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Projectable Non-linear Connections and Foliations[1]

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Abstract. The aim of this paper is to study some classes of foliations on a Lagrangian manifold, which are related to the Lagrange metric structure. Some sufficient conditions on the Lagrange metric to extend to a bundle-like metric for the lifted foliation on the tangent space, are given.

The Riemann foliations are of great interest, but their analogous for Finsler and Lagrange metrics is not yet studied. A major difficulty is that the Cartan-Kern connection (the analogous of Levi-Civita connection in the Lagrangian case) is not good for projection on leafs or on the transverse bundle of the foliation.

The aim of this paper is to study some classes of foliations on a Lagrangian manifold, which are related to the Lagrange metric structure. Every foliation $\mathcal{F}$ on a manifold M has a natural lift to a foliation $\mathcal{F}^T$ on the total space TM of the tangent bundle τM. Some sufficient conditions on the Lagrange metric to extend to a bundle-like metric for the lifted foliation $\mathcal{F}^T$ are given.

The content of the paper is as follows. The first section contains the definition of the projectable non-linear connection, using the paper [7]. The second section contains the main results of the paper. The adapted and totally adapted Lagrangian and the adapted non-linear connections for a foliation on a manifold are defined. The following two results are proved:

- The Kern non-linear connection of an adapted Lagrangian to a regular foliation is an adapted non-linear connection for the foliation (Theorem 1).
- If the semi-spray defined by the Lagrangian is basic for the foliation, then the lifted foliation is a Riemannian foliation (Theorem 2).

[1]The paper is in final form and no other version has been submitted for publication elsewhere.

1. Projectable non-linear connections

The definition of a projectable non-linear connection is given in [7] using:

Proposition 1. *Let $\xi = (E, \pi, M)$ and $\xi' = (E', \pi', M)$ be two vector bundles and $\xi \xrightarrow{P'} \xi'$ be an epimorphism of vector bundles $\xi = (E, \pi, M)$. Consider also a non-linear connection C' on ξ' and a right splitting S of the induced epimorphism $\tau E \xrightarrow{\tau P'} (P')^* \tau E'$ of vector bundles over the base E.*

Then there is a unique non-linear connection C on the vector bundle ξ which projects by $\tau P'$ the fibres of the horizontal bundle corresponding to C isomorphically on the fibres of the horizontal subbundle corresponding to C'.

Definition 1. *If $\xi \xrightarrow{f} \xi'$ is an epimorphism of vector bundles, we say that a non-linear connection C on ξ is projectable on ξ' if there exists a non-linear connection C' on ξ' and a right splitting of the induced morphism $\tau E' \xrightarrow{(P')^* f} (P')^* \xi'$ which induce, according to Proposition 1, the non-linear connection C.*

Consider some local coordinates (x^i, y^α, y^u) on E, (x^i, y^u) on E', $(x^i, y^\alpha, y^u, X^j, Y^\beta, Y^v)$ on TE, (x^i, y^u, X^j, Y^v) on TE' (around the corresponding points) which we call *adapted coordinates*. Notice that in this section $i, j, \ldots \in \{1, \ldots, m = \dim M\}$, $\alpha, \beta, \ldots \in \{1, \ldots, n\}$, $u, v, \ldots \in \{1, \ldots, n'\}$, where n and n' are the dimension of the fibres of ξ and ξ' respectively. Using these local coordinates, let us denote as (N_j^α, N_j^u) and $(\widetilde{N}_j^u)$ the local components of the non-linear connections C and C' respectively.

Proposition 2. *Let $\xi \xrightarrow{P'} \xi'$ be an epimorphism of vector bundles and C, C' be non-linear connections on ξ and on ξ' respectively. Then the non-linear connection C is projectable on the non-linear connection C' iff the relation*

$$N_j^u(x^i, y^\alpha, y^u) = \widetilde{N}_j^u(x^i, y^u) \tag{1}$$

holds between the local components of the non-linear connections C and C', in an adapted system of coordinates.

2. Lagrange spaces and foliations

Let us consider a Lagrange space $(M, \mathcal{L})$, where M is a smooth manifold and $\mathcal{L} : TM \to \mathbb{R}$ is a regular Lagrangian on M. According to a result of J. Kern (see [1, Theorem 4.1, p. 120]), there is a non-linear connection N on M which depends on the Lagrangian $\mathcal{L}$. In the sequel i, j, k, $\ldots \in \{1, \ldots, m = \dim M\}$.

Let us consider local coordinates (x^i) on M and some coordinates (x^i, y^j) on TM, adapted to the vector bundle structure. Then the Lagrange metric g has as local components $(g_{ij} = \frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j})$. The semi-spray $G : TM \to TTM$, which has the local form $(x^i, y^j, y^j, G^k(x^i, y^j))$, where:

$$G^i = \frac{1}{4} g^{ij} \left(\frac{\partial^2 \mathcal{L}}{\partial y^j \partial x^k} y^k - \frac{\partial \mathcal{L}}{\partial x^j} \right), \tag{2}$$

defines the Kern non-linear connection N, which has as local components $(N_j^i(x^k,y^l))$, where $N_j^i = \dfrac{\partial G^i}{\partial y^j}$.

Let us suppose now that a (regular) foliation $\mathcal{F}$, which has the dimension p and the codimension $q = m - p$, is given on the manifold M. Let us consider some local coordinates $(x^i) = (x^u, x^{\bar{u}})$ which are *adapted* to the foliation $\mathcal{F}$ $(u, v, \ldots \in \{1, \ldots, p\}$, $\bar{u}, \bar{v}, \ldots \in \{p + 1, \ldots, p + q = m\})$. It means that the change rule of these coordinates has the form:

$$x^{u'} = x^{u'}(x^u, x^{\bar{u}}),\ x^{\bar{u}'} = x^{\bar{u}'}(x^{\bar{u}}). \tag{3}$$

There are local maps

$$U \to \bar{U},\ (x^u, x^{\bar{u}}) \to (x^{\bar{u}}),$$

called *local projections*, and

$$\bar{U} \to U,\ (x^{\bar{u}}) \xrightarrow{s} (s^u(x^{\bar{u}}), x^{\bar{u}}),$$

called *local basic sections*, where U is the domain of a local adapted chart on M and $\bar{U}$ is the corresponding transverse domain. A *basic function* is a function $f \in \mathcal{F}(M)$ such that $X(f) = 0$, $(\forall) X \in \tau\mathcal{F}$, i.e. X is tangent to the leafs of $\mathcal{F}$. A (local) *basic vector field* is a vector field $X \in \mathcal{X}(M)$ which enjoys the property that $[X, Y] \in \tau\mathcal{F}$, $(\forall) Y \in \tau\mathcal{F}$, where $[\cdot,\cdot]$ denotes the Lie bracket. Notice that the basic vector fields are locally spanned by the vector fields $\left\{\dfrac{\partial}{\partial x^{\bar{u}}}\right\}$.

Consider a non-linear connection C on M. For every U, $\bar{U}$ and local section s as above, a non-linear connection $C_{\bar{U}}$ is induced on the vector bundle $s^*\tau U$ over the base $\bar{U}$. We say that a non-linear connection C on M is *adapted* if every local non-linear connection $C_{\bar{U}}$ is projectable (according to Definition 1) on a non-linear connection $\bar{C}_{\bar{U}}$ on the vector bundle $\tau\bar{U}$, for every local basic section s. The non-linear connection C has as local adapted components $\{N_v^u, N_v^{\bar{u}}, N_{\bar{v}}^u, N_{\bar{v}}^{\bar{u}}\}$, each of them having as variables $(x^u, x^{\bar{u}}, y^v, y^{\bar{v}}\}$. The non-linear connection $C_{\bar{U}}$ has as local components $\{N_v^u, N_v^{\bar{u}}, N_{\bar{v}}^u, N_{\bar{v}}^{\bar{u}}\} \circ (s^u(x^{\bar{u}}), x^{\bar{u}}, y^v, y^{\bar{v}})$. Using Proposition 2 and the definition of an adapted non-linear connection, it follows that C is a projectable non-linear connection iff the local functions $\{N_{\bar{v}}^{\bar{u}}\}$ do not depend on (y^v).

The foliation $\mathcal{F}$ can be lifted to a foliation $\mathcal{F}^T$ on TM , which has as adapted coordinates $(x^u, y^v, x^{\bar{u}}, y^{\bar{v}})$. The change rule of these coordinates are

$$x^{u'} = x^{u'}(x^u, x^{\bar{u}}),\ x^{\bar{u}'} = x^{\bar{u}'}(x^{\bar{u}}),$$

$$y^{u'} = \frac{\partial x^{u'}}{\partial x^u} y^u + \frac{\partial x^{u'}}{\partial x^{\bar{u}}} y^{\bar{u}},\ y^{\bar{v}'} = \frac{\partial x^{\bar{v}'}}{\partial x^{\bar{v}}} y^{\bar{v}}.$$

Notice that the transverse coordinates are $(x^{\bar{u}}, y^{\bar{v}})$. We denote as $V\mathcal{F}^T$ the integrable distribution on TM obtained as the intersection of the distributions $\tau\mathcal{F}^T$ (tangent to the foliation $\mathcal{F}^T$) and $V\tau M$ (the vertical distribution).

We say that a Lagrangian $\mathcal{L}$ on M is *adapted* to the foliation if the vertical basic vector fields of the foliation span a distribution which is orthogonal to $V\mathcal{F}^T$ in $V\tau M$. Using local coordinates adapted to the foliation, the vertical basic vector fields are local generated by the vector fields $\{\frac{\partial}{\partial y^{\bar{u}}}\}$. Then $\mathcal{L}$ is an adapted Lagrangian iff the following relation holds:

$$g_{\bar{u}u} = \frac{\partial^2 \mathcal{L}}{\partial y^{\bar{u}} \partial y^u} = 0. \tag{4}$$

We have the following result:

Theorem 1. *The Kern non-linear connection of an adapted Lagrangian to a regular foliation is an adapted non-linear connection for the foliation.*

Proof. We use local coordinates adapted to the foliation. The relation (4) implies that $\frac{\partial}{\partial y^u} g_{\bar{u}\bar{v}} = \frac{\partial^3 \mathcal{L}}{\partial y^u \partial y^{\bar{u}} \partial y^{\bar{v}}} = 0$, thus the local functions $\{g_{\bar{u}\bar{v}}\}$ and $\{g^{\bar{u}\bar{v}}\}$ do not depend on the tangent variables $\{y^u\}$. Using the relations (2) we have $\frac{\partial}{\partial y^u} N^{\bar{v}}_{\bar{u}} = \frac{\partial^2 G^{\bar{v}}}{\partial y^u \partial y^{\bar{u}}} = \frac{\partial^2 G^{\bar{u}}}{\partial y^{\bar{u}} \partial y^u} = 0$, then the local functions $\{N^{\bar{v}}_{\bar{u}}\}$ do not depend on $\{y^u\}$, thus the non-linear connection is locally projectable. □

Notice that using the same local coordinates, it can be proved in the same manner that the local functions $\{g_{uv}\}$, $\{g^{uv}\}$ and $\{N^v_u\}$ do not depend on the variables $\{y^{\bar{u}}\}$.

We say that $\mathcal{L}$ is a *basic* Lagrangian if the vertical sections orthogonal to the distribution $V\mathcal{F}^T$ are basic (or foliated) for the lifted foliation $\mathcal{F}^T$. Using adapted local coordinates, this condition becomes:

$$g_{u\bar{u}} = \frac{\partial^2 \mathcal{L}}{\partial y^u \partial y^{\bar{u}}} = 0, \ \frac{\partial}{\partial y^u} g_{\bar{u}\bar{v}} = \frac{\partial^3 \mathcal{L}}{\partial y^u \partial y^{\bar{u}} \partial y^{\bar{v}}} = 0, \ \frac{\partial}{\partial x^u} g_{\bar{u}\bar{v}} = \frac{\partial^3}{\partial x^u \partial y^{\bar{u}} \partial y^{\bar{v}}} = 0.$$

Thus a basic Lagrangian is as well adapted to the foliation.

The non-linear connection N defines a local base of its horizontal vector fields given by the formulas:

$$\frac{\delta}{\delta x^u} = \frac{\partial}{\partial x^u} - N^v_u \frac{\partial}{\partial y^v} - N^{\bar{v}}_u \frac{\partial}{\partial y^{\bar{v}}}, \ \frac{\delta}{\delta x^{\bar{u}}} = \frac{\partial}{\partial x^{\bar{u}}} - N^v_{\bar{u}} \frac{\partial}{\partial y^v} - N^{\bar{v}}_{\bar{u}} \frac{\partial}{\partial y^{\bar{v}}}.$$

Proposition 3. *Consider the semi-spray $G : TM \to TTM$ defined by an adapted Lagrangian to a foliation $\mathcal{F}$, the tangent bundle $\tau\mathcal{F}^T$ of the lifted foliation $\mathcal{F}^T$ and the horizontal distribution $H\tau M$ of the non-linear connection defined by the semi-spray G. Let us suppose that G is a foliated (or basic) field for the foliation $\mathcal{F}^T$.*

Then the intersections $\tau\mathcal{F}^T \cap H\tau M$ and $\tau\mathcal{F}^T \cap V\tau M$ are supplementary distributions in $\tau\mathcal{F}^T$ or, equivalent, the above intersections induce non-linear connections on leafs.

Proof. The condition that G be foliated means:

$$N^{\bar{u}}_u = \frac{\partial G^{\bar{u}}}{\partial y^u} = 0, \ \frac{\partial G^{\bar{u}}}{\partial x^u} = 0, \tag{5}$$

but we use only the first relation. It follows that $\frac{\delta}{\delta x^u} = \frac{\partial}{\partial x^u} - N_u^v \frac{\partial}{\partial y^v}$, thus $\{\frac{\delta}{\delta x^u}\}$ is a local base in $\Gamma(\tau\mathcal{F}^T \cap H\tau M)$. Taking into account that $\{\frac{\partial}{\partial y^u}\}$ is a local base in $\Gamma(\tau\mathcal{F}^T \cap V\tau M)$, the conclusion follows. $\square$

It is well-known (see [1]) that a Lagrangian $\mathcal{L}$ canonically defines a Riemannian metric h on the fibres of τTM. In local coordinates, this metric has the form:

$$h\left(X^i \frac{\delta}{\delta x^i} + Y^j \frac{\partial}{\partial y^j}, Z^k \frac{\delta}{\delta x^k} + W^h \frac{\partial}{\partial y^h}\right) = g_{ik} X^i Z^k + g_{jh} Y^j W^h.$$

A Riemannian metric g on M (on the fibres of τM) is *bundle-like* for a foliation $\mathcal{F}$ on M if it has the locally property that for every basic local fields X and Y on M, orthogonal to the leafs (i.e. to $\tau\mathcal{F}$), then $g(X,Y)$ is a real basic function. According to [3], a foliation which allows a bundle-like metric is called a *Riemannian foliation.*

Theorem 2. *Let us suppose that the semi-spray $G : TM \to TTM$, associated with the adapted Lagrangian $\mathcal{L}$, is a foliated field for the foliation $\mathcal{F}^T$.*

Then the lifted foliation $\mathcal{F}^T$ is a Riemannian foliation and the canonical metric h of $\mathcal{L}$ on TM is a bundle-like metric.

Proof. We use local coordinates and the above notations. It suffices to prove that $\left\{\frac{\delta}{\delta x^{\bar u}}\right\}$ are local basic sections and $\{g_{\bar u \bar v}\}$ are basic functions. Since $\left\{\frac{\delta}{\delta x^{\bar u}}, \frac{\partial}{\partial y^{\bar v}}\right\}$ is a local base of basic sections (all for $\mathcal{F}^T$), $h\left(\frac{\delta}{\delta x^{\bar u}}, \frac{\delta}{\delta x^{\bar v}}\right) = h\left(\frac{\partial}{\partial y^{\bar u}}, \frac{\partial}{\partial y^{\bar v}}\right) = g_{\bar u \bar v}$ and $h\left(\frac{\delta}{\delta x^{\bar u}}, \frac{\partial}{\partial y^{\bar v}}\right) = 0$, the bundle-like condition follows.

We have already seen that for an adapted Lagrangian the functions $\{g_{\bar u \bar v}\}$ are basic. Using that G is foliated, the relations (5) hold. We have $0 = \frac{\partial G^{\bar u}}{\partial y^u} = \frac{\partial}{\partial y^u}(g^{\bar u \bar v}(\frac{\partial^2 \mathcal{L}}{\partial y^{\bar v} \partial x^i} y^i - \frac{\partial \mathcal{L}}{\partial x^{\bar v}})) = g^{\bar u \bar v}(\frac{\partial^2 \mathcal{L}}{\partial y^{\bar v} \partial x^u} - \frac{\partial^2 \mathcal{L}}{\partial y^u \partial x^{\bar v}})$. Thus $N_{\bar u}^u = \frac{\partial G^u}{\partial y^{\bar u}} = \frac{\partial}{\partial y^{\bar u}}(g^{uv}(\frac{\partial^2 \mathcal{L}}{\partial y^v \partial x^i} y^i - \frac{\partial \mathcal{L}}{\partial x^v})) = g^{uv}(\frac{\partial^2 \mathcal{L}}{\partial y^v \partial x^{\bar u}} - \frac{\partial^2 \mathcal{L}}{\partial y^{\bar u} \partial x^v}) = 0$.

The second relation (5) gives $\frac{\partial^2 G^{\bar v}}{\partial x^u \partial y^{\bar u}} = 0$, thus $N_{\bar u}^{\bar v}$ does not depend on (x^u).

We have $\frac{\partial N_{\bar u}^{\bar v}}{\partial y^u} = \frac{\partial^2 G^{\bar v}}{\partial y^u \partial y^{\bar u}} = 0$. Thus $\left\{\frac{\delta}{\delta x^{\bar u}}\right\}$ are local basic sections. $\square$

The metric g on the fibres of the vertical bundle has as components those of the Hessian of $\mathcal{L}$ restricted to the vertical bundle, which can be called a *vertical Hessian* of $\mathcal{L}$. In fact we can define also the *vertical differential* of the real function $\mathcal{L}$ (defined on M) as the section $d_v\mathcal{L} \in \Gamma(V^*\tau M)$ (the dual of the vertical bundle of M) defined by the formula:

$$d_v X = X(\mathcal{L}),\ (\forall) X \in \Gamma(V\tau M).$$

The local decomposition $U \cong V \times \bar{U}$ given by a local chart that defines the foliation $\mathcal{F}$, induces a local inclusion of a local leaf (slice) $i_F : V \to U$ and the projection $\pi_T : U \to \bar{U}$ (on the transverse model). They induce the π_F-morphism $\tau\pi_F : \tau U \to \tau V$, the $\tau\pi_F$-morphism $p_F : V\tau U \to V\tau V$, the π_T-morphism $\tau\pi_T : \tau U \to \tau\bar{U}$ and the $\tau\pi_T$-morphism $p_T : V\tau U \to V\tau\bar{U}$, using the canonical isomorphism of the vertical bundle and the induced bundles. A $\tau\pi_F$-comorphism $p_F^* : V^*\tau V \to V^*\tau U$ and a $\tau\pi_T$-comorphism $p_T^* : V^*\tau\bar{U} \to V^*\tau U$ follow, which are injective on fibres, since π_F and π_T are projections. We say that a vertical form $\omega \in \Gamma(V^*\tau U)$ is *locally $\mathcal{F}$-decomposable* if every point of M belongs to a domain $U \cong V \times \bar{U}$ of a foliated chart as above, and there are two local forms $\omega_F \in \Gamma(V^*\tau V)$ and $\omega_T \in \Gamma(V^*\tau\bar{U})$, such that $\omega = p_F^*\omega_F + p_T^*\omega_T$ on U. In local coordinates the condition of local $\mathcal{F}$-decomposability of the vertical form $\omega = \omega_u dy^u + \omega_{\bar{u}} dy^{\bar{u}}$ becomes $\omega_u = \omega_u(x^v, y^w)$ and $\omega_{\bar{u}} = \omega_{\bar{u}}(x^{\bar{v}}, y^{\bar{w}})$.

We say that a Lagrangian $\mathcal{L}$ on M is *totally adapted* to the foliation $\mathcal{F}$ if its vertical differential $d_v\mathcal{L}$ is locally $\mathcal{F}$-decomposable. Notice that using the local calculus it follows that a totally adapted Lagrangian is as well adapted.

Notice also that a natural situation when the totally adaptability occurs is when the foliation $\mathcal{F}$ allows a complementary foliation $\mathcal{F}'$, i.e. τM allows a Whitney sum decomposition $\tau M = \tau\mathcal{F} \oplus \tau\mathcal{F}'$. In this case the coordinates on the base M change according the rules $x^{u'} = x^{u'}(x^u)$, $x^{\bar{u}'} = x^{\bar{u}'}(x^{\bar{u}})$ and on TM change according the rules: $y^{u'} = \dfrac{\partial x^{u'}}{\partial x^u} y^u$, $y^{\bar{v}'} = \dfrac{\partial x^{\bar{v}'}}{\partial x^{\bar{v}}} y^{\bar{v}}$.

Corollary 1. *A Lagrangian $\mathcal{L}$ which is totally adapted to the foliation $\mathcal{F}$ has its canonical metric h on TM bundle-like for the lifted foliation $\mathcal{F}^T$, which is a Riemannian foliation.*

Proof. It suffices to prove that a totally adapted Lagrangian is basic and has a foliated semi-spray.

We use local coordinates and the above notations.

Since $\mathcal{L}$ is adapted we have that $\dfrac{\partial \mathcal{L}}{\partial y^u}$ and $\dfrac{\partial \mathcal{L}}{\partial y^{\bar{u}}}$ depend only on the variables (x^v, y^w) and $(x^{\bar{v}}, y^{\bar{w}})$ respectively. It follows easily that the local functions $g_{\bar{u}\bar{v}} = \dfrac{\partial^2 \mathcal{L}}{\partial y^{\bar{u}} \partial y^{\bar{v}}}$ are basic and $g_{u\bar{v}} = 0$, thus $\mathcal{L}$ is totally adapted. Using the above remarks, it follows easily that $\dfrac{\partial G^{\bar{u}}}{\partial y^u} = 0$ and $\dfrac{\partial G^{\bar{u}}}{\partial x^v} = 0$, thus the semi-spray G is foliated. Using Theorem 2, the conclusion follows. □

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