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The Lagrangian Version of a Theorem of J. B. Meusnier[1]

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1. Introduction

Let $\phi : M^n \longrightarrow \tilde{M}^m$ be an isometric immersion of a Riemannian n-manifold M into a Kaehler manifold $\tilde{M}$ of complex dimension m. The submanifold M is called *totally real* (or *isotropic* in symplectic geometry) if the almost complex structure J of $\tilde{M}$ carries each tangent space of M into its corresponding normal space. A totally real submanifold M of $\tilde{M}$ is said to be *Lagrangian* if $n = m$.

Lagrangian submanifolds in Kaehler manifolds have been classically studied from a symplectic point of view and are of a great interest in Physics. From a Riemannian point of view, Lagrangian submanifolds are far from trivial. This paper is devoted to the family of Lagrangian surfaces in complex Euclidean plane, although most of the results in Section 2 are true in arbitrary dimension. Using several joint results of the author with F.Urbano, we first introduce in Section 2 the Lagrangian version of geometric notions like umbilicity or constant mean curvature, both concepts classically studied in any kind of submanifolds. In addition, we pay our attention to the minimal case and the well known Willmore problem in the Lagrangian setting.

In this way, two very regular examples of Lagrangian surfaces appear in this context: the Whitney sphere and the Lagrangian catenoid. Thanks to different local and global characterizations described in Section 2, we can say that they play the role of the round sphere and the classical catenoid in Euclidean 3-space. It is specially important for us that

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both examples can be considered, in a certain sense explained in detail in Section 3, as (Lagrangian) surfaces of revolution.

The aim of this article is then a new characterization of the Lagrangian catenoid. It is a well known fact that the catenoid is the only minimal surface of revolution in $\mathbb{R}^3$. This result (in the above formulation) was proved by J.B. Meusnier in 1785 (cf. [M]). In Theorem 6, we conclude the Lagrangian version of the above classical theorem.

2. Characterizations of the Whitney sphere and the Lagrangian catenoid

Let $\phi : \Sigma \longrightarrow \mathbb{C}^2$ be an immersion of a orientable surface into complex Euclidean plane endowed with Euclidean metric $\langle , \rangle$. The most important two classes of this kind of surfaces are the Lagrangian (or totally real) surfaces and the complex surfaces. ϕ is called *Lagrangian* (resp. *complex*) if the complex structure J of $\mathbb{C}^2$ carries each tangent plane to Σ into its corresponding normal plane (resp. into itself).

In this article we are interested in Lagrangian surfaces and the basic properties of this family are:

(i) The trilinear form C associated to the second fundamental form σ

$$C(v, w, u) = \langle \sigma(v, w), Ju \rangle$$

is totally symmetric.

(ii) If H denotes the mean curvature vector of ϕ, the 1-form $\alpha_H = \langle JH, - \rangle$ dual to the tangent vector field JH, usually known as the *Maslov form*, is closed.

The simplest examples of Lagrangian surfaces in $\mathbb{C}^2$ are the Lagrangian planes, which are of course the totally geodesic ones. If we look for *totally umbilical* surfaces, i.e. $\sigma(v, w) = \langle v, w \rangle H$, the above property (i) easily leads to the totally geodesic case. In this way, it is natural to ask for a Lagrangian version of umbilicity that provides non-trivial examples. Taking (i) into account, in the following result the easiest examples – from the point of view of the second fundamental form– are locally characterised.

Theorem 1. [CU1], [RU] *Let $\phi : \Sigma \longrightarrow \mathbb{C}^2$ be a Lagrangian immersion. The second fundamental form of ϕ is given by*

$$\sigma(v, w) = \frac{1}{2} \left(\langle v, w \rangle H + \langle H, Jw \rangle Jv + \langle H, Jv \rangle Jw \right)$$

for any tangent vectors v, w, if and only if either ϕ is totally geodesic or $\phi(\Sigma)$ is an open set in a Whitney sphere.

The *Whitney spheres* are an important family of genus zero Lagrangian surfaces that, up to translations and dilations in $\mathbb{C}^2$, can be described by the Whitney immersion

$$\omega : \mathbb{S}^2 \longrightarrow \mathbb{C}^2, \quad \omega(x_1, x_2, x_3) = \frac{1 + ix_3}{1 + x_3^2}(x_1, x_2). \tag{1}$$

By an interpolation argument, we conclude from Theorem 1 the following corollary which shows that the Whitney spheres play the role of the umbilical spheres in Euclidean 3-space $\mathbb{R}^3$.

Corollary 1. [CU1] *Let $\phi : \Sigma \longrightarrow \mathbb{C}^2$ be a Lagrangian immersion of an orientable surface. Then $2K \leq |H|^2$ and the equality holds if and only if $\phi(\Sigma)$ is an open set in a (Lagrangian) plane or in the Whitney sphere.*

It is often that totally umbilical surfaces are the simplest examples of *constant mean curvature* in a given target manifold. But Whitney spheres have no parallel mean curvature vector, property that is usually taken as the version in higher codimension of constant mean curvature. However, the Maslov form α_H is conformal for the Whitney sphere (that is, JH is a conformal vector field). We can show that this property is a good version of constant mean curvature in the Lagrangian setting, thanks to the following theorem that can be considered as the Lagrangian version of a famous Hopf's theorem.

Theorem 2. [CU1] *The only genus zero Lagrangian surface in $\mathbb{C}^2$ with conformal Maslov form is the Whitney sphere.*

On the other hand, the *Willmore functional* for a surface in Euclidean space, $\phi : \Sigma \longrightarrow \mathbb{R}^4$, is defined by

$$\mathcal{W}(\phi) = \int_\Sigma |H|^2 dA.$$

This functional has been studied by several authors (we refer to [W]) and the critical points of $\mathcal{W}$, known as *Willmore surfaces*, are of special interest.

In this context we are also able to obtain new global characterizations of the Whitney sphere as an absolute minimum for this functional in the Lagrangian context.

Theorem 3. [CU1], [CU4] *Let $\phi : \mathbb{S}^2 \longrightarrow \mathbb{C}^2$ be a Lagrangian immersion of a (topological) sphere in $\mathbb{C}^2$. Then*

$$\mathcal{W}(\phi) \geq 8\pi$$

and the Whitney sphere is the only one that reaches the equality.

In addition, the Whitney sphere is the only Lagrangian Willmore sphere in $\mathbb{C}^2$.

If we think about the *minimal* case, that is $H \equiv 0$, we must point out that B.Y. Chen and J.M. Morvan proved in 1987 (cf. [CM]) that an orientable minimal surface Σ in $\mathbb{C}^2 \equiv \mathbb{R}^4$ is Lagrangian with respect to an orthogonal almost complex structure on $\mathbb{R}^4$ if and only if it is complex with respect to some other orthogonal almost complex structure on $\mathbb{R}^4$. In this context, we pay our attention to the following example:

$$M_0 = \{(z, \frac{1}{z}) \in \mathbb{C}^2 \, : \, z \in \mathbb{C}^*\},$$

described by D. Hoffman and R. Osserman in [HO] where they show that M_0 is a minimal surface in $\mathbb{R}^4$ with finite total curvature -4π. Later, in their classical paper about calibrations, R. Harvey and H.B. Lawson give a different description of M_0 in the following way:

$$M_0 \equiv \{(x, y) \in \mathbb{C}^2 \equiv \mathbb{R}^2 \times \mathbb{R}^2 \, : \, |x|y = |y|x; 2|x||y| = 1\}$$

and state that it is an area-minimizing minimal surface invariant under the diagonal action of $SO(2)$.

Our contribution for this example is the following local characterization that allow us to refer M_0 as the *Lagrangian catenoid.*

Theorem 4. [CU3] *Let $\phi : \Sigma \longrightarrow \mathbb{C}^2$ be a minimal (non-flat) Lagrangian immersion. Then Σ is foliated by pieces of circles of $\mathbb{C}^2$ if and only if $\phi(\Sigma)$ is (up to dilations) an open subset of the Lagrangian catenoid.*

We can ask if some relation exists between the Whitney sphere and the Lagrangian catenoid. There is an affirmative answer. If we consider the inversion of $\mathbb{C}^2$ from the only double point of the Whitney sphere (the origin in the representation of (1)) given by

$$\mathcal{F} : \mathbb{C}^2 - \{0\} \longrightarrow \mathbb{C}^2 - \{0\},\ \mathcal{F}(p) = \frac{p}{|p|^2},$$

it happens that $\mathcal{F}(M_0) \equiv \omega(\mathbb{S}^2) - \{0\}$, that is, the Whitney sphere is the compactification by the inversion of the Lagrangian catenoid. Precisely, in this direction we characterize the Lagrangian catenoid globally.

Theorem 5. [CU4] *Let M^2 be a complete minimal (non-flat) surface with finite total curvature immersed in Euclidean space $\mathbb{R}^4$.*

The compactification by the inversion of M^2 is Lagrangian for a certain orthogonal complex structure on $\mathbb{R}^4$ if and only if M is (up to dilations) the Lagrangian catenoid.

3. The Lagrangian catenoid as a surface of revolution

In the latter section, we have shown that the Whitney sphere and the Lagrangian catenoid are both regular examples of Lagrangian surfaces of $\mathbb{C}^2$, thanks to their geometric properties that characterize them, locally and globally, inside different families of Lagrangian surfaces.

We now describe explicit parametrizations of both examples in terms of a plane curve. We follow a particular case of a construction introduced in [CU2], that admits a general version for arbitrary dimension in [RU].

Given a regular plane curve $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{C}^*$, we consider the map

$$\varphi_\alpha : I \times \mathbb{R} \longrightarrow \mathbb{C}^2$$

given by

$$\varphi_\alpha(t,s) = \frac{\alpha(t)}{\sqrt{2}}(e^{is}, e^{-is}).$$

We can call φ_α the *Lagrangian surface of revolution generated by* α because of the following two reasons:

1. The curve $s \mapsto \frac{1}{\sqrt{2}}(e^{is}, e^{-is})$ can be interpretated as a horizontal lift to $\mathbb{S}^3$ of a great circle of $\mathbb{S}^2$ via the Hopf fibration $\Pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ (see [CU2]). In particular, φ_α is always a Lagrangian immersion.
2. Besides φ_α is constructed using a *generatrix* plane curve α, φ_α is invariant by the one-parameter group of holomorphic isometries of $\mathbb{C}^2$ given by $\{\mathrm{diag}(e^{i\theta}, e^{-i\theta})\,;\, \theta \in \mathbb{R}\} \subset SU(2)$.

Remark 1. Any Lagrangian surface of revolution in $\mathbb{C}^2$ is H-umbilical in the sense of [Ch]. In fact, up to a change of coordinates in $\mathbb{C}^2$, φ_α can be written as $\alpha(t)(\cos s, \sin s)$, and so φ_α can be seen as the complex extensor of α via the inclussion $\mathbb{S}^1 \hookrightarrow \mathbb{R}^2$. In other words, φ_α is the easiest possible construction described in Definition 1 of [RU] taking $n = 2$ and ϕ the totally geodesic immersion of $\mathbb{S}^1$ in $\mathbb{CP}^1 \equiv \mathbb{S}^2$. We must point out that if one consider the similar construction for $n > 2$, one loses the desired property commented above in 2.

It is straightforward to check (or see Lemma 1 of [CU2] or Proposition 3 in [RU]) that the fundamental forms of the Lagrangian immersion $\varphi_\alpha : I \times \mathbb{R} \longrightarrow \mathbb{C}^2$ are given by:

$$\langle , \rangle = |\alpha'|^2 dt^2 + |\alpha|^2 ds^2, \tag{2}$$

$$\begin{aligned} C(\partial_t, \partial_t, \partial_t) &= \langle \alpha'', J\alpha' \rangle = |\alpha'|^3 k_\alpha, \\ C(\partial_t, \partial_t, \partial_s) &= C(\partial_s, \partial_s, \partial_s) = 0, \\ C(\partial_t, \partial_s, \partial_s) &= \langle \alpha', J\alpha \rangle, \end{aligned} \tag{3}$$

where J also denotes the complex structure in $\mathbb{C}$ (i.e. the rotation of angle $+\pi/2$) and k_α is the signed curvature of α.

We remark that, up to congruences in $\mathbb{C}^2$, φ_α is well defined up to rotations or dilations of α (but not up to translations of α).

Now we can see the Whitney sphere and the Lagrangian catenoid as the Lagrangian surfaces of revolution generated by the lemniscata (in polar coordinates, $r^2 = \cos 2\theta$) and the equilateral hyperbola (in polar cordinates, $r^2 \cos 2\theta = 1$) respectively.

3.1. The Whitney sphere

We take

$$\alpha(t) = \frac{\cosh t + i \sinh t}{\cosh 2t}, \ t \in \mathbb{R},$$

and then $\varphi_\alpha = \varphi_\alpha(z = t + is)$, $z \in \mathbb{C}$, is a conformal Lagrangian immersion which is singly-periodic. Via the exponential map $\xi = e^z : \mathbb{C} \longrightarrow \mathbb{C}^*$, we can see

$$\varphi_\alpha : \mathbb{C}^* \subset \bar{\mathbb{C}} \longrightarrow \mathbb{C}^2$$

given by

$$\varphi_\alpha(\xi) = \frac{(|\xi|^2 + 1) + i(|\xi|^2 - 1)}{\sqrt{2}(1 + |\xi|^4)}(\xi, \bar{\xi})$$

and it admits a regular extension to 0 and ∞ corresponding to the poles of $\mathbb{S}^2$. Via the inverse of stereographic projection, $\bar{\mathbb{C}} \longrightarrow \mathbb{S}^2$, we finally arrive at (1).

3.2. The Lagrangian catenoid

We now take the inverse of the lemniscata from the origin, that is,

$$\alpha(t) = \cosh t + i \sinh t, \ t \in \mathbb{R},$$

obtaining in this way a conformal Lagrangian immersion

$$\varphi_\alpha(t,s) = \frac{1}{\sqrt{2}}(\cosh t + i \sinh t)(e^{is}, e^{-is}).$$

Up to congruences in $\mathbb{C}^2$, we can rewrite φ_α as

$$\varphi_\alpha(t,\zeta) = (e^t\zeta, e^{-t}\bar{\zeta}),\ \zeta = e^{is} \in \mathbb{S}^1,$$

and then it is clear that we can conclude that

$$\varphi_\alpha(\mathbb{R}\times\mathbb{S}^1) = M_0.$$

4. A new characterization of the Lagrangian catenoid

In this section we are going to prove the main theorem of this article. For this purpose, we look for minimal Lagrangian surfaces of revolution in $\mathbb{C}^2$.

If we consider the orthonormal basis (see (2)) given by $\{\partial_t/|\alpha'|, \partial_s/|\alpha|\}$, it is easy to get from (3) the mean curvature vector of φ_α:

$$H = \frac{1}{2|\alpha'|^4|\alpha|^2}(|\alpha|^2\langle\alpha'', J\alpha'\rangle + |\alpha'|^2\langle\alpha', J\alpha\rangle)J(\varphi_\alpha)_t. \tag{4}$$

There is no restriction if we suppose that α is parametrized in such a way that $|\alpha'| = |\alpha|$, which implies that φ_α is, in addition, a conformal immersion. Then, from (4), the minimality of φ_α is equivalent to

$$\langle\alpha'', J\alpha'\rangle + \langle\alpha', J\alpha\rangle = 0. \tag{5}$$

If we put $\alpha' = a\,\alpha + b\,J\alpha$, with $a = \frac{\langle\alpha',\alpha\rangle}{|\alpha|^2} = (\log|\alpha|)'$ and $b = \frac{\langle\alpha', J\alpha\rangle}{|\alpha|^2}$, then an easy computation leads to $a^2 + b^2 = 1$ and

$$a' = b(b - k_\alpha|\alpha|), \quad b' = -a(b - k_\alpha|\alpha|); \tag{6}$$

moreover, we can write $\alpha(t) = |\alpha(t)|e^{i\int^t b}$.

Then from (5) we deduce that $|\alpha|k_\alpha + b = 0$, and so (6) becomes in $a' = 2b^2$, $b' = -2ab$. Thus

$$a' = 2(1-a^2),\ b' = -2ab. \tag{7}$$

The trivial solution of (7), $a = \pm 1$, $b = 0$, leads to a straight line α passing through the origin and then φ_α would be a Lagrangian plane. The non-trivial solution of (7), $a(t) = \tanh 2t$, $b(t) = \pm 1/\cosh 2t$, after an easy computation, gives $\alpha(t) = \cosh t + i \sinh t$, which is the equilateral hyperbola generatrix of the Lagrangian catenoid.

As a summary, we have proved the following new local characterization of the Lagrangian catenoid.

Theorem 6. *The only minimal (non-flat) Lagrangian surface of revolution in $\mathbb{C}^2$ is (an open subset of) the Lagrangian catenoid.*

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