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Degeneracy of Some Lie Algebras[1]

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Abstract. We consider real Lie algebras with only zero and two-dimensional coadjoint orbits and having a nontrivial center. We show that, apart from the already known cases (studied for example by Dufour and Weinstein), such Lie algebras are degenerate in both the smooth and analytic category.

1. Introduction

A Poisson structure $\{,\}$ on a manifold M is a Lie algebra structure on $C^\infty(M)$ satisfying the Leibniz identity:

$$\{fg,h\} = f\{g,h\} + \{f,h\}g, \qquad \forall f,g,h \in C^\infty(M).$$

Alternatively it can be given by a contravariant skew-symmetric 2-tensor P such that $[P,P]=0$, where $[,]$ stands for the Schouten bracket. In local coordinates the Poisson tensor P can be written in the form:

$$P = \sum_{1\le i<j\le n} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

Using Weinstein's splitting theorem ([7]), the local study of P can be reduced to zero rank points, which translates into the following local expression for P:

$$P = \sum_{1\le i<j\le n} \sum_{k=1}^{n} C_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \text{ higher order terms.}$$

[1]This is an original research article and no version has been submitted for publication elsewhere.

The numbers C_{ij}^k are the structure constants of a Lie algebra and they sometimes determine the possibility of bringing P to a (local) linear form:

$$\sum_{1\leq i<j\leq n}\sum_{k=1}^{n} C_{ij}^k y_k \frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j},$$

through a smooth or analytic change of coordinates. P is then said to be linearizable. In this context a Lie algebra is said to be (smoothly or analytically) nondegenerate if every Poisson tensor associated with it in the above way, is (smoothly or analytically) linearizable. For example, semisimple Lie algebras are analytically nondegenerate (see [7] and [2]).

In this result we consider real Lie algebras of dimension at least four, whose connected and 1-connected Lie group has only zero and two-dimensional coadjoint orbits (such Lie algebras will be called *nice* and an exhaustive list of them can be found in [1]). We restrict to the case where the Lie algebra has a nontrivial center although the result is valid without this assumption (see [3]). We conclude that, apart from the already known cases (those to which the results of Weinstein in [7] and of Dufour in [5] apply), all these Lie algebras are degenerate in both the smooth and analytic category. Our proof is constructive, i.e., we associate a nonlinearisable Poisson structure to every Lie algebra $\mathfrak{g}$ being studied. This is done by perturbing the Lie-Poisson tensor in $\mathfrak{g}^*$ with second order terms in such a way that higher dimensional symplectic leaves appear around the singular point. This technique was used by A. Weinstein in [8] to prove that noncompact semisimple Lie algebras of real rank at least two are smoothly degenerate.

Notation. We follow the notation in [1] for the nice Lie algebras. These are, up to a direct sum with a central ideal:

1. type (i) – $\mathfrak{so}(3)$ or $\mathfrak{sl}(2,\mathbb{R})$;
2. type (ii) – $\mathbb{R}T +_E \mathfrak{a}$, where $\mathfrak{a}$ is an abelian ideal and the action of T on $\mathfrak{a}$ is by an endomorphism E of $\mathfrak{a}$;
3. type (iii) – $\mathbb{R}T + \mathfrak{h}$, where $\mathfrak{h}$ is the three-dimensional Heisenberg algebra spanned by X, Y, Z with $[X,Y] = Z$ and either:

$$[T,X] = Y,\ [T,Y] = -X,\ [T,Z] = 0$$

   or:

$$[T,X] = X,\ [T,Y] = -Y,\ [T,Z] = 0;$$

4. type (iv) – $\mathfrak{g}$ is six-dimensional with basis $X_i, Y_i,\ 1\leq i\leq 3$ and the nonvanishing brackets are:

$$[X_1,X_2] = Y_3,\ [X_2,X_3] = Y_1,\ [X_3,X_1] = Y_2;$$

5. type (v) – $\mathfrak{g}$ is five-dimensional with basis $X_i,\ 1\leq i\leq 3,\ Y_j,\ 1\leq j\leq 2$ and the multiplicative law reads:

$$[X_1,X_2] = X_3,\ [X_1,X_3] = Y_1,\ [X_2,X_3] = Y_2.$$

The Lie-Poisson tensors in the dual of any of these Lie algebras will be denoted by *Lie-Poisson tensor of type* (i)–(v).

The main result is the following:

Theorem 1. *Let $\mathfrak{g}$ be a nice Lie algebra (with $\dim \mathfrak{g} \geq 4$) with a nontrivial center. Then $\mathfrak{g}$ is smoothly and analytically degenerate except if $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathbb{R}$ or $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$.*

Remark 1. The condition on the center of $\mathfrak{g}$ means that Lie algebras of type (ii) will not be under consideration. Nevertheless, Theorem 1 holds without this assumption. The proof in that case envolves the use of Jordan normal form for the endomorphism E of $\mathfrak{a}$ (see [3] for more details).

Remark 2. The result is no longer valid if we restrict to the Poisson-Lie group case: Poisson-Lie groups associated with nice Lie algebras are linearizable (Mohammed Sbai, personal communication).

The proof of this theorem can be found in Section 4.

2. Raising the rank of Poisson structures

Definition 1. *A Lie algebra is said to be* nice *if the coadjoint orbits of its connected and 1-connected Lie group have dimension zero or two.*

Definition 2. *Let P be a Poisson tensor on a manifold M. Then P is said to be* nice *if its symplectic leaves have dimension zero or two and* not nice at a singular point *if it has symplectic leaves of dimension at least four, in some set whose closure contains the singular point.*

We start with a nice Lie algebra $(\mathfrak{g}, [\,,])$, or equivalently with a nice Lie-Poisson tensor P on $V = \mathfrak{g}^*$. We want to perturb P with second order terms so that symplectic leaves of higher dimension appear in any neighbourhood of the origin. Let $(x_1, \ldots, x_n)$ (with $n \geq 4$) be linear coordinates on V and P be a linear Poisson tensor on V. Then the expression of P in the basis:

$$\left\{ \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right\}_{1 \leq i < j \leq n}$$

is linear. Let Q be an alternating contravariant 2-tensor whose expression in the above basis is quadratic. Then $P' = P + Q$ is said to be a *quadratic perturbation of P*. Such P' will be a Poisson tensor if and only if:

$$[P + Q, P + Q] = 0,$$

where $[,]$ stands for the Schouten bracket. Equivalently:

$$[P, Q] = 0 \qquad \text{and} \qquad [Q, Q] = 0. \tag{1}$$

The last equation means that Q itself is a Poisson tensor. Our goal is then to find a quadratic Poisson tensor Q such that $[P, Q] = 0$ and $P + Q$ is not nice at the origin.

3. Nice tensors admitting Casimir functions

Let be a nice Lie algebra with a nontrivial center. This means that the Lie-Poisson tensor P on its dual space admits a linear coordinate function as a Casimir function. This is the case of the tensors of type (iii)–(v). Assume that x_n is such a Casimir function for P. We will consider quadratic perturbations $P+Q$ of P by taking Q to be of the form $x_n L$, with L a Lie-Poisson tensor.

Lemma 1. *Let P and L be nice Lie-Poisson tensors on V and x_n a Casimir function for P. Then $P + x_n L$ is a Poisson tensor if and only if $[P, L] = 0$.*

Proof. Using properties of the Schouten bracket we can write:

$$[P + x_n L, P + x_n L] = [P, P] + 2[P, x_n L] + [x_n L, x_n L].$$

Now $[P, P] = 0$ since P is a Poisson tensor and:

$$[x_n L, x_n L] = x_n^2 [L, L] - 2x_n L^\sharp(dx_n) \wedge L.$$

Again $[L, L] = 0$ and the fact that L is nice implies that $L^\sharp(dx_n) \wedge L = 0$, so that:

$$\begin{aligned} [P + x_n L, P + x_n L] &= 2[P, x_n L] \\ &= 2x_n[P, L] - 2P^\sharp(dx_n) \wedge L. \end{aligned}$$

The conclusion follows using the fact that x_n is a Casimir function for P. □

We consider now the problem of raising the rank of P.

Lemma 2. *Let P and L be as in the previous lemma and suppose that:*

1. $[P, L] = 0$;
2. *there is a subset U of V, whose closure contains the origin, where the following conditions hold:*
   (a) $im(P^\sharp) \cap im(L^\sharp) = \{0\}$;
   (b) $\ker(P^\sharp) \neq \ker(L^\sharp)$.

Then the tensor $P + x_n L$ is not nice at the origin.

Proof. First we remark that, if L is any nontrivial Lie-Poisson tensor, then the set:

$$M_0(L) = \{p \in V : \mathrm{rank}(L)_p = 0\}$$

is an hyperplane of codimension at least one. Since both P and L are in these conditions, and furthermore they are nice, then this implies that in any neighbourhood of the origin there is a point p such that:

$$\mathrm{rank}(P)_p = 2 \quad \text{and} \quad \mathrm{rank}(L)_p = 2.$$

Furthermore we can choose p such that $x_n(p) \neq 0$. The hypothesis on the image of P and L then implies that:

$$\ker(P^\sharp + x_n L^\sharp)_p = \ker P^\sharp_p \cap \ker L^\sharp_p.$$

Since both $\ker P^\sharp_p$ and $\ker L^\sharp_p$ have codimension two, then the hypothesis on the kernel of $P^\sharp$ and $L^\sharp$ implies that $\ker(P^\sharp + x_n L^\sharp)_p$ has codimension four, which concludes the lemma. □

Our goal is now to find L such that conditions 1, 2a and 2b of Lemma 2 hold.

3.1. Choice of L

We will choose L from the Lie-Poisson tensors of type (ii), as this will give us some freedom to choose (by choosing the endomorphism E). We will use the notation L_E for the Lie-Poisson tensor on the dual of the Lie algebra $\mathfrak{g}_E = \mathbb{R}T +_E \mathfrak{a}$.

We first write L_E in a coordinate free way. Let V be the given vector space (base space for the Poisson tensor P). Then a Lie algebra $\mathfrak{g}_E$ of type (ii) on V^* is determined by $\alpha \in V$, $z \in V^*$ (with $z(\alpha) \neq 0$) and E a nonzero endomorphism of $\ker(\alpha)$, in the following sense:

$$\mathfrak{g}_E = \mathbb{R}z + \ker(\alpha), \tag{2}$$

where the action of z on $\ker(\alpha)$ is by the endomorphism E. Our goal is to show that there exist α, z and E such that the Lie-Poisson tensor L_E satisfies conditions 1, 2a and 2b. Let $\beta = \{X_1, \dots, X_n\}$ be a basis for V and $(x_1, \dots, x_n)$ be coordinates in that basis. We write α as $\alpha_1 X_1 + \cdots + \alpha_n X_n$ and assume that $\alpha_1 \neq 0$. Assuming furthermore that $z(\alpha) = 1$, the expression of L_E in x-coordinates is given by:

$$L_E(dx_1, dx_i) = E(\alpha_1 x_i - \alpha_i x_1)$$

and

$$L_E(dx_i, dx_j) = \frac{\alpha_i}{\alpha_1} E(\alpha_1 x_j - \alpha_j x_1) - \frac{\alpha_j}{\alpha_1} E(\alpha_1 x_i - \alpha_i x_1),$$

where $j > i > 1$. Now let u and v be two generators for the image of $P^\sharp$. Condition 2a is equivalent to saying that u and v form a free system together with the vector fields:

$$u' = \alpha_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_n \frac{\partial}{\partial x_n} \text{ and } v' = E(x_1 - \alpha_1 z) \frac{\partial}{\partial x_1} + \cdots + E(x_n - \alpha_n z) \frac{\partial}{\partial x_n}$$

(see Lemma 3 in Appendix for the details).

We now remark that for fixed α and z it is easy to find an endomorphism E such that equation $[P, L] = 0$ holds. Conditions 2a and 2b merely restrict the field of those solutions. We choose $\alpha = X_1$ and $z = x_1$ so that $L_E(dx_i, dx_j)$ will be zero for all $j > i > 1$ and u' and v' will be given by:

$$u' = \frac{\partial}{\partial x_1} \text{ and } v' = E(x_2) \frac{\partial}{\partial x_2} + \cdots + E(x_n) \frac{\partial}{\partial x_n}.$$

Using these simplifications the problem of finding L_E is easily solved.

In Table 1 we present (up to an isomorphism of coordinates in the base space V) the generators for the image and kernel of the nice Lie-Poisson tensors of types (ii)–(v). For the tensor of type (ii) we are using the just described choice of α and z and we denote by E_i the function $E(x_i)$. We have also assumed that $E_2 \neq 0$. This can always be achieved by permuting the coordinates $(x_2, \dots, x_n)$, unless $E = 0$. From this table it is easy to see that, taking L_E to be of type (ii), condition 2b holds automatically. The choice of E that forces L_E to satisfy conditions 1 and 2a can be found in Table 2. In that table we present the functions $E_2, \dots, E_n$ which determine the endomorphism E, where in (iv) one of E_4, E_5 or E_6 is nonzero and in (v) one of E_4 or E_5 is nonzero. This endomorphism, together with the just described choice of α and z, determines the Lie algebra $\mathfrak{g}_E$, and therefore the tensor L_E. We can then conclude that:

Type of Tensor	**Generators for**	
	Image	**Kernel**
(ii)	$\frac{\partial}{\partial x_1}, \quad E_2\frac{\partial}{\partial x_2}+\cdots+E_n\frac{\partial}{\partial x_n}$	$E_2dx_i - E_idx_2, i=3,\ldots,n$
(iii)	$x_2\frac{\partial}{\partial x_1}-x_4\frac{\partial}{\partial x_2}, \quad x_3\frac{\partial}{\partial x_1}-x_4\frac{\partial}{\partial x_3}$	$dx_4, \quad x_4dx_1+x_2dx_2+x_3dx_3$
	$x_3\frac{\partial}{\partial x_1}-x_4\frac{\partial}{\partial x_2}, \quad x_2\frac{\partial}{\partial x_1}-x_4\frac{\partial}{\partial x_3}$	$dx_4, \quad x_4dx_1+x_3dx_2+x_2dx_3$
(iv)	$x_6\frac{\partial}{\partial x_1}-x_4\frac{\partial}{\partial x_3}, \quad x_6\frac{\partial}{\partial x_2}-x_5\frac{\partial}{\partial x_3}$	$dx_4, \quad dx_5, \quad dx_6,$ $x_4dx_1+x_5dx_2+x_6dx_3$
(v)	$x_4\frac{\partial}{\partial x_1}+x_5\frac{\partial}{\partial x_2}, \quad x_3\frac{\partial}{\partial x_1}-x_5\frac{\partial}{\partial x_3}$	$dx_4, \quad dx_5,$ $x_5dx_1-x_4dx_2+x_3dx_3$

Table 1: Generators for the image and kernel of some nice Lie-Poisson tensors

Theorem 2. *Let P be one of the tensors of type* (iii), (iv) *or* (v) *as denoted in* [1]. *Then there is a nice tensor L_E of type* (ii) *such that $P + x_nL_E$ is not nice at the origin.*

Type of P	**Endomorphism E determining L_E**
(iii)	$E_2 = a_2x_2 + a_3x_3 + a_4x_4$ $E_3 = b_2x_2 + b_3x_3 + b_4x_4$ $E_4 = (a_2 + b_3)x_4 \neq 0$
(iv)	$E_2 = a_2x_2 + \cdots + a_6x_6$ $E_3 = b_2x_2 + \cdots + b_6x_6$ $E_4 = (a_2 + b_3)x_4$ $E_5 = c_4x_4 + c_5x_5 + c_6x_6$ $E_6 = d_4x_4 + d_5x_5 + d_6x_6$
(v)	$E_2 = a_2x_2 + \cdots + a_5x_5$ $E_3 = b_2x_2 + \cdots + b_5x_5$ $E_4 = c_4x_4 + c_5x_5$ $E_5 = (a_2 + b_3)x_5$

Table 2: Tensor L_E of type (ii) to be associated with P

4. Proof of the main theorem

As proved in Theorem 2, the Lie algebras of type (iii), (iv) and (v) are degenerate in both the smooth and analytic category. In fact that theorem shows that it is possible to perturb the Lie-Poisson tensor on the dual of these Lie algebras with second order terms, in such a way that the perturbed tensor is no longer nice.

Because abelian Lie algebras of dimension at least two are degenerate in any category, then taking their direct sum with central ideals will always produce a degenerate Lie algebra. This leaves us to classify:

$$\mathfrak{so}(3) \oplus \mathbb{R} \text{ and } \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}.$$

The Lie algebra $\mathfrak{so}(3) \oplus \mathbb{R}$ is smoothly and analytically nondegenerate as a consequence of the theorem in [5]. Using the same result one concludes that $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$ is analytically nondegenerate. It is, however, degenerate in the smooth sense since the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is smoothly degenerate (see [7]). This concludes the proof of Theorem 1. □

Appendix: Coordinate changes in the base space

Let $\phi : M \longrightarrow N$ be a diffeomorphism between manifolds M and N, and suppose that P is a Poisson tensor on M. Then there exists a unique Poisson tensor Q on N making ϕ into a Poisson diffeomorphism. Such tensor Q is given by $\phi_* P$, the pushforward of P by ϕ. Furthermore the image of $Q^\sharp$ is just the pushforward by ϕ of the image of $P^\sharp$. In the case we are interested in, ϕ is an automorphism of a vector space V. If ϕ is represented by the matrix A and the tensor matrix for P is M, then the tensor matrix for Q is just $N = AMA^T$. Furthermore $im(Q^\sharp) = A(im(P^\sharp))$.

Lemma 3. *Let V be a real vector space and let L_E denote the Lie-Poisson tensor on the dual of the Lie algebra:*

$$\mathfrak{g}_E = \mathbb{R}z +_E \ker(\alpha).$$

Then there exist coordinates $x = (x_1, \ldots, x_n)$ in V such that:

$$L_E(dx_1, dx_j) = E(\alpha_1 x_j - \alpha_j x_1), \ \textit{for } j > 1$$

and

$$L_E(dx_i, dx_j) = \frac{\alpha_i}{\alpha_1} E(\alpha_1 x_j - \alpha_j x_1) - \frac{\alpha_j}{\alpha_1} E(\alpha_1 x_i - \alpha_i x_1) \ \textit{for } j > i > 1.$$

Furthermore the image of $L_E^\sharp$ is spanned by the vector fields:

$$u = \alpha_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_n \frac{\partial}{\partial x_n} \ \textit{and } v = E(x_1 - \alpha_1 z) \frac{\partial}{\partial x_1} + \cdots + E(x_n - \alpha_n z) \frac{\partial}{\partial x_n}.$$

Proof. Let $(x_1, \ldots, x_n)$ be coordinates in V such that $x_1(\alpha) \neq 0$. Then the following is a basis for $\ker(\alpha)$:

$$\{y_2, \ldots, y_n\} = \{\alpha_1 x_2 - \alpha_2 x_1, \ldots, \alpha_1 x_n - \alpha_n x_1\},$$

where α_i stands for $x_i(\alpha)$. We complete this basis with $y_1 = z_1 x_1 + \ldots + z_n x_n$ to get a basis for $\mathfrak{g}_E$. In y-coordinates L is represented by the matrix:

$$M = \begin{pmatrix} 0 & -E(y_2) & \cdots & -E(y_n) \\ E(y_2) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ E(y_n) & 0 & \cdots & 0 \end{pmatrix}.$$

The change of coordinates that takes us back to x coordinates is given by A the inverse matrix of:

$$\begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_n \\ -\alpha_2 & \alpha_1 & 0 & \cdots & 0 \\ -\alpha_3 & 0 & \alpha_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\alpha_n & 0 & 0 & \cdots & \alpha_1 \end{pmatrix}.$$

This is just:

$$A = \begin{pmatrix} \alpha_1 & -z_2 & -z_3 & \cdots & -z_n \\ \alpha_2 & \frac{1-\alpha_2 z_2}{\alpha_1} & \frac{-\alpha_2 z_3}{\alpha_1} & \cdots & \frac{-\alpha_2 z_n}{\alpha_1} \\ \alpha_3 & \frac{-\alpha_3 z_2}{\alpha_1} & \frac{1-\alpha_3 z_3}{\alpha_1} & \cdots & \frac{-\alpha_3 z_n}{\alpha_1} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_n & \frac{-\alpha_n z_2}{\alpha_1} & \frac{-\alpha_n z_3}{\alpha_1} & \cdots & \frac{1-\alpha_n z_n}{\alpha_1} \end{pmatrix}$$

and a tedious but straight forward calculation will complete the proof of the lemma. □

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