \Omega_1) \times H^1_{\Gamma_2,0}(\Omega_2)$ such that,
$\forall (q_1, v_2) \in H(div, \Omega_1) \times H^1_{\Gamma_2,0}(\Omega_2)$,

$$B((p_1, u_2); (q_1, v_2)) = -\int_{\Omega_1} f \ divq_1 dx + \int_{\Omega_2} f v_2 dx \tag{3}$$

where

$$\begin{array}{rcl} B((p_1, u_2); (q_1, v_2)) & = & \int_{\Omega_1} (p_1.q_1 + divp_1 \ divq_1)dx - < q_1.n_1, u_2 >_\Sigma \\ & + & \int_{\Omega_2} \{ \ \mathbf{grad} \ u_2. \ \mathbf{grad} \ v_2 + u_2 v_2\}dx + < p.n_1, v_2 >_\Sigma \end{array}$$

Concerning the existence and uniqueness results, we have

Theorem 2.1 *There exists a unique solution (p_1, u_2) of problem (2). Moreover,*

$$\begin{array}{r} p_1 = \ \mathbf{grad} \ u_{|\Omega_1}, \\ u_2 = u_{|\Omega_2}, \end{array} \tag{4}$$

where u is the weak solution of the Helmholtz problem (2.1).

Proof: First remark that the bilinear form B(.;.) is $H(div, \Omega_1) \times H^1_{0,\Gamma_2}(\Omega_2)$-elliptic. We prove easily the continuity of this form. So by Lax-Milgram theorem, problem (2) has a unique solution. The second part of the theorem is obtained by a slight modification of standard arguments.

The Approximated Problem

For its numerical solution, the variational problem (2) must first be approximated by a problem with a finite number of unknowns [Cia91]. In the finite element or spectral methods context, this approximation is realized by replacing the space $H(div, \Omega_1) \times H^1_{0,\Gamma_2}(\Omega_2)$ by a finite dimensional space. In this method, we want to approximate separately the spaces $H(div,\Omega_1)$ and $H^1_{0,\Gamma_2}(\Omega_2)$. Therefore, we introduce finite dimensional spaces :

$$V_{h_1} \subset H(div,\Omega_1), \quad \dim V_{h_1} < +\infty$$
$$V_{h_2} \subset H^1_{\Gamma_2,0}(\Omega_2), \quad \dim V_{h_2} < +\infty.$$

The classical conforming Galerkin approximation of (2) is
Find $(p_{h_1}, u_{h_2} \in V_{h_1} \times V_{h_2}$ such that ,
$\forall (q_{h_1}, v_{h_2}) \in V_{h_1} \times V_{h_2}$,

$$B((p_{h_1}, u_{h_2}); (q_{h_1}, v_{h_2})) = -\int_{\Omega_1} f \; divq_{h_1} dx + \int_{\Omega_2} f v_{h_2} dx. \tag{5}$$

Similarly to problem (2), problem (5) has a unique solution. Moreover, it is possible to prove the following

Theorem 2.2 *Let $(p_1,u_2) \in H(div,\Omega_1) \times H^1_{0,\Gamma_2}(\Omega_2)$ be the solution of (2) and let V_{h_1} and V_{h_2} defined as above. Problem (5) has a unique solution (p_{h_1}, u_{h_2}) and there exists a constant C which does not depend on dimensions of V_{h_1} and V_{h_2} such that*

$$\begin{aligned} \|p_1 - p_{h_1}\|_{H(div,\Omega_1)} &+ \|u_2 - u_{h_2}\|_{1,\Omega_2} \leq \\ C &\inf_{(q_{h_1},v_{h_2}) \in V_{h_1} \times V_{h_2}} \{\|p_1 - q_{h_1}\|_{H(div,\Omega_1)} + \|u_2 - v_{h_2}\|_{1,\Omega_2}\}. \end{aligned}$$

Proof: Follows easily from Lax-Milgram Theorem and Céa Lemma.

An Example of Discretization

A basic choice of V_{h_1} and V_{h_2} in the spectral methods context consists of introducing:

$$V_{h_1} = RT_{N1}(\Omega_1) = P_{N1,N1-1}(\Omega_1) \times P_{N1-1,N1}(\Omega_1)$$

and

$$V_{h_2} = Q_{N_2}(\Omega_2) \cap H^1_{0,\Gamma_2}(\Omega_2).$$

With this choice, a consequence of (2.2) is the following

Theorem 2.3 *Assume that the solution (p_1,u_2) of problem (2) is such that, p_1 and $divp_1$ belong to $(H^{\sigma_1}(\Omega_1))^d$ and $H^{\sigma_1}(\Omega_1)$ for a real number $\sigma_1 > 0$, and u_2 belongs to $H^{\sigma_2}(\Omega_2)$ for a real number σ_2, $1 < \sigma_2$. Then the following estimate holds*

$$\begin{aligned} \|p_1 - p_{h_1}\|_{H(div,\Omega_1)} &+ \|u_2 - u_{h_2}\|_{1,\Omega_2} \leq \\ C_\epsilon &\{N_1^{-\sigma_1+\epsilon}(\|p_1\|_{\sigma_1,\Omega_1} + \|divp_1\|_{\sigma_1,\Omega_1}) + N_2^{-\sigma_2+1}\|u_2\|_{\sigma_2,\Omega_2}\}. \end{aligned}$$

for all $\epsilon > 0$.

By post-processing, we can easily obtain an approximation of u_1. More precisely, if we set

$$u_{N_1} = \Pi_{N_1-1}(divp_{N_1} + f)$$

where Π_{N_1-1} is the projection operator defined from $L^2(\Omega_1)$ onto $P_{N_1-1}(\Omega_1)$, we have

$$\|u_1 - u_{N_1}\|_{0,\Omega_1} \leq C_\epsilon\{N_1^{-\sigma_1+\epsilon}(\|p_1\|_{\sigma_1,\Omega_1} + \|divp_1\|_{\sigma_1,\Omega_1}) + N_2^{-\sigma_2+1}\|u_2\|_{\sigma_2,\Omega_2}\}.$$

for all $\epsilon > 0$.
These results are illustrated in the following figure.

Figure 2 We consider here the solution of problem (2.1) on the domain $\Omega = (-1, 1)^2$. A spectral element discretization is used on each subdomain. The right-hand side f is given by the exact solution $u_{ex}(x, y) = \sin(\pi x)\, \sin(\pi x)$. We plot in **(a)** the H^1 - error of u in Ω_2 as a function of related polynomial order; and in **(b)** the $L^2 - error$ of u in Ω_1, obtained by post-processing. The error decreases exponentially fast as would be expected for spectral approximation of a smooth solution.

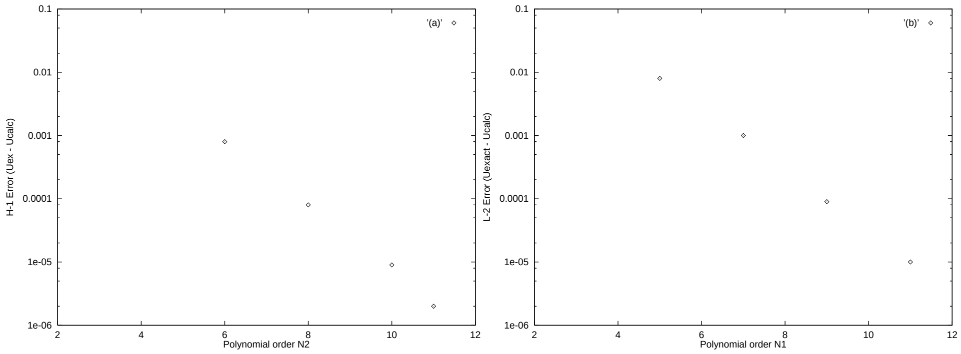
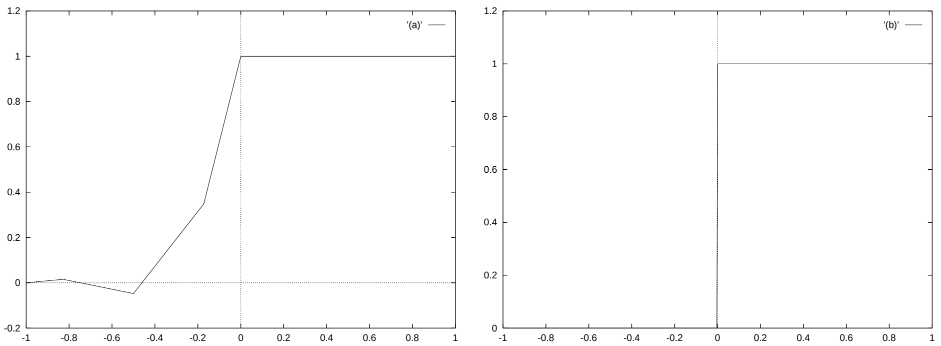


Figure 3 Case of the operator $-$ div $(\nu(.)$ **grad** $)$ with discontinuous coefficients. We consider here the numerical approximation of the solution of the Helmholtz equation $-\nu\Delta u(x, y) + u(x, y) = f(x, y)$. The domain is split into two physical subdomains: The diffusivity parameter varies from one subdomain to other; $\nu_{|\Omega_1} \gg \nu_{|\Omega_2}$: **(a)** A conforming spectral element method is used with degree polynomial $N = 5$. Obviously, one needs more refinement to has good approximation. However this objective is achieved in **(b)** with a least polynomial degree using the spectral element version of the present approach.



3 Extension to Other Cases

Heterogeneous Domain Decomposition

Heterogeneous domain decomposition have broad applications in engineering and in natural science. In this section, we give an extension of our ideas to the heterogeneous domain decomposition methods.

As an example, we consider the coupling between elliptic diffusion equations and hyperbolic convection (transport) ([AL94b],[AD96]). The idea of this procedure is that in convection-diffusion problems where the convection is dominant, the diffusion terms play a role only in the vicinity of boundary or internal layers. From the physical information, these regions can be detected a priori, it is logical to suppose that only there the complete equations have to be solved, whereas elsewhere the reduced equations can serve as a correct model. Here, we consider the following problem.

$$
\begin{aligned}
\beta.\ \mathbf{grad}\, u_1 + u_1 &= f_1 \text{ in } \Omega_1,\\
-\Delta u_2 + u_2 &= f_2 \text{ in } \Omega_2,\\
u_1 &= u_2, \text{ on } \Sigma,\\
\frac{\partial u_2}{\partial n_2} + \beta.n_1 u_1 &= 0 \text{ on } \Sigma,\\
u_1 &= 0 \text{ on } \Gamma_1^-,\\
u_2 &= 0 \text{ on } \Gamma_2,
\end{aligned}
\tag{6}
$$

where

$$\Gamma_1^- = \{x \in \Gamma_1; \beta.n_1 < 0\}$$

and

$$\beta \in W^{1,+\infty}(\Omega).$$

we assume that

$$\beta(x).n_1(x) < 0 \quad \text{a.e} \quad x \in \Sigma \quad , \quad div\beta = 0.$$

Using similar arguments as in [AL94b], we can state that this problem has a unique solution.

First, as in the elliptic case, we transform (6) into an equivalent problem

$$
\begin{aligned}
\beta.\ \mathbf{grad}\, u_1 + u_1 &= f_1 \text{ in } \Omega_1,\\
p_2 &= \mathbf{grad}\, u_2 \text{ in } \Omega_2,\\
-div p_2 + u_2 &= f_2 \text{ in } \Omega_2,\\
u_1 &= u_2 \text{ on } \Sigma,\\
p_2.n_2 + \beta.n_2 u_1 &= 0 \text{ on } \Sigma,\\
u_1 &= 0 \text{ on } \Gamma_1^-,\\
u_2 &= 0 \text{ on } \Gamma_2
\end{aligned}
\tag{7}
$$

Let us now set $\mathcal{T}_h$ a regular triangulation of the domain Ω_1 with triangular ($d = 2$) or tetrahedral ($d = 3$) finite elements whose diameters are less or equal to h, and let k and N be positive integers. We define the finite dimensional spaces V_h and V_N by

$$V_h = \{v_h \in \mathcal{C}^0(\overline{\Omega}_1); \forall T \in \mathcal{T}_h, v_{h|T} \in P_k(T)\},$$

and

$$V_N = RT_N(\Omega_2).$$

The discrete problem is now

$$\begin{aligned}
&\text{Find } (u_h, p_N) \in V_h \times V_N, \text{ such that}\\
&\forall q_N \in V_N,\ \int_{\Omega_2} (p_N.q_N + divp_N\ divq_N)dx - < q_N.n_2, u_h >_{0,\Sigma} = -\int_{\Omega_2} f\ divq_N dx,\\
&\forall v_h \in V_h,\ \sum_{T\in_h} \int_T (\beta.\ \mathbf{grad}\ u_h + u_h)(v_h + \delta h\beta.\ \mathbf{grad}\ v_h)dx + \int_\Sigma |\beta.n_1|u_h v_h d\sigma\\
&\quad + \int_\Sigma p_N.n_2 v_h d\sigma + \int_{\Gamma_1^-} |\beta.n_1|u_h v_h d\sigma = \sum_{T\in\mathcal{T}_h} \int_T (h\beta.\ \mathbf{grad}\ v_h + v_h) f dx.
\end{aligned} \tag{8}$$

where δ is a stabilization parameter.

We have the following

Theorem 3.1 *The problem (8) has a unique solution $(u_h, p_N) \in V_h \times V_N$. Moreover if the solution (u_1, p_2) of problem (6) is such that, p_2 and $divp_2$ belong to $(H^{\sigma_2}(\Omega_2))^d$ and $H^{\sigma_2}(\Omega_2)$, respectively, for a real number $\sigma_2 > 0$, and u_1 belongs to $H^{\sigma_1}(\Omega_1)$ for a real number σ_1, $1 < \sigma_1 \le k+1$, then the following estimate holds*

$$\begin{aligned}
\|p_2 - p_N\|_{H(div,\Omega_2)} &+ \|u_1 - u_h\|_{0,\Omega_1} + h^{\frac{1}{2}}\|\beta\ \mathbf{grad}\ (u_1 - u_h)\|_{0,\Omega_1} \le\\
&C_\epsilon\ \{N^{-\sigma_2+\epsilon}(\|p_2\|_{\sigma_2,\Omega_2} + \|divp_2\|_{\sigma_2,\Omega_2}) + h^{\sigma_1+\frac{1}{2}}\|u_1\|_{\sigma_1,\Omega_1}\}.
\end{aligned}$$

for all $\epsilon > 0$.

Partial Differential Equations in Nonstationary Invariant Geometries

The so-called sliding schemes have been already presented in either a finite difference, [Gil88, Rai87], or mortar element framework [Ana91]. Sample candidate applications include rotating machinery and turbomachinery.

For the sake of simplicity, the method is presented for the following model problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f & \text{in } \Omega\times]0,T[\\ u(.,t=0) = u_0 & \text{in } \Omega \end{cases}$$

where $\Omega = \Omega(t)$ is a nonstationary domain, f is a given force that may depend on time, and u_0 is a given initial condition. It is obvious that we are not interested here in numerical simulation of physical situation, in that problem (3) does not take into account the equation of motion of the fluid medium. Our intent is to present the formulation of sliding interfaces problem that couples primal and equilibrium variables. We shall also focus our presentation on the simple case where $\Omega(t)$ is decomposed into two subdomains, one sliding with respect to the other along an interface $\Gamma(t)$:

$$\begin{aligned}
\Omega(t) &= \Omega_1 \cup \Omega_2(t),\\
\Gamma(t) &= \overline{\Omega}_1 \cap \overline{\Omega}_2(t),
\end{aligned}$$

Figure 4 One dimensional example: The domain $\Omega = (-1., 1.)^2$ is split into two physical subdomains. A spectral method is used to discretize the elliptic equation on $\Omega_1 = (-1., 0.)$, while a stabilized finite element method is used for the hyperbolic equation on $\Omega_2 = (0., 1.)$. β is constant and the right-hand side f_2 on Ω_2 is piecewise-constant. **(a)** The solution is discontinuous on the interface, and the non physical oscillations on the hyperbolic domain do not affect the elliptic domain. The continuity of the fluxes could also be illustrated. **(b)** A best approximation can be recovered on the hyperbolic solution using an adaptive finite element method based on a posteriori error estimates established for this one-dimensional problem.

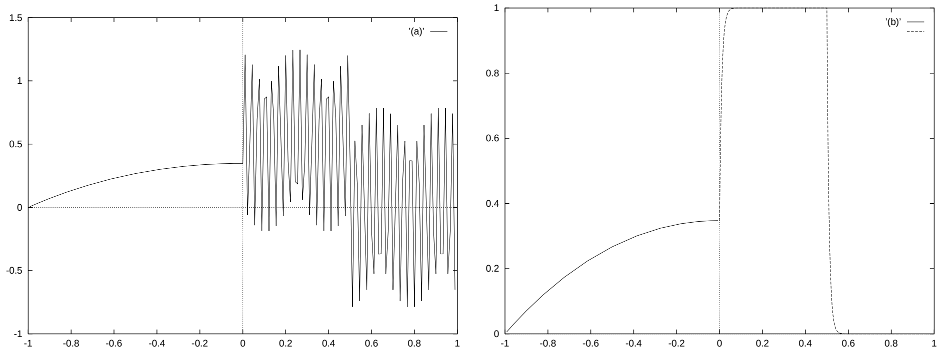
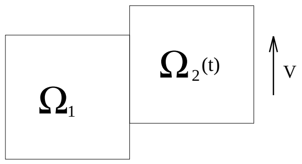


Figure 5



as illustrated in Figure 5, where $\Gamma(t)$ is a segment.

The basic formulation is of spectral element type, but the methodology we introduce is appropriate to finite elements as well. We just point out the fact that since no matching conditions are imposed on the meshes, in case of complicated geometry, one does not have to exhibit $\mathcal{C}^\infty$ mappings to use isoparametric elements. This method could then be a tool for analysing fluid flows in truly complex moving geometries, where the moving interfaces are in general curvilinear.

The scheme presented here is locally conservative, and the aliasing errors induced by numerical quadrature have no effect on the stability. This remark is mainly of interest for the implementation issue.

This variational method also preserves element-based locality, and the flexibility is evident in the treatment of mesh refinement or moving boundary problems by sliding meshes that do not introduce any mesh distortion or expensive interpolation.

Let us denote by V the velocity of $\Omega_2(t)$ and suppose it constant in time. We introduce the Lagrangian variable X in $\Omega_2(t)$ by

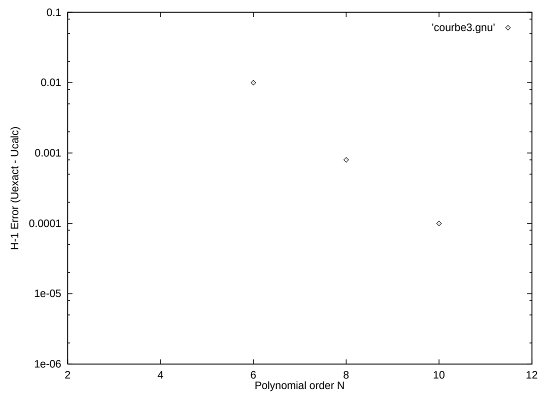
$$\begin{cases} \dfrac{\partial X}{\partial t}(x,t;\tau) &= V(X(x,t;\tau)) \\ X(x,t;\tau) &= x. \end{cases}$$

Problem (3) is then expressed as,

$$\begin{cases} \forall x \in \Omega_2(t), & \dfrac{\partial[u(X(x,t;\tau),t)]}{\partial t} - \Delta u(X(x,t;\tau),t) - V.\nabla u(X(x,t;\tau),t) = f(x,t), \\ \forall x \in \Omega_1, & \dfrac{\partial u}{\partial t}(x,t) \quad - \quad \Delta u(x,t) = f(x,t). \end{cases}$$

In practice, we can also perform internal decompositions of Ω_1 and $\Omega_2(t)$. We can

Figure 6 Plot of discretization error $\|u-u_h\|_{H^1}$ as a function of polynomial order for the diffusion equation on the domain given by Figure 5. The exact solution is given by $u(x,y,t) = exp(-2\pi^2 t)\sin(\pi)\sin(\pi y)$. The simulation is carried out to a final time $T_f = .05$, with Δt insuring stability.



choose any temporal discretization. For sake of simplicity, we deal here with a simple implicit scheme for the treatment of diffusion term, and an explicit (for example Adams-Bashforth) for the convection term. With the superscript ^{n} referring to the time $t^n = n\,\Delta t$, and u^n denoting $u(t^n,.)$, the semi-discrete problem states now as

$$\begin{cases} \forall x \in \Omega_2(t^{n+1}), & u^{n+1} - \Delta u^{n+1} &= \Delta t\, f^{n+1} + u^{n+1} + \Delta t(V.\nabla u^n), \\ \forall x \in \Omega_1, & u^{n+1} - \Delta u^{n+1} &= \Delta t\, f^{n+1} + u^n. \end{cases}$$

The functional framework introduced in the previous section completes the discretization.

The proposed scheme for the approximation of problem (3) in the case of a first order time discretization reads now as follows

Find $p_{N_1}^{n+1} \in RT_{N_1}(\Omega_1)$, $u_{N_2}^{n+1} \in Q_{N_2}(\Omega_2^{n+1})$, $u_{N_2}^{n+1}|_{\partial\Omega_2\setminus\Gamma^{n+1}} = 0$ such that
$\forall q^{n+1} \in RT_{N_1}(\Omega_1)$,

$$\int_{\Omega_1} (p_{N_1}^{n+1}\, q^{n+1} + \Delta t \nabla . p_{N_1}^{n+1}\, \nabla . q^{n+1})\, dx - \Delta t \int_{\Gamma^{n+1}} q^{n+1} . n\, u_{N_2}^{n+1} d\Gamma =$$
$$- \int_{\Omega_1} (\Delta t . f^{n+1} + u_{N_1}^{n})\, \nabla . q^{n+1}\, dx$$

$\forall v^{n+1} \in Q_{N_2}(\Omega_2^{n+1})$, $v^{n+1}|_{\partial\Omega_2\setminus\Gamma^{n+1}} = 0$,

$$\Delta t \int_{\Omega_2} \nabla u_{N_2}^{n+1} . \nabla v^{n+1}\, dx + \int_{\Omega_2} u_{N_2}^{n+1}\, v^{n+1}\, dx + \Delta t \int_{\Gamma^{n+1}} p_{N_1}^{n+1} . n\, v^{n+1}\, d\Gamma =$$
$$\int_{\Omega_2} (\Delta t\, f^{n+1} + u_{N_2}^{n} + \Delta t\, (V . \nabla u_{N_2}^{n})) v^{n+1}\, dx.$$

From the analysis of the previous section, we deduce that this discretization generates a unique sequence $(u_N^n)_n$ of solutions.

The analysis of the discrete problem and stability analysis of this scheme give that the error in (u, p) is bounded by a temporal error of the scheme order and spatial errors as in the Helmholtz equation. The related details of approximation results are left to a forthcoming report.

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