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Minimal submanifolds of Kähler-Einstein manifolds without complex directions and of half dimension[1]

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Abstract. We study minimal real $2n$-submanifolds $F : M \to N$ without complex directions immersed into a Kähler-Einstein manifold of complex dimension $2n$. We find conditions, for example on the curvature of M and N, or equal Kähler-angles, or broad-pluriminimality of F, to conclude that F must be a Lagrangian submanifold. Our main tool is a formula of the Laplacian of a certain symmetric function of the Kähler angles of F.

1. Introduction

This paper is a summary of part of research results, [5] and [6], started two years ago, on the study of immersions into Kähler-Einstein manifolds through its Kähler angles, and that is a generalization of part of the work of Wolfson in [7] to higher dimensions.

Let (N, J, g) be a Kähler manifold of complex dimension $2n$, with complex struture J and Riemannian metric g, and let $F : M \to N$ be an immersed submanifold of real dimension

[1]This is a summary of research results which have been submitted to publication elsewhere.
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$2n$. We denote by ω the Kähler form of N, $\omega(X,Y) = g(JX,Y)$. On M we take the induced metric $g_M = F^*g$. N is Kähler-Einstein if its Ricci tensor is a multiple of the metric, $Ricci^N = Rg$. We start by recalling a result of Wolfson:

Theorem 1.1 [7] *If M is a real compact orientable surface and N is a complex Kähler-Einstein surface with $R < 0$, and if F is minimal with no complex points, then F is Lagrangian.*

The main idea of the proof is the following. Since $n = 1$, $F^*\omega$ is a multiple of the volume element of M, $F^*\omega = aVol_M$, with $a : M \to [-1,1]$ a real smooth map. Then $a = \cos\theta$, where θ is called the Kähler angle of the surface. If F is minimal without complex points, that is, $|\cos\theta| \neq 1$, then

$$\triangle(\kappa(\cos\theta)) = -R\cos\theta$$

for some convenient real map $\kappa : (-1,1) \to I\!R$. If $R < 0$, an application of the maximum and minimum principle implies $\cos\theta = 0$ everywhere. In some cases (see Sections 2, 3) we generalize this result to higer dimensions.

First we recall the concept of Kälher angles of F, introduced by Chern and Wolfson [1] for surfaces, as some functions that at each point p of M measure the deviation of the tangent plane T_pM of M from a complex subspace or a Lagrangian subspace of $T_{F(p)}N$. At each base point $p \in M$, we identify $F^*\omega$ with a skew-symmetric operator of T_pM by $g_M(F^*\omega(X), Y) = F^*\omega(X,Y)$. The polar decomposition of $F^*\omega$ is given by

$$F^*\omega = J_\omega\,\tilde{g}, \tag{1.1}$$

where $J_\omega : T_pM \to T_pM$ is a (unique) partial isometry with the same kernel $\mathcal{K}_\omega$ as of F^*w, and where $\tilde{g}$ is the positive semidefinite operator $\tilde{g} = |F^*\omega| = \sqrt{-(F^*\omega)^2}$. It turns out that $J_\omega : \mathcal{K}_\omega^\perp \to \mathcal{K}_\omega^\perp$ defines a complex structure on $\mathcal{K}_\omega^\perp$, the orthogonal complement of $\mathcal{K}_\omega$ in T_pM. We denote by Ω^0_{2k} the set of interior points in M where F^*w has constant rank $2k$, $0 \leq k \leq n$. Then $\mathcal{K}_\omega^\perp$ is a smooth sub-vector bundle of TM on Ω^0_{2k}. Moreover, $\tilde{g}$ and J_ω are both smooth on those open sets, and $(\mathcal{K}_\omega^\perp, J_\omega, g_M)$ constitutes a smooth Hermitean vector bundle. We should note that on Ω^0_{2n}, the open set where $F^*\omega$ is non-degenerate, the almost complex struture J_ω is not integrable in general. The positive semidefinite tensor $\tilde{g}$ is in fact continuous in all M and locally Lipschitz. At each point p we may take $\{X_\alpha, Y_\alpha\}_{1\leq\alpha\leq n}$ a g_M-orthonormal basis of T_pM that diagonalizes $F^*\omega$ at p. By this one means that, in that basis, $F^*\omega$ is matricially a sum of block matrices of the form

$$F^*\omega = \bigoplus_{0\leq\alpha\leq n}\begin{bmatrix} 0 & -a_\alpha \\ a_\alpha & 0 \end{bmatrix},$$

where $a_1, a_2, \ldots, a_n$ are real numbers, also called the singular values of $F^*\omega$. From

$$|a_\alpha| = |F^*\omega(X_\alpha, Y_\alpha)| = |g(JdF(X_\alpha), dF(Y_\alpha))| \leq 1$$

we have $a_\alpha = \cos\theta_\alpha$, for some angle θ_α. We may assume $\cos\theta_1 \geq \ldots \geq \cos\theta_n \geq 0$, by interchanging X_α with Y_α, and reordering the basis when necessary.

Definition 1.1 *These angles θ_α, $1 \leq \alpha \leq n$, are called the Kähler angles of F at p.*

Then, $\forall\alpha$, $F^*\omega(X_\alpha) = \cos\theta_\alpha Y_\alpha$, $F^*\omega(Y_\alpha) = -\cos\theta_\alpha X_\alpha$, and, if $2k$ is the rank of $F^*\omega$ at p with $k \geq 1$, $J_\omega X_\alpha = Y_\alpha$, $\forall\alpha \leq k$. Since the map $A \rightarrow |A|$ is a Lipschitz map in the space of normal operators, the Weyl's pertubation theorem applied to the eigenvalues of the symmetric operator $|F^*\omega|$ shows that, ordering the $\cos\theta_\alpha$ in the above decreasing way, for each α the map $p \rightarrow cos\theta_\alpha(p)$ is locally Lipschitz on M. A *complex direction* of F is a real two-plane P of T_pM such that $dF(P)$ is a complex line of $T_{F(p)}N$, that is, $JdF(P) \subset dF(P)$. Similarly, P is called a *Lagrangian direction* of F if ω vanishes on $dF(P)$, that is, $JdF(P) \perp dF(P)$. F has no complex directions iff $\cos\theta_\alpha < 1$ $\forall\alpha$. We note that for $n = 1$ our definition of Kähler angles may be slightly different from the original one in [1], because we demanded $\cos\theta \geq 0$. Our $\cos\theta$ may not be smooth near a point where it vanishes. We have chosen this definition because in higher dimensions we do not have a preferential orientation assigned to the real planes $P_\alpha = span\{X_\alpha, Y_\alpha\}$, even if M is orientable.

Let us consider the complex vectors of the complexified tangent space of M at p, $T^c_p(M)$, where $\{X_\alpha, Y_\alpha\}_{1\leq\alpha\leq n}$ diagonalizes $F^*\omega$ at p,

$$Z_\alpha = \frac{X_\alpha - iY_\alpha}{2} = \text{“}\alpha\text{”}, \qquad Z_{\bar\alpha} = \overline{Z_\alpha} = \frac{X_\alpha + iY_\alpha}{2} = \text{“}\bar\alpha\text{”}.$$

We extend g_M and the curvature tensors of M and N to the complexified tangent spaces by $\mathbb{C}$-multilinearity. Let $NM = (dF(TM))^\perp$ denote the normal bundle of F, and $(\ )^\perp$ denote the orthogonal projection of $F^{-1}TN$ onto the normal bundle. If F is any immersion of real dimension $2n$ and no complex directions, then $\{dF(Z_\alpha),\, dF(Z_{\bar\alpha}),\, (JdF(Z_\alpha))^\perp,\, (JdF(Z_{\bar\alpha}))^\perp\}_{1\leq\alpha\leq n}$ constitutes a complex basis of $T^c_{F(p_0)}N$. On M the Ricci tensor of N can be described by the following expression, for $U, V \in T_{F(p)}N$,

$$Ricci^N(U,V) = \sum_{1\leq\mu\leq n} \frac{4}{\sin\theta_\mu} R^N(U, JV, dF(\mu), (JdF(\bar\mu))^\perp),$$

where R^N denotes the Riemannian curvature tensor of N. An application of the Codazzi's equation to the above expression leads the first result:

Proposition 1.1 [5] *If F is a totally geodesic map without complex directions and N is Kähler-Einstein with non-zero Ricci tensor, then F is Lagrangian.*

We consider the following morphism of vector bundles

$$\begin{array}{rccc} \Phi: & TM & \rightarrow & NM \\ & X & \rightarrow & (JdF(X))^\perp. \end{array}$$

Both TM and NM are real vector bundles of the same dimension $2n$. If F has no complex directions, then Φ is an isomorphism. Moreover, the tensor $\hat{g}$ given by

$$\hat{g}(X,Y) = g_M(X,Y) - g(F^*\omega(X), F^*\omega(Y))$$

defines a smooth Riemannian metric on M, and $\hat{g} = g_M - \tilde{g}^2 = \sum_\alpha \sin^2\theta_\alpha Z_*^\alpha \odot Z_*^{\bar{\alpha}}$, where $\tilde{g}$ is given by (1.1). With this metric, $\Phi : (TM, \hat{g}) \to (NM, g)$ is an isometry, that is, Φ is an isomorphism of Riemannian vector bundles. Let us denote by

∇	the Levi-Civita connection of (M, g_M)
$\hat{\nabla}$	the Levi-Civita connection of $(M, \hat{g})$
$\nabla^\perp$	the usual connection of NM induced by the Levi-Civita connection of N
∇'	the connection on TM that makes the isomorphism Φ parallel.

We will also denote by ∇ the Levi-Civita connection of N. Thus, if U is a smooth section of $NM \subset F^{-1}TN$ and X is a vector field on M,

$$\nabla_X^\perp U = (\nabla_X U)^\perp$$

and $\nabla' = \Phi^{-1*}\nabla^\perp$, that is, for X, Y smooth vector fields of M,

$$\Phi(\nabla'_X Y) = \nabla_X^\perp(\Phi(Y)).$$

The connections ∇ and $\hat{\nabla}$ have no torsion because they are Levi-Civita, but ∇' may have non-zero torsion T'. Since both $\hat{\nabla}$ and ∇' are Riemannian connections of TM for the same Riemannian metric $\hat{g}$, $T' = 0$ iff $\hat{\nabla} = \nabla'$. Let us denote by $\nabla_X dF(Y) = \nabla dF(X, Y)$ the second fundamental form of F, which is a symmetric tensor on M that takes values on the normal bundle NM.

Lemma 1.1 [6] *If $\{X_\alpha, Y_\alpha\}$ is a diagonalizing g_M-orthonormal basis of $F^*\omega$ at p, then at p*

$$\begin{aligned}\Phi(T'(Z_\alpha, Z_{\bar{\beta}})) &= i(\cos\theta_\alpha + \cos\theta_\beta)\nabla_{Z_\alpha} dF(Z_{\bar{\beta}})\\ \Phi(T'(Z_\alpha, Z_\beta)) &= i(\cos\theta_\alpha - \cos\theta_\beta)\nabla_{Z_\alpha} dF(Z_\beta).\end{aligned}$$

Proof. Note that, if $X \in TM$, $JdF(X) - dF(F^*\omega(X)) \in NM$. Thus

$$\Phi(X) = JdF(X) - dF(F^*\omega(X)).$$

Then

$$\begin{aligned}\Phi(\nabla'_X Y) &= \nabla_X^\perp(\Phi(Y)) = \big(\nabla_X(\Phi(Y))\big)^\perp\\ &= \big(\nabla_X(JdF(Y) - dF(F^*\omega(Y)))\big)^\perp\\ &= \big(J\nabla_X dF(Y) + JdF(\nabla_X Y) - \nabla_X dF(F^*\omega(Y))\big)^\perp.\end{aligned}$$

Therefore, using the symmetry of ∇dF and the fact that ∇ is torsionless,

$$\begin{aligned}\Phi(T'(X, Y)) &= \Phi(\nabla'_X Y - \nabla'_Y X - [X, Y])\\ &= -\nabla_X dF(F^*\omega(Y)) + \nabla_Y dF(F^*\omega(X)).\end{aligned} \tag{1.2}$$

The lemma now follows immediately. □

This lemma will be useful in the following. It also shows that T' vanishes in certain directions for broadly-pluriminimal immersions or for immersions with equal Kähler angles.

2. A formula

For a minimal immersion F with no complex directions, we consider the map

$$\kappa = \sum_{1\leq\alpha\leq n} \log\left(\frac{1+\cos\theta_\alpha}{1-\cos\theta_\alpha}\right) = \frac{1}{2}\log\left(\frac{\det(g_M+\tilde{g})}{\det(g_M-\tilde{g})}\right).$$

This map is non-negative, and vanishes at a point p iff all $\cos\theta_\alpha = 0$ at that point. It is smooth on each Ω^0_{2k}, continuous on all M, and locally a Lipschitz map. We can compute $\triangle\kappa$ on those open sets where κ is smooth, but this gives a very long computation. In [5], [6] we derive an expression for $\triangle\kappa$ at $p_0 \in \Omega^0_{2k}$, namely

$$\begin{aligned}
\triangle\kappa \;=\;& 4i\sum_\beta Ricci^N(JdF(\beta), dF(\bar\beta)) \qquad (2.1)\\
&+\sum_{\beta,\mu}\frac{32}{\sin^2\theta_\mu} Im\Big(R^N(dF(\beta),dF(\mu),dF(\bar\beta),JdF(\bar\mu)+i\cos\theta_\mu dF(\bar\mu))\Big)\\
&-\sum_{\beta,\mu,\rho}\frac{32(\cos\theta_\mu+\cos\theta_\rho)}{\sin^2\theta_\mu\sin^2\theta_\rho} Re\Big(g(\nabla_\beta dF(\mu),JdF(\bar\rho))g(\nabla_{\bar\beta}dF(\rho),JdF(\bar\mu))\Big)\\
&+\sum_{\beta,\mu,\rho}32\Big(\frac{1}{\sin^2\theta_\rho}-\frac{1}{\sin^2\theta_\mu}\Big) Im\Big(\langle\nabla_\beta\mu,\rho\rangle g(\nabla_{\bar\beta}dF(\bar\rho),JdF(\bar\mu))\Big)\\
&+\sum_{\beta,\mu,\rho}32\Big(\frac{1}{\sin^2\theta_\rho}-\frac{1}{\sin^2\theta_\mu}\Big) Im\Big(\langle\nabla_{\bar\beta}\mu,\rho\rangle g(\nabla_\beta dF(\bar\rho),JdF(\bar\mu))\Big)\\
&+\sum_{\beta,\mu,\rho}\frac{32(\cos\theta_\mu+\cos\theta_\rho)}{\sin^2\theta_\mu}\Big(|\langle\nabla_\beta\mu,\rho\rangle|^2+|\langle\nabla_{\bar\beta}\mu,\rho\rangle|^2\Big),
\end{aligned}$$

where $\{X_\alpha, Y_\alpha\}_{1\leq\alpha\leq n}$ is a g_M-orthonormal local frame of M, with $Y_\alpha = J_\omega X_\alpha$ for $\alpha\leq k$, $\{X_\alpha, Y_\alpha\}_{\alpha\geq k+1}$ any g_M-orthonormal frame of $\mathcal{K}_\omega$, and which at p_0 diagonalizes $F^*\omega$. There are two cases where the above equation is simplified. The first one is when F is broadly-pluriminimal, and the second one when F has equal Kähler angles. We will treat these cases in Sections 3 and 4, respectively.

3. Broadly-pluriminimal submanifolds

Definition 3.1 [5] *F is said to be broadly-pluriminimal if*
(i) *F is minimal,*
(ii) *For each $p\in\Omega^0_{2k}$, $k\geq 1$, F is pluriharmonic with respect to any g_M-orthogonal complex structure $\tilde{J} = J_\omega \oplus J'$ on T_pM, where J' is any g_M-orthogonal complex structure of $\mathcal{K}_\omega$, that is, $(\nabla dF)^{(1,1)} = 0$.*

In (ii), the (1,1)-part of ∇dF is just given by

$$(\nabla dF)^{(1,1)}(X,Y) = \frac{1}{2}\Big(\nabla dF(X,Y)+\nabla dF(\tilde{J}X,\tilde{Y})\Big) \quad \forall X,Y\in T_pM.$$

If $\mathcal{K}_\omega = 0$, (ii) means that F is pluriharmonic with respect to the almost complex structure J_ω (see e.g. [4]). Pluriharmonic maps are trivially minimal. Products of minimal real surfaces of

Kähler surfaces, minimal Lagrangian submanifolds, and complex submanifolds are examples of pluriminimal submanifolds. For F with equal Kähler angles Ω^0_{2k} is only considered and $\tilde{J} = J_\omega$.

If F is broadly-pluriminimal, we get the following very simple final expression for $\triangle\kappa$ on Ω^0_{2k},

$$\triangle\kappa = 4i \sum_{1\leq\beta\leq k} Ricci^N(JdF(\beta), dF(\bar{\beta})).$$

Consequentely,

Proposition 3.1 [5] *If N is Kähler-Einstein and F is broadly-pluriminimal without complex directions, then, on each Ω^0_{2k}*

$$\triangle\kappa = -2R\Big(\sum_{1\leq\beta\leq n} \cos\theta_\beta\Big).$$

The main problem in applying the maximum principle to this equation at a maximum point p_0 of κ for $n \geq 2$, as Wolfson did in [7] for the case $n = 1$, is that p_0 may not lie on a open set where κ is smooth. Since we can write $\kappa = \kappa_1 + \kappa_2$, where $\kappa_1 = \sum_{1\leq\alpha\leq 2k} \log\left(\frac{1+\cos\theta_\alpha}{1-\cos\theta_\alpha}\right)$ (with $2k$ the rank of $F^*\omega$ at p_0) is the (smooth) piece of κ defined by the angles whose cosine is not zero near p_0 , and κ_2 with the remaining angles, κ_1 and κ_2 having at p_0 a maximum and a minimum respectively, we can prove that κ is differentiable at p_0, but we do not know if it is C^2 on a neighbourhood of that point. The following lemma is an immediate conclusion from the above formula and will lead to the next theorem.

Lemma 3.1 [5] *If N is Kähler-Einstein with $R < 0$, and if F is broadly-pluriminimal but not Lagrangian and has no complex directions, then p_0 is not in Ω^0_{2k} $\forall 0 \leq k \leq n$. That is, if the rank of $F^*\omega$ is $2k$ at p_0, then there exists a sequence $p_m \to p_0$ such that the rank of $F^*\omega$ at p_m is $> 2k$.*

Theorem 3.1 [5] *Assume N is Kähler-Einstein with $R < 0$, and M is compact.*

(i) *If F is broadly-pluriminimal, then F either has complex or Lagrangian directions.*

(ii) *If F is broadly-pluriminimal without complex directions, and $F^*\omega$ has constant rank or rank ≤ 2, then F is Lagrangian.*

(iii) *If $n = 2$, that is, M has real dimension 4 and N complex dimension 4, and if M is orientable and F is broadly-pluriminimal without complex directions, then F is Lagrangian.*

To prove (iii) we replace the continuous map κ by a smooth map $\tilde{\kappa}$ which coincides with κ on an open set that has p_0 at the boundary and allows p_0 as a maximum as well. Namely,

$$\tilde{k} = \log\left(\frac{1+\cos\theta_1}{1-\cos\theta_1}\right) + \log\left(\frac{1+s_2}{1-s_2}\right),$$

where $s_2(p) = \epsilon(p)\cos\theta_2(p)$ for p near p_0, with $\epsilon(p)$ equal to $+1$ or -1 according as X_1, Y_1, X_2, Y_2 is a direct or inverse basis, respectively. We can prove that s_2 is smooth by using the smoothness of $F^*\omega \wedge F^*\omega$. Unfortunately, a similar argument does not seem to work in higher dimensions.

4. Immersions with equal Kähler angles

We say that F *has equal Kähler angles* if $\theta_\alpha = \theta\ \forall\alpha$. In this case

$$F^*\omega = \cos\theta\, J_\omega \qquad \text{and} \qquad \hat{g} = \sin^2\theta\, g_M,$$

with $\cos\theta$ a locally Lipschitz map on M, smooth on the open set where it does not vanish, and $\Omega^0_{2k} = \emptyset\ \ \forall k \neq 0, n$. On the open set $\Omega^0_{2n} = \cos\theta^{-1}(\mathbb{R}\backslash\{0\})$, J_ω defines a smooth almost complex structure g_M-orthogonal. On the open set $\cos\theta^{-1}(\mathbb{R}\backslash\{1\})$, $\hat{g}$ is a smooth metric conformally equivalent to g_M. Thus, if $n \geq 2$, $\hat{\nabla} = \nabla$ iff θ is constant. Note that in this case any local orthonormal frame of the type $\{X_\alpha, Y_\alpha = J_\omega X_\alpha\}$ diagonalizes $F^*\omega$ on the whole set it is defined. Moreover, from (1.2) of Lemma 1.1, we get

$$\Phi(T'(X,Y)) = 2\cos\theta(\nabla dF)^{(1,1)}(J_\omega X, Y). \tag{4.1}$$

This shows that

Proposition 4.1 [6] *If F is a minimal immersion with equal Kähler angles and without complex directions, then $T' = 0$, that is, $\nabla' = \hat{\nabla}$ iff Φ is parallel iff F is pluriminimal.*

We may extend $\Phi : T^cM \to NM^c$ by $\mathbb{C}$-linearity to the complexified spaces, and we define $Re(u + iv) = u$, for $u, v \in NM$.

Proposition 4.2 [6] *If F is any immersion with equal Kähler angles and no complex directions, then*

$$\Phi\left(\tfrac{1-n}{2}\nabla\log(\sin^2\theta)\right) = \frac{4\cos\theta}{\sin^2\theta} Re\left(i \sum_{\beta,\mu}\Big(g(\nabla_{\bar{\mu}} dF(\mu), JdF(\beta)) - g(\nabla_{\bar{\mu}} dF(\beta), JdF(\mu))\Big)\Phi(\bar{\beta})\right).$$

where $\nabla\log(\sin^2\theta)$ is the gradient with respect to g_M.

Since $\{\Phi(\beta), \Phi(\bar{\beta}) = \overline{\Phi(\beta)}\}_{1\leq\beta\leq n}$ multiplied by $\frac{\sqrt{2}}{\sin\theta}$ constitutes an unitary basis of NM^c, we immediately conclude

Corollary 4.1 [6] *Let F be an immersion with equal Kähler angles, no complex directions, and $n \geq 2$. Then θ is constant iff*

$$\sum_\mu g(\nabla_{\bar{\mu}} dF(\mu), JdF(\beta)) = \sum_\mu g(\nabla_{\bar{\mu}} dF(\beta), JdF(\mu)) \qquad \forall\beta. \tag{4.2}$$

In particular, if F is a pluriminimal immersion, then $\nabla = \hat{\nabla} = \nabla'$ and $\theta =$ constant.

To prove Proposition 4.2 we relate the three connections of M, ∇, $\hat{\nabla}$, and ∇'. Any local g_M-orthonormal frame of the form $\{e_1, \ldots, e_{2n}\} = \{X_\mu, Y_\mu = J_\omega X_\mu\}_{1\leq\mu\leq n}$ diagonalizes $F^*\omega$. Let

$$S'(X,Y) = \nabla'_X Y - \hat{\nabla}_X Y.$$

Then $S'(X,Y) - S'(Y,X) = T'(X,Y)$ and so $S' = 0$ iff $\nabla' = \hat{\nabla}$ iff S' is symmetric.
In [6] we prove

$$\sum_i \hat{g}(S'(e_i, e_i), X) = -\sum_i \hat{g}(T'(e_i, X), e_i). \tag{4.3}$$

We may compute

$$\frac{(1-n)}{4} \nabla \log \sin^2 \theta = \sum_\mu \hat{\nabla}_{\bar{\mu}} \mu - \nabla_{\bar{\mu}} \mu.$$

Now

$$\Phi(\nabla'_X \mu) = \Big((J - i\cos\theta) \nabla_X dF(\mu) \Big)^\perp + \Phi(\nabla_X \mu).$$

It follows that

$$\Phi(\sum_\mu \hat{\nabla}_{\bar{\mu}}\mu - \nabla_{\bar{\mu}}\mu) = \Big((J - i\cos\theta)\frac{H}{4}\Big)^\perp - \sum_\mu \Phi(\nabla'_{\bar{\mu}}\mu - \hat{\nabla}_{\bar{\mu}}\mu) = \frac{1}{4}\Big(2n(JH)^\perp - \Phi(Trace_{g_M} S')\Big),$$

where $H = \frac{1}{2n}\sum_i \nabla dF(e_i, e_i) = \frac{2}{n}\sum_\mu \nabla dF(\bar{\mu}, \mu)$ is the mean curvature of F. On the other hand, using (4.3) and Lemma 1.1, we have

$$\Phi(Trace_{g_M} S') = \frac{4}{\sin^2\theta} \sum_{\mu,\beta} 2i\cos\theta \Big(g(\nabla_{\bar{\mu}} dF(\beta), JdF(\mu))\Phi(\bar{\beta}) - g(\nabla_\mu dF(\bar{\beta}), JdF(\bar{\mu}))\Phi(\beta) \Big).$$

With this, and writing $(JH)^\perp$ in terms of $\Phi(\beta)$ and $\Phi(\bar{\beta})$, we prove Proposition 4.2. □

Note that (4.2) of Corollary 4.1 is a sort of symmetry property (we may commute μ with β, and sum over μ), and the first term is just $\frac{n}{2} g(H, JdF(\beta))$. In [6] we compute the divergences of $F^*\omega$ and of J_ω considering $F^*\omega$ as an operator of TM. It is particulary interesting the case $n = 2$.

Proposition 4.3 *Let F be an immersion with equal Kähler angles and $\nabla \cos\theta$ denote the gradient w.r.t g_M.*

(i) *For $n = 1$, $\delta J_\omega = 0$ (obviously!). Moreover, $\delta(F^*\omega) = 0$ iff θ is constant.*

(ii) *For $n = 2$, $\delta(F^*\omega) = 0$ and $J_\omega(\nabla \cos\theta) = \cos\theta(\delta J_\omega)$. Hence, $\delta J_\omega = 0$ iff θ is constant.*

(iii) *For $n \neq 1, 2$, $\delta(F^*\omega) = (n-2) J_\omega(\nabla\cos\theta) = \frac{n-2}{n-1}\cos\theta(\delta J_\omega)$. Hence, $\delta(F^*\omega) = 0$ iff $\delta J_\omega = 0$ iff θ is constant iff (4.2) holds.*

If N is Kähler-Einstein, expression (2.1) for the Laplacian of κ, for F minimal with equal Kähler angles at a maximum point p_0 of κ with $\cos\theta(p_0) \neq 0$ and $\cos\theta(p_0) \neq 1$, can be simplified to

$$\triangle\kappa = \cos\theta\Big(-2nR + \frac{32}{\sin^2\theta}\sum_{\beta,\mu} R^M(\beta, \mu, \bar{\beta}, \bar{\mu}) + A + B\Big), \tag{4.4}$$

where R^M is the Riemannian curvature of M, and

$$A = \frac{4}{\sin^2\theta}\|\nabla J_\omega\|^2 \qquad B = \frac{8(n-1)}{\sin^2\theta}\|\nabla\cos\theta\|^2.$$

Then $A, B \geq 0$, and $A = 0$ on an open set of M iff (M, J_ω, g) is Kähler on that set. If $B = 0$ on an open set, then the equal Kähler angle is constant. The curvature term of (4.4)

$$\sum_{\beta,\mu} R^M(\beta, \mu, \bar{\beta}, \bar{\mu}) \tag{4.5}$$

is a hermitian trace of the curvature of M restricted to a maximal isotropic subspace of $T^c M$. To require it to be ≥ 0 seems to be strictly weaker than the non-negative isotropic sectional curvature defined by Micallef and Moore in [3], namely

$$K(\sigma) = \frac{g_M(\mathcal{R}(z \wedge w), \bar{z} \wedge \bar{w})}{||z \wedge w||^2} \geq 0,$$

where $\sigma = span_{\mathbb{C}}\{z, w\}$ is a totally isotropic two-plane in $T^c M$, that is, $u \in \sigma \Rightarrow g_M(u, u) = 0$, and where $g_M(\mathcal{R}(x \wedge y), u \wedge v) = R^M(x, y, u, v)$. Finally, we present the following result.

Theorem 4.1 [6] *Let F be minimal with equal Kähler angles, M compact orientable, and N Kähler-Einstein with $R \leq 0$. If $n = 2$ or $\theta =$ constant, and M has non-negative isotropic sectional curvature, then one and only one of the following cases holds:*

(i) *M is a complex submanifold of N.*

(ii) *M is a Lagrangian submanifold of N.*

(iii) *$R = 0$, $\cos\theta =$constant $\neq 0, 1$, and J_ω is a complex integrable structure, with (M, J_ω, g_M) a Kähler manifold.*

To prove the theorem for $n = 2$, we use the Weitzenböck formula applied to $F^*\omega$ (see [2]). In this case $F^*\omega$ is a harmonic 2-form, because it is closed and co-closed (by Proposition 4.3). Moreover, since F has equal Kähler angles, the curvature term (4.5) can be also expressed as a multiple of $< SF^*\omega, F^*\omega >$, where S is the Ricci operator applied to forms. We may then conclude that $\nabla F^*\omega = 0$, and so the Kähler angle is constant, and if $\cos\theta \neq 0$, (M, J_ω, g_M) is a Kähler manifold. If $n \neq 2$, we assume the Kähler angle is constant. Then, in both cases, equation (4.4) is valid on all M, with $\triangle\kappa = 0$. Of course, we may replace the condition on the curvature of M by the weaker condition (4.5) ≥ 0.
This theorem can for instance be applied to flat minimal tori immersed in Calabi-Yau manifolds.

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