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Selected Topics in Singularities of Space-Times

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Abstract. The aim of this paper is to summarise briefly the problems arising within the General Theory of Relativity once singularities are considered. We will review the difficulties of providing a general definition of singularities, state the famous Singularity Theorem of Hawking and Penrose, as well as a generalisation of this and comment on some of the still unsolved problems.

The basic concepts needed to formulate and understand the aforementioned theorems will be illustrated, along with sketches of proofs when it is considered a deeper insight into the matter is necessary. Alternatively references will be provided in order for the reader to pursue a matter in greater detail.
None of the results presented in the following are due to the authors. The selection of material has been done in accordance with the activities of researchers at the TU Berlin under the guidance of the second author.

1. What is a singularity in General Relativity?

The space-times within Einstein's General Theory of Relativity are four-dimensional, connected, time-orientable Lorentzian manifolds; manifolds furnished with a non-degenerate metric tensor of index one. Throughout this paper the space-times will be denoted by M and only when different metrics are considered we write (M, g), where M is the manifold and g denotes the Lorentzian metric that is always assumed to be of appropriate differentiable order.

Gravity is given a geometrical interpretation as one postulates that the world-lines of freely falling particles (particles influenced by no other force than Gravity itself) are the geodesics of the space-time. Thus, in contrary to the various field theories describing the other fundamental forces, one is not a priori given a fixed background metric but rather has to deal with the metric tensor as physical field itself.

It is this feature of General Relativity, which distinguishes Gravity from the other fundamental forces and causes difficulties, when one attempts to treat problems in a conceptually similar way. For instance, when considering a singularity of the gravitational field, one is tempted to argue by analogy with electrodynamics that the singularity is merely a point of the manifold, where the metric tensor is either undefined or just not suitably differentiable. (It is worthwhile mentioning that the concept of singularities in General Relativity seems unavoidable as singular states develop even for some of the simplest space-time models.) By simply removing those points from the manifold one would, according to this definition, form a perfectly non-singular space-time. This would, however, be unsatisfactory, as particles, the world-lines of which have passed these former singularities, would now fall off the non-singular space-time.

In fact, this consideration suggests a way in which one may define a singularity; namely in terms of some kind of incompleteness of the space-time under consideration. In order to avoid the detection of unreal coordinate singularities one first maximally extends the space-time. Note that the question of what is a real singularity crucially depends on the criteria chosen for extendibility ([TCE]).
Unlike the positive definite metric tensor of a Riemannian manifold, the non-degenerate indefinite metric tensor of a Lorentzian manifold does not give rise to a metric function on the manifold, relative to which *m(etric)-completeness* may be defined. Nevertheless, there is another notion of completeness of semi-Riemannian manifolds which coincides with the m-completeness in the Riemannian case: A semi-Riemannian manifold is called *g(eodesically)-complete* iff all its affinely parameterised maximal geodesics can be defined in all of M.

In the presence of a Lorentzian metric, all geodesics fall into one of three classes; space-like-, time-like-, and null geodesics, according to whether their tangent vectors have positive, negative, or vanishing Lorentzian lengths.

A space-time is called *time-like* (null, space-like, respectively) *geodesically complete* iff all its time-like (null, space-like, respectively) affinely parametrised maximal geodesics can be defined in all of M. This definition is particularly interesting, as the affine parameter of a time-like geodesic represents the proper time of a (non-accelerated) observer moving on the geodesic. Thus a time-like incomplete space-time admits observers whose histories would either end after or only go back a finite interval of proper time – a feature which should let us think of the space-time as being singular!

It is, however, interesting, that the existence of incomplete maximal geodesics of either type does not imply the existence of incomplete maximal geodesics of any other type [G1]. The null geodesics of a space-time still represent world-lines of (zero rest mass) particles and although the affine parameter of such a geodesic does not represent a proper time, one has reason to regard null incomplete space-times as being singular also. An example of a time-like complete but null incomplete space-time, that should be classified as singular, is the Reissner Nordstroem solution.

Thus it is generally agreed that null or time-like incompleteness are sufficient conditions for a space-time to be classified as singular. (Note that the singularities which will occur according to the theorems we are going to consider in the following are of that type.) That these, however, are not necessary conditions has been observed by Geroch ([G1]).

In order to identify this spacetime as singular, one needs a generalised notion of completeness which allows the identification of arbitrary incomplete curves. The so-called bundle completeness may be achieved by defining a generalised affine parameter for arbitrary curves. A space-time is then said to be *b(undle)-complete*, iff for any non-extensible curve such a parameter can be defined in all of M.

This notion, however, allows a mathematically more elegant description in terms of the principal bundle of linear frames $L(M)$ on the space-time M with structure group $Gl(4,\mathcal{R})$ [Sch]: A suitable Riemannian metric g_L can be defined on $L(M)$, which gives rise to a metric function on a connected component $L'(M)$ of $L(M)$ and thus to a completion $\overline{L'(M)}$ of this connected component. (In fact, a whole class of metrics can be defined in a similar way, but all of them induce the same uniform structure on a connected component $L'(M)$ of $L(M)$.) This completion again induces a completion $\overline{M} = M \cup \dot{M}$ of the manifold M itself, such that incomplete maximal curves in M terminate on $\dot{M}$ in $\overline{M}$. Furthermore $\overline{M}$ is naturally endowed with the final topology relative to the natural projection $\pi : \overline{L'(M)} \longrightarrow \overline{M}$, $\overline{L'(M)}$ equipped with the metric topology induced by g_L.

Points on this so-called *b(undle)-boundary* $\dot{M}$ represent the singularities of M. Thus the b-boundary construction suggests a way of how to examine the space-time's behaviour near the singularities. Unfortunately further investigations showed that it has to be modified by either choosing different topologies on $L'(M)$ or a different metric h_L on $L(M)$ in order to rule out topological difficulties and identification problems that arise when calculating the original b-boundary for certain exact solutions of the Einstein field equations. (See [TCE] and the literature quoted there for further details.) The basic idea, that is to attach a boundary to the manifold, such that at least some points of the boundary correspond to the singularities of the space-time, has been motivating a number of different constructions.

Amongst those the *c(ausal)-boundary* construction that uses the causal properties of space-time in order to attach the boundary is for many purposes the most promising and therefore also worth to study in some detail ([HE]). For a systematic approach to this construction see [GKP] and the detailed exposition in [R1]. A topological insufficiency of this construction has been exhibited in [R2].

Considering a space-time as singular, one intuitively expects the gravitational field, and therefore the curvature, to become (in some sense) unboundedly large by approaching the singularity. As the metric and the Riemann curvature are tensors, one can always choose appropriate coordinates in order to make their components as large or small as one desires. Therefore one has to give attention to the scalar functions on the manifold related to the metric and/or curvature instead. One way of doing that is to consider the scalar polynomials in the metric, the Riemann curvature and the volume element. A b-incomplete curve in M is said to correspond to a scalar polynomial curvature singularity, if one of those scalar polynomials is unbounded along the curve. (These scalar polynomials, however, do not fully describe the curvature tensor.)

Another type of curvature singularity is related to a parallelly propagated basis along a b-incomplete curve. In case any of the Riemann curvature components relative to such a basis are unbounded along the curve, one says, that the b-incomplete curve corresponds to a curvature singularity relative to the parallelly propagated basis.

For the sake of completeness it should be mentioned that the Taub-NUT space provides an example for a b-incomplete space-time, which does not admit either of these two possible types of curvature singularities. It can be shown that the so-called imprisoned b-incomplete curves, along which all the scalar polynomials and curvature components relative to any parallelly propagated basis remain bounded, only arise in space-times which violate the strong causality assumption ([HE]). For further investigations on different types of curvature singularities see [TCE] and [Cla].

2. Singularity Theorems

2.1. Sets of Particular Interest

The former discussion was based upon the expectation, that in presence of a singularity something goes wrong according to our intuitive imagination of the physical world. The interpretation of time-like- and null curves as being world-lines of appropriate material- and zero rest mass particles played a fundamental role along those lines. It also implies the information about which points of the space-time are chronologically and causally related to each other and thus motivates a number of definitions that will be used frequently in the following.

At each point of the space-time the set of non-space-like non-vanishing tangent vectors consists of two connected components. The time-orientability required ensures that one can label these connected components continuously throughout the tangent bundle TM of M. Therefore all causal vectors fall into one of two classes called future- and past directed and so do all causal curves.

As a consequence of the interpretation of causal curves as being the possible world-lines of particles, an observer can travel from an event $p \in M$ to an event $q \in M$ if and only if there exists a time-like future directed curve from p to q. q is then said to belong to the *chronological future* $I^+(p)$ of p. (Note that all of the definitions using the term "future" have an obvious analogue for the "past", replacing "+" by "−" in the notations.) For a subset S of M the set $I^+(S) := \bigcup_{p\in S} I^+(p)$ denotes the chronological future of S. Sometimes one also needs a local version of the chronological future of a set $S \subset M$. One says that an event $q \in U \subset M$ belongs to the chronological future $I^+(S,U)$ of S relative to U if there is a future directed time-like curve from S to q contained in U.

The *causal future* $J^+(p)$ of an event $p \in M$ is defined as the union of $\{p\}$ with the set of all points, that can be reached from p by a causal curve. Physically spoken this is the set of events, which can be influenced by p. Again for a subset S of M the set $J^+(S) := \bigcup_{p \in S} J^+(p)$ denotes the causal future of S.

The *future horismos* $E^+(S)$ of a set $S \subset M$ is defined to be $E^+(S) := J^+(S) \setminus I^+(S)$. It can be shown, that a point $p \in E^+(S)$ must lie on a future directed null geodesic from S to p, and for a convex space-time the future horismos $E^+(p)$ of a single point p consists of the future directed null geodesics from p.

Note that a point p must not necessarily belong to its own future horismos $E^+(p)$. If this is not the case p must belong to its own chronologically future and therefore to its own chronologically past, i.e. $p \in I^+(p) \cap I^-(p)$. The set $V := \bigcup_{p \in M} I^+(p) \cap I^-(p)$ is called the *chronology violating set* and can be written as the disjoint union of sets of the form $I^+(p) \cap I^-(p)$. For $V = \emptyset$, i.e. if there are no closed time-like curves in M, one says that the *chronology condition* holds on M, whereas otherwise the space-time M is said to be chronology violating. A set $A \subset M$ which meets no time-like curve more than once is called *achronal*. Obviously one has $V \cap A = \emptyset$ for any achronal set A. The subset $edge(A) \subset \bar{A}$ of all points $r \in \bar{A}$ such that for any neighbourhood U_r of r and points $p \in I^-(r, U_r)$, $q \in I^+(r, U_r)$ there exists a time-like curve from p to q contained in U not intersecting A is called the *edge of the achronal set* A.

Given a subset S of M one can ask for the region of the space-time that is fully determined by a maximal set of data on S. This region $D^+(S)$ is obviously the set of all points $p \in M$ such that any past-inextendible causal curve through p meets S and is referred to as *domain of dependence* [G2] or *future Cauchy development*. The union of the future Cauchy development and the analogue past Cauchy development is referred to as Cauchy development of S and will be denoted by $D(S)$. Also of interest in many respects is the limit of the space-time region which is fully determined by data on S. It is the so-called *future Cauchy horizon* $H^+(S)$ of S which is defined via $H^+(S) := \overline{D^+(S)} \setminus I^-(D^+(S)) = \{p \in \overline{D^+(S)} \mid I^+(p) \cap D^+(S) = \emptyset\}$. For a closed achronal set A one proves $edge(H^+(A)) = edge(A)$ [HE].

2.2. Causality Conditions

Above we defined the chronology condition to hold on a space-time if it admits no closed time-like curves. In the same way one says that the *causality condition* holds on M, if M does not admit closed causal curves. Chronology violation, however, implies causality violation and as causality violation without chronology violation is no stable property. One can further strengthen these conditions by demanding that the space-time should not contain causal curves that return arbitrarily close to points they passed or approach other causal curves that then approach points the first curve already passed and so on.

Amongst those conditions the mathematically strongest is the stable causality condition: Consider the set $Lor^0(M)$ of all C^0-Lorentzian metrics on M with the Whitney-C^0-topology W^0. The *stable causality condition* is said to hold on (M, g), if there is a neighbourhood U_g of g in $(Lor^0(M), W^0)$, such that for any $h \in U_g$ the chronology condition holds on (M, h). It is remarkable that this condition is equivalent to the existence of a (global time) function on M with time-like gradient everywhere on M [H4].

Mathematically weaker but often more convenient is the *strong causality condition*, which is said to hold at a point $p \in M$ provided that given any neighbourhood U_p of p there exists a neighbourhood $V_p \subset U_p$ of p such that every causal curve segment with endpoints in V_p lies entirely in U_p, i.e., every neighbourhood of p contains a neighbourhood of p, that no causal curve intersects more than once. A subset S of a space-time M is *strongly causal* if the strong causality condition holds at every point of S. Note that on a strongly causal compact subset of M one can always choose a finite covering of so-called *local causality neighbourhoods* , i.e. a finite covering of convex normal neighbourhoods U_i with compact closure $\overline{U_i}$, such that no causal curve intersects any U_i more than once.

A strongly causal subset $S \subset M$ is said to be *globally hyperbolic* if for any two points $p, q \in S$, $J^+(p) \cap J^-(q)$ is compact and contained in S. If this condition is imposed on the whole space-time a rather restricted set of models will be the result (see [Di]). Nevertheless it holds for the fairly wide class of sets of the form $int(D(A)) \neq \emptyset$ where A is an arbitrary closed achronal set in M.

2.3. Energy Conditions

Considering that freely falling particles are presumed to move on the geodesics of the space-time it is clear that the matter, being the source of gravity, must couple to the space-time metric. This coupling is given through the Einstein field equations

$$G \;=\; Ric \;-\; \frac{1}{2} S\, g = 8\pi\, T \;,$$

where Ric denotes the *Ricci* tensor of the metric, S its scalar curvature tensor and T the energy momentum tensor of the matter distribution under consideration. Thus physically reasonable energy conditions imposed on the energy momentum tensor become conditions on the *Ricci* curvature tensor of the Lorentzian metric. For a detailed discussion of the various energy conditions the reader is referred to [HE, chapter 4], where he can also find a physical justification for the so called *strong energy condition* which is said to hold on a space-time if for any causal vector $k \in TM$ $(g(k,k) \leq 0, k \neq 0)$ the inequality $Ric(k,k) \geq 0$ holds.

The set of C^2-metrics on the Lorentz manifold M that satisfy the strong energy condition is denoted by $\mathcal{E}$, while $\mathcal{SE}$ denotes the set of C^2-metrics that satisfy the slightly stronger condition $g(k,k) \leq 0 \Longrightarrow Ric(k,k) > 0$. (In [L] this was called "the strong energy condition" while the slightly weaker condition that is now generally said to be "the strong energy condition" was just referred to as "energy condition".) With respect to the Whitney-C^2-Topology W^2

on $Lor^2(M)$ one has for the interior of $\mathcal{E}$ $int_{W^2}(\mathcal{E}) = \mathcal{SE}$ [L]. Thus requiring that amongst the set of physically realistic space-times the strong energy condition should be W^2-stable one can restrict one's considerations on $\mathcal{SE}$. As the topology W^0 is coarser than W^2 one has $int_{W^2}(\mathcal{E}) \subset int_{W^2}(\mathcal{E})$. Thus requiring W^0-stability as in the case of stable causality is even a further restriction.

It is easy to verify that for any metric $g \in \mathcal{SE}$ the *generic condition* is satisfied, i.e. for all causal geodesics γ on M there exists a point on γ at which

$$\dot{\gamma}_{[a}R_{b]cd[e}\dot{\gamma}_{f]}\dot{\gamma}^c\dot{\gamma}^d \neq 0.$$

Hence this is no further physical restriction.

Interesting with respect to the singularity theorems is the following

Lemma 1. *If the strong energy condition and the generic condition hold on a space-time M, and if M is causal geodesically complete, then*

i) *every causal geodesic γ contains two points being conjugate to each other along γ;*

ii) *the chronology condition on M is equivalent to the strong causality condition.*

The formulation given here is not the strongest possible but sufficient for our purposes. Once again the reader is referred to [HE, chapter 4 and 6] for further details.

2.4. The Theorem of Hawking and Penrose

As soon as Albert Einstein had published his general theory of relativity in 1915 it turned out that singular states would develop even for the simplest space-time models like the Schwarzschild solution and the Robertson-Walker space-times. These singular states, however, were believed to arise only as a consequence of the high symmetry of the solutions under consideration. In 1965 Penrose proved the first singularity theorem not relying on any symmetry assumptions ([P1]) that encouraged many scientists (most successful Hawking, Geroch and Penrose himself) to seek for stronger results. In 1973 Hawking and Penrose finally proved the following

Theorem 1. [HP] *A space-time M is not causal geodesically complete if*

i) *the strong energy condition holds on M;*

ii) *the generic condition holds on M;*

iii) *the chronology condition holds on M;*

iv) *there exists at least one of the following:*

- **a)** *a compact achronal set without edge,*
- **b)** *a closed trapped surface (i.e. a closed space-like 2-manifold S such that $tr(\chi_1)$ and $tr(\chi_2)$ are negative everywhere, where χ_1 and χ_2 denote the two null second fundamental forms of S),*
- **c)** *a point p such that on every past (or every future) null geodesic from p the divergence θ of the null geodesics from p becomes negative.*

Some motivations for the assumptions iv) a)–c):
a) being a global condition is surely not experimentally verifiable but still interesting as spatially closed solutions of the Einstein field equations (like the Robertson-Walker space-times with positive curvature) satisfy this assumption. Closed trapped surfaces as required in b) occur for instance in the Schwarzschild solution and are therefore expected to surround sufficiently dense stars. Assumption c) finally seems to hold for our point of the universe [HP, appendix], such that the theorem claims that there was a big bang in the past or that there will be a big crunch in the future. The detailed proof of the theorem can be found in [HE, chapter 8].

With the Lemma above it is easy to verify that Theorem 1 follows from the

Proposition 1. *The following conditions cannot all hold on a spacetime M:*

I) *every non-extendible causal geodesic contains a pair of conjugate points;*

II) *M is strongly causal;*

III) *there is an achronal set A, such that $E^+(A)$ or $E^-(A)$ is compact;*

once one has shown, that each of the conditions in iv) together with i) imply III) [HE].

The proof of this proposition proceeds as follows:

- Taking $E^+(S)$ to be compact one establishes first that $H^+(E^+(S))$ is either noncompact or empty.
- It then follows that any time-like vector field on M admits an integral curve $\tilde{\gamma} : \tilde{I} \longrightarrow M$ that intersects $E^+(S)$ but not $H^+(E^+(S))$ (for otherwise $H^+(E^+(S)) \neq \emptyset$ can be shown to be compact) and $\gamma : I \longrightarrow M$ with $\gamma = \tilde{\gamma}|_I$, $I := \{t \in \tilde{I} \mid \tilde{\gamma}(t) \in I^+(S)\}$ yields a future non-extendible time-like curve contained in $D^+(E^+(S))$.
- For $F := E^+(S) \cap \overline{J^-(\gamma)}$ one next determines $E^-(F)$ to be compact (this argument requires the existence of conjugate points along non-extendible causal geodesics as assumed in I)) and concludes along the same lines as above that there exists a past non-extendible time-like curve λ contained in $D^-(E^-(F))$.
- Finally the existence of the future non-extendible time-like curve γ in $D^+(E^+(S))$ and the past non-extendible time-like curve λ contained in $D^-(E^-(F))$ is shown to imply the existence of an non-extendible causal geodesic without conjugate points (contained in $D(E^+(S))$) in contradiction to I).

2.5. Singularities and Causality Violation

In Theorem 1 the space-time M is assumed to be chronological. This requirement, however, being a global one cannot be verified by experiments of local character. Thus the lack of experience cannot be taken as a physical justification for the absence of causality violations. This lack of evidence encouraged scientists to examine the relation of causality violations to the existence of singularities ([New], [T]) and finally led Kriele to seek generalisations of Theorem 1 and another singularity theorem [H3], that apply to non-chronological space-times ([K2], [K3],[K4]).

Assumption iii) in Proposition 1 required the existence of a so-called trapped set $S \neq \emptyset$, i.e. a set such that $E^+(S)$ or $E^-(S)$ is compact. The chronology condition imposed assured that $S \subset E^{\pm}(S) \neq \emptyset$. A generalised future horismos $e^+(S)$ satisfying $S \subset e^+(S)$ even in the case of chronology violations and preserving the non-global features of a horismos can be defined as follows.

Consider a compact set D that is achronal in some neighbourhood U_D of D which means that given two points $p, q \in D$ there is no time-like curve joining p and q that is contained in U_D. As a generalisation of the notion of focal points of space-like surfaces one says that a point p on a future directed null geodesic β starting in $r \in D$ is a *focal point* with base r, if for all $q \in J^+(p)$ along β there exists a time-like curve from D to q arbitrarily close to β but not for any q on β before p.

For a point $p \in D$ let β_p be the maximal geodesic prolongation of the generator of $E^+(D, U_D)$ with $\beta(0) = p$ that has no focal point to the future of p. The set $e^+(D, M) = e^+(D) := \overline{D \cup \{\beta_p(t) \mid p \in D,\, t > 0\}}$ is called the *generalised future horismos* of D while $cl(D, M, +) := \{q \in e^+(D, M) \mid q \text{ is the future endpoint of some generator of } e^+(D, M)\}$ is referred to as its *cut locus*.

Studying the relation of causality violations to the existence of singularities the existence of a chronological neighbourhood of $e^+(S)$ turned out to provide essential help in order to generalise Theorem 1 to the case of causality violations. This insight led to assumptions on the cut locus of the generalised future horismos. In order to f ormulate these assumptions we do need two further definitions.

Let h be some arbitrary Riemannian metric on M. For a curve α on M $\tilde{\alpha} : I \longrightarrow M$ denotes a unit speed reparametrisation of α $(h(\tilde{\alpha}', \tilde{\alpha}') = 1)$. α is said to be *almost close* if there exists an $X \in \overline{\{\tilde{\alpha}'(t) \mid t \in I\}}$ such that for every neighbourhood U_X of X in TM there exists a closed curve $\gamma : I \longrightarrow M$ with $\gamma|_{I_{\neg U_X}} = \tilde{\alpha}$ and $\gamma'(I_{U_X}) \subset U_X$ where I_{U_X} and $I_{\neg U_X}$ are defined via $I_{U_X} := \{t \in I \mid \tilde{\alpha}(t) \in \pi_{TM}(U)\}$ and $I_{\neg U_X} := I \setminus I_{U_X}$. Note that this definition is independent of the specific Riemannian metric h chosen.

A causal curve λ is said to be a *limit curve* of a sequence of curves $\{\lambda_n\}_n$ if there is a subsequence $\{\lambda'_n\}_n$ of $\{\lambda_n\}_n$ such that for every point p on λ the sequence $\{\lambda'_n\}_n$ converges to p, i.e., for any neighbourhood U_p of p almost all λ'_n intersect U_p. A limit curve of a sequence of segments of some curve γ is simply called a limit curve of γ.

With these notions we can finally state the generalisation of Theorem 1:

Theorem 2. [K2], [K3], [K4] *A space-time M is not causal geodesically complete if*

i) *The strong energy condition holds on M;*

ii) *the generic condition holds on M;*

iii) *there exists at least one of the following:*

- **a)** *a compact achronal set without edge,*
- **b)** *a closed locally space-like (but not necessarily achronal) trapped surface S,*

c) *a point p such that on every past (or every future) null geodesic from p the divergence θ of the null geodesics from p becomes negative.*

iv) *Neither $cl(S, M, +)$ (respectively $cl(\{p\}, M, +)$) nor any $cl(D, M, -)$, where D is a compact topological submanifold (possibly with boundary) with $D \cap S \neq \emptyset$ (respectively $p \in D$) contains any almost closed causal curve that is a limit curve of a sequence of closed time-like curves.*

The basic idea of the proof of this theorem is to show that a space-time satisfying the assumptions of Theorem 2 is locally isometric to one satisfying the assumptions of Theorem 1. By construction of the space-time M^* that is locally isometric to M it then follows that the incomplete maximal geodesic that exists in M^* due to Theorem 1 projects on one in M which is inextendible as well ([K3]).

Theorem 2 is a fairly strong generalisation of Theorem 1 in the sense that it allows causality violation almost everywhere in M except from the cut loci of appropriate sets. Causality violation in these regions, however, seems to prevent the space-time from having singularities. It is now natural to ask whether – the other way around – causality violation can produce singularities under certain circumstances. This question has been answered by another theorem of Kriele:

Theorem 3. [K1] *Let M be a space-time with chronology violating set V and $W \neq \emptyset$ be a compact connected component of the boundary $\dot{V}$ of V.*
If $Ric(k, k) \geq 0$ $\forall$ null vectors k in $\pi_{TM}^{-1}(W)$ and the generic condition holds in W, then every generator in W has a limit curve that is an almost closed but incomplete null geodesic generator.

It follows that chronology violating sets that do not extend to infinity must contain singularities.

Not only Theorem 3 but also Theorem 2 has been generalised further in [MI]. Moreover, in a series of papers Kriele and other authors studied space-times where the current Lorentzian metric had been Riemannian in the past, the signature change behaving conveniently at a suitable hypersurface. But also such a change does not prevent that singularities may occur in the Riemannian "past". Such models have been developed in [KM].

2.6. A Classification of Non-Curvature Singularities

Singularities are associated with some kind of incompleteness of the space-time. In fact all the singularities predicted by the theorems above correspond to a geodesic incompleteness of the space-time under consideration.

Given a specific coordinate representation of a solution (M, g) of Einstein's field equations that admits incomplete causal curves the question arises whether these curves correspond to proper curvature singularities or whether there exists some extension $\tilde{M}$ of M such that the incomplete curves in M extend to complete curves in $\tilde{M}$. In the case that such an extension is C^∞ the singularities in M are just artefacts of a bad choice of coordinates. These singularities are referred to as *apparent singularities*. Extendibility in general, however, crucially depends on the criteria chosen for extendibility, and even in the case that there does not exist a C^∞-extension of M there still might be an extension of lower differentiable order. In that case the singularities in M are said to be *weak singularities*.

The most popular example of an apparent singularity is of course the $r = 2m$ singularity in the static, spherically symmetric Schwarzschild space-time. In [Lim] Lim considered the class of static, spherically symmetric space-times (M, g) with metric

$$g \;=\; -F(u)\,dt\otimes dt \;+\; G(u)\,du\otimes du \;+(u+r_0)^2\left(d\theta\otimes d\theta \;+\; sin^2\theta\,d\phi\otimes d\phi\right), \qquad (1)$$

where $r := u + r_0$ denotes the area-radius of the spheres of symmetry and $grad(r)$ is assumed to be space-like. (It can be shown that with $grad(r)$ being space-like it is always possible to choose the coordinate function t such that F and G are positive.) The functions F and G are considered as being of generalised power series type: $F = u^\alpha f(u)$ and $G(u) = u^\beta g(u)$ with $f, g \in C^\infty(M)$ bounded, and bounded away from zero at $u = 0$, $\alpha, \beta \in \mathcal{R}$.

In addition to that appropriate energy conditions were imposed that restrict the class of space-times further to those with physically reasonable matter models. These energy conditions, appearing in the theorem below, are the weak energy condition (WEC), the strong energy condition (SEC) and the dominant energy condition (DEC). With $\gamma := \frac{1}{2}(-\alpha + \beta + 2)$ and $\sqrt{\frac{g(u)}{f(u)}} =: \sum_{i=0}^{n} c_i u^i + u^{n+1}\tilde{c}(u)$, where $c_i \in \mathcal{R}$ and $\tilde{c}$ is a smooth function, Lim derived the following remarkable classification of extendibility of the space-times being considered:

Theorem 4. [Lim] *Assume that $c_{-\gamma} = 0$ when $-\gamma \in \mathcal{N}\backslash\{0\}$, $r_0 > 0$, and let $k \in \mathcal{N}\backslash\{0\}$. All spacetimes of the form (M, g) with g as given in (1) that have bounded curvature and satisfy an appropriate energy assumption (either* WEC, SEC *or* DEC*) fall in one of the following three classes:*

i) *The metric is smoothly extendible.*
This is the case if and only if $\alpha = -\beta \in \mathcal{N} \setminus \{0\}$.

ii) *The metric is C^k-extendible but not C^{k+1}-extendible.*
This is the case if and only if $\alpha = -\beta$ and $k + 1 < \alpha < k + 2$.

iii) *The metric is not even C^1-extendible.*
This is the case if and only if $\alpha > -\beta > 2$.

2.7. Cosmic Censorship

The incompleteness theorems above merely indicate the existence of singularities, whereas it is the exact solutions that give an idea of what those do look like. Already in 1969 the examination of some exact solutions led Penrose to write “We are thus presented with what is perhaps the most fundamental unanswered question of general-relativistic collapse theory, namely: does there exist a “**cosmic censor**” who forbids the appearance of naked singularities, clothing each one in an absolute event horizon?” Since then various formulations of cosmic censorship have been proposed and discussed and a division of weak and strong cosmic censorship has been established (see for instace [TCE]).

To obtain a notion of strong cosmic censorship there are basically two different approaches. While the one is based on considering the evolution of given initial data on a partial Cauchy surface (PDE-approach), the other one is a more geometric approach, which is based on imposing certain causality and energy conditions. The latter one might be presented by the

Conjecture 1. (*geometric strong cosmic censorship*)
Let (M, g) be a Lorentzian manifold and assume that the space of Lorentzian metrics on M, $Lor(M)$, is endowed with a topology τ. Assume further that

i) *(M, g) is chronological,*

ii) *the strong energy condition holds,*

iii) *the dominant energy condition holds,*

iv) *there exists a closed trapped surface $\mathcal{T}$.*

Then in each τ-neighbourhood U_g of g in $Lor(M)$, there is a metric $\tilde{g}$ satisfying i),ii) *and* iii) *such that each future inextensible, incomplete causal geodesic $\tilde{\gamma} \subset I^+(\mathcal{T})$ has a future endpiece $\tilde{\gamma}_f$ which is contained in some globally hyperbolic set.*

In [K5] Kriele presented a class of space-times that provides counterexamples to this conjecture for various topologies chosen on $Lor(M)$. As a consequence the future of the singularities associated with the $\tilde{\gamma}_f$ would not be predictable using the regular initial data for a partial Cauchy surface located in their past. This stable class of space-times contains singularities that are visible from future null infinity $\mathcal{J}^+$, i.e. singularities such that there exists a past inextensible, incomplete causal curve from $\mathcal{J}^+$ terminating at the singularities. Those singularities are referred to as *naked singularities*.

The counterexamples mentioned show that more detailed assumptions on the matter model than those required in Conjecture 1 must be made in order to ensure cosmic censorship.

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