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ASYMPTOTIC BEHAVIOR OF ORTHOGONAL POLYNOMIALS PRIMITIVES

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En recuerdo de nuestro cariño y amistad con Chicho

Abstract. We study the zero location and the asymptotic behavior of the primitives of the standard orthogonal polynomials with respect to a finite positive Borel measure concentrate on $[-1, 1]$.

1. Introduction

Let μ be a finite positive Borel measure with $\operatorname{supp}(\mu) = \Delta \subsetneqq [-1, 1]$, such that it contains an infinite number of points. Let us consider $L_n(z) = z^n + \cdots$ the nth monic (i.e. its leading coefficient is equal to one) orthogonal polynomial with respect to μ, that is

$$\int_\Delta L_n(x)\, x^k\, d\mu(x) = 0\,, \quad k = 0, 1, 2, \ldots, n-1\,. \tag{1}$$

Let us consider a monic polynomial $P_n(x)$ of degree n and a complex number ζ fixed, such that

$$(n+1)\, L_n(z) = ((z-\zeta)\, P_n(z))' = P_n(z) + (z-\zeta)P_n'(z)\,. \tag{2}$$

Note that $\Lambda(z) = (z-\zeta)\, P_n(z)$ is a monic polynomial primitive of $(n+1)\, L_n(z)$, normalized by $\Lambda(\zeta) = 0$. A direct consequence of (1)–(2) is that $P_n(z)$ satisfy the orthogonality relations

$$\int_\Delta [P_n(x) + (x-\zeta)P_n'(x)]\, x^k\, d\mu(x) = 0\,, \quad k = 0, 1, 2, \ldots, n-1\,. \tag{3}$$

The location of critical points of polynomials has many physical and geometrical interpretations. Let us consider , for instance, a field of forces given by a system of n masses m_j, $1 \le j \le n$, at the fixed points z_j, $1 \le j \le n$, that repels a movable unit mass at z according to the law of repulsion being the inverse distance law.

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Let $Q_m(z)$, where $m = m_1 + m_2 + \cdots + m_n$, be the polynomial $(z - z_1)^{m_1} \cdot (z - z_2)^{m_2} \cdots (z - z_n)^{m_n}$. The logarithmic derivative of $Q_m(z)$ is

$$\frac{d(\log(Q_m(z)))}{dz} = \frac{Q'_m(z)}{Q_m(z)} = \frac{m_1}{(z - z_1)} + \frac{m_2}{(z - z_2)} + \cdots + \frac{m_n}{(z - z_n)}. \tag{4}$$

The conjugate of $\dfrac{m_j}{(z - z_j)}$ is a vector whose direction (including sense) is the direction from z_j to z, so this vector represents the force at the movable unit mass z due to a single fixed particle at z_j. Every multiple zero (but no simple zero) of $Q_m(z)$ is a zero of $Q'_m(z)$; every other zero of $Q'_m(z)$ is by (4) a position of equilibrium in the field of force; every position of equilibrium is by (4) a zero of $Q'_m(z)$. This result is known as Gauss‘s theorem (1816).

Now, we consider an inverse problem, let $z'_1, z'_2, \ldots, z'_n$ be the zeros of the orthogonal polynomial L_n and the equilibrium positions of a field of forces with $n+1$ units masses, one of which ζ is given. What is the location of the remaining masses?

By (2),

$$\frac{(n+1)\, L_n(z)}{(z - \zeta) P_n(z)} = \frac{1}{z - \zeta} + \frac{P'_n(z)}{P_n(z)} = \frac{\Lambda'(z)}{\Lambda(z)}. \tag{5}$$

Then, according with (5) and the above interpretation of the logarithmic derivative, the location of the remaining units masses are the zeros of the polynomial $P_n(z)$ defined in (2).

The main purpose of this paper is to study some of the algebraic and analytic properties of the orthogonal polynomials primitives.

2. Localization of zeros

It is well know that the zeros of $L_n(z)$ are simple, using (2) is easy to see that the zeros of $P_n(z)$ have at most multiplicity two. Nevertheless the zeros of $P_n(z)$ need not to be simple as we can see in the following example

Let μ be the Lebesgue measure in $[-1, 1]$ and set in (3) $\zeta = \frac{2\sqrt{3}}{3}$ or $\zeta = -\frac{2\sqrt{3}}{3}$. The corresponding monic polynomials of degree two defined by (2) are $P_2(z) = z^2 + \frac{2\sqrt{3}}{3} z + \frac{1}{3}$ or $P_2(z) = z^2 - \frac{2\sqrt{3}}{3} z + \frac{1}{3}$ respectively. Note that $z = -\frac{\sqrt{3}}{3}$ or $z = \frac{\sqrt{3}}{3}$ are zeros of multiplicity two of the corresponding polynomials $P_2(z)$.

Our next propose is to prove that all the zeros of the polynomials of the sequence $\{P_n(z)\}_{n=0}^{\infty}$ are contained in a disc which radius is independent of n. First, let us rewrite the polynomials P_n and L_n in terms of $(z - \zeta)$, that is

$$P_n(z) = \sum_{k=0}^{n} a_k\, (z - \zeta)^k, \qquad L_n(z) = \sum_{k=0}^{n} b_k\, (z - \zeta)^k. \tag{6}$$

Lemma 1. *The coefficients a_k of P_n and b_k of L_n in (6) are related by*

$$a_k = \frac{n+1}{k+1}\, b_k\,. \tag{7}$$

Proof. Replacing (6) in (2). □

The proof of the next result is based in the following Szegő's theorem (see [5] or [2, page 23]).

Lemma 2. *Given the polynomials*

$$f(z) = \sum_{k=0}^{n} \alpha_k \binom{n}{k} z^k, \quad \alpha_n \neq 0 \quad \text{and} \quad g(z) = \sum_{k=0}^{n} \beta_k \binom{n}{k} z^k, \quad \beta_n \neq 0\,,$$

let us construct a third polynomial as $h(z) = \sum_{k=0}^{n} \alpha_k \beta_k \binom{n}{k} z^k$.

If all the zeros of $f(z)$ lie in a closed disk $\bar{D}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the zeros of $g(z)$. Then every zero of $h(z)$ has the form $\lambda_k \gamma_k$, where $\gamma_k \in \bar{D}$.

Then we have that

Theorem 1. *All the zeros of P_n are contained in the closed disk $\mathbf{D}$, where*

$$\mathbf{D} = \{z \in \mathbb{C} : |z| \leq 2 + 3\,|\zeta|\}. \tag{8}$$

Proof. Let us write $w = z - \zeta$, hence

$$f(w) = \sum_{k=0}^{n} b_k\, w^k = L_n(z)\,, \qquad h(w) = \sum_{k=0}^{n} \frac{n+1}{k+1}\, b_k\, w^k = P_n(z)$$

and

$$g(w) = \sum_{k=0}^{n} \frac{n+1}{k+1} \binom{n}{k} w^k = \frac{(1+w)^{n+1} - 1}{w} = \frac{(1+z-\zeta)^{n+1} - 1}{z - \zeta}\,.$$

If z_0 is a zero of L_n, it is well known that $-1 \leq z_0 \leq 1$, hence $w_0 = z_0 - \zeta$ is a zero of $f(w)$ and lie in a closed disk $\bar{D} = \{|w + \zeta| \leq 1\}$. On the other hand, if w_1 is a zero of $g(w)$ then $|1 + w_1| = 1$.

Finally, by Lemma 2, if $h(w_3) = 0$ we have that $|w_3| \leq 2 + 3|\zeta|$ and then the theorem is proved. □

3. Auxiliary results

In order to obtain the asymptotic behaviour of the sequence $\{P_n\}$ we need some general results that we will discuss in what follows.

If $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of measures on a compact set, we say that μ_n converges weakly to the measure μ as $n \to \infty$ if

$$\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu$$

for every continuous function f on $\mathbb{C}$ having compact support. In this case, we write $\mu_n \xrightarrow{*} \mu$, or $d\mu_n \xrightarrow{*} d\mu$, or if μ is absolutely continuous, $d\mu_n(x) \xrightarrow{*} \mu'(x)dx$.

For any polynomial q of degree exactly n, we consider

$$\nu_n(q) := \frac{1}{n} \sum_{j=1}^{n} \delta_{z_j}\,,$$

where $z_1, \ldots, z_n$ are the zeros of q repeated according to their multiplicity, and δ_{z_j} is the Dirac measure with mass one at the point z_j. This is the so called *normalized zero counting measure associated with* q.

Let $\|\cdot\|_\Delta$ denotes the supremum norm on Δ and $\mathrm{Cap}(\Delta)$ the logarithmic capacity of a set Δ. Another result needed is

Lemma 3 ([1], Theorem 2.1 and Corollary 2.1)**.** *Let $\Delta \subset \mathbb{C}$ be a compact set with empty interior, connected complement and positive logarithmic capacity. If $\{P_n\}_{n=0}^{\infty}$ is a sequence of monic polynomials,* $\deg(P_n) = n$, *such that*

$$\overline{\lim_{n\to\infty}} \|P_n\|_\Delta^{\frac{1}{n}} \leq \mathrm{Cap}(\Delta)\,,$$

then

$$\nu_n(P_n) \xrightarrow{\ *\ } \omega_\Delta\,,$$

where ω_Δ is the equilibrium measure of Δ.

Finally, we have the following useful result

Lemma 4 ([3], Lemma 3)**.** *Let $\{P_n\}$ be a sequence of polynomials. Then, for all $j \in \mathbb{Z}_+$,*

$$\overline{\lim_{n\to\infty}} \left(\frac{\|P_n^{(j)}\|_\Delta}{\|P_n\|_\Delta}\right)^{1/n} \leq 1\,. \tag{9}$$

For $\Delta = [-1,1]$ is well known that $\mathrm{Cap}(\Delta) = \frac{1}{2}$ and the equilibrium measure on Δ is the so-called arcsin measure given by

$$\mu_\Delta(B) = \int_B \frac{\arcsin'(x)\,dx}{\pi} = \frac{1}{\pi}\int_B \frac{dx}{\sqrt{1-x^2}}\,, \tag{10}$$

where B is a Borel set in $[-1,1]$.

4. Asymptotic behavior

Let us set $\varphi(z) = z + \sqrt{z^2-1}$, $z \in \mathbb{C}\setminus[-1,1]$. φ is a conformal map of $\mathbb{C}\setminus[-1,1]$ onto $\{z \in \mathbb{C} : |z| > 1\}$. Here the branch of the square root is chosen so that $|z + \sqrt{z^2-1}| > 1$ for $z \in \mathbb{C}\setminus[-1,1]$. Let $\zeta \in \mathbb{C}\setminus[-1,1]$ be a fixed point, $\Omega = \mathbb{C}\setminus \mathbf{D}$ and $\Delta = [-1,1]$.

Theorem 2. *With the previous conditions it holds, for all $j \in \mathbb{Z}_+$,*

- *the sequence $\{P_n^{(j)}\}_{n=0}^{\infty}$ verifies*

$$\lim_{n\to\infty} \|P_n^{(j)}\|_\Delta^{\frac{1}{n}} = \frac{1}{2}\,; \tag{11}$$

- *$\nu_{n,j}(P_n^{(j)})$ converges to the* arcsin *measure in the sense of the weak-$*$ topology of measures, that is*

$$\lim_{n\to\infty} \frac{1}{n-j}\sum_{k=1}^{n-j} f(x_{n,k}^{(j)}) = \frac{1}{\pi}\int_{-1}^{1} f(x)\,\frac{dx}{\sqrt{1-x^2}}\,, \tag{12}$$

for every continuous function on $\mathbf{D}$*, where* $\{x_{n,k}^{(j)}\}_{k=1}^{n-j}$ *is the set of zeros of* $P_n^{(j)}(z)$*.*

Proof. Let us prove first that

$$\lim_{n\to\infty} \|P_n\|_\Delta^{\frac{1}{n}} = \frac{1}{2}\,. \tag{13}$$

If $x \in \Delta$, integrating in (2) we have

$$(n+1)\int_\zeta^x L_n(t)\,dt = (x-\zeta)P_n(x),$$

by taking absolute values both sides, we obtain

$$M(n+1)\|L_n(x)\|_\Delta \ge (n+1)\left|\int_\zeta^x L_n(t)\,dt\right| = |x-\zeta||P_n(x)|, \ge m|P_n(x)|\,,$$

where $m = \inf\limits_{x\in\Delta}|x-\zeta|$ and $M = \sup\limits_{x\in\Delta}|x-\zeta|$. Then

$$M(n+1)\|L_n(x)\|_\Delta \ge m\|P_n\|_\Delta \ge m\|T_n\|_\Delta\,,$$

where T_n is the n-th Chebyshev polynomial in $[-1,1]$.

It is well known, for general theory of orthogonal polynomials, that

$$\lim_{n\to\infty}\|L_n\|_\Delta^{\frac{1}{n}} = \lim_{n\to\infty}\|T_n\|_\Delta^{\frac{1}{n}} = \frac{1}{2}\,,$$

hence we have (13).

By Lemma 4 and (13),

$$\overline{\lim_{n\to\infty}}\,\|P_n^{(j)}\|_\Delta^{\frac{1}{n}} = \overline{\lim_{n\to\infty}}\,\frac{\|P_n^{(j)}\|_\Delta^{\frac{1}{n}}}{\|P_n\|_\Delta^{\frac{1}{n}}}\,\|P_n\|_\Delta^{\frac{1}{n}} \le \frac{1}{2} = \operatorname{Cap}(\Delta). \tag{14}$$

But

$$\underline{\lim_{n\to\infty}}\,\|P_n^{(j)}\|_\Delta^{\frac{1}{n}} \ge \overline{\lim_{n\to\infty}}\,\|T_{n-j}\|_\Delta^{\frac{1}{n}} = \frac{1}{2} = \operatorname{Cap}(\Delta) \tag{15}$$

and then (14) and (15) implies (11).

Finally, by Lemma 3 we deduce that (11) implies (12). □

Theorem 3. *With the above assumptions, it holds:*

- *For all* $j \in \mathbb{Z}_+$,

$$\frac{P_n^{(j+1)}(z)}{nP_n^{(j)}(z)} \underset{n}{\rightrightarrows} \frac{1}{\sqrt{z^2-1}} \tag{16}$$

  uniformly on compact subsets of Ω.

- *(Relative Asymptotic) For all* j_1, $j_2 \in \mathbb{Z}_+$,

$$n^{j_2-j_1}\,\frac{L_n^{(j_1)}(z)}{P_n^{(j_2)}(z)} \underset{n}{\rightrightarrows} \frac{z-\zeta}{\sqrt{z^2-1}}\left(\sqrt{z^2-1}\right)^{j_2-j_1} \tag{17}$$

  uniformly on compact subsets of Ω.

Proof. Let $x^j_{n,k}$, $k = 1,\dots,n-j$, denote the $n-j$ zeros of the polynomial $P_n^{(j)}$. It is known that all the critical points of a non-constant polynomials P_n and it's derivatives lied in the convex hull of his zeros, then by theorem 1 $x^j_{n,k} \in \mathbf{D} = \{z : |z| \le 2 + 3|\zeta|\}$, $k = 1,\dots,n-j$. Using the decomposition in simple fractions and the definition of $\nu_{n,j}(P_n^{(j)})$, we obtain

$$\frac{P_n^{(j+1)}(z)}{n\,P_n^{(j)}(z)} = \frac{1}{n}\sum_{k=1}^{n-j}\frac{1}{z - x^j_{n,k}} = \frac{n-j}{n}\int\frac{d\nu_{n,j}(x)}{z-x}\,. \tag{18}$$

Therefore, the family of functions

$$\left\{\frac{P_n^{(j+1)}(z)}{n\,P_n^{(j)}(z)}\right\}, \quad n \in \mathbb{Z}_+\,, \tag{19}$$

is uniformly bounded on each compact subset of $\Omega = \mathbb{C}\setminus\mathbf{D}$.

On the other hand, all the measures $\nu_{n,j}, n \in \mathbb{Z}_+$, are supported in $\mathbf{D}$ and for $z \in \Omega$ fixed, the function $(z-x)^{-1}$ is continuous on $\mathbf{D}$ with respect to x. Therefore, from (12) and (18), we find that any subsequence of (19) which converges uniformly on compact subsets of Ω converges pointwise to $\int (z-x)^{-1}\,d\omega_\Delta(x)$. Finally, by (10), the Cauchy's formula and the residue Theorem,

$$\int_{-1}^{1}\frac{d\omega_\Delta(x)}{(z-x)} = \frac{1}{\pi}\int_{-1}^{1}\frac{1}{(z-x)}\frac{dx}{\sqrt{1-x^2}} = \frac{1}{\sqrt{z^2-1}}\,.$$

Thus, the whole sequence converges uniformly on compact subsets of Ω to this function as stated in (16).

For $j_1 = j_2 = j$, the proof of (17) is a direct consequence of the j-th derivative of (2) and (16), that is

$$\frac{n+1}{n}\frac{L_n^{(j)}(z)}{P_n^{(j)}(z)} = \frac{j+1}{n} + (z-\zeta)\frac{P_n^{(j+1)}(z)}{nP_n^{(j)}(z)} \underset{n}{\rightrightarrows} \frac{z-\zeta}{\sqrt{z^2-1}} \tag{20}$$

uniformly on compact subsets of Ω.

Assume without loss of generality that $j_2 < j_1$, hence

$$\frac{1}{n^{j_1-j_2}}\frac{L_n^{(j_1)}(z)}{P_n^{(j_2)}(z)} = \frac{L_n^{(j_1)}(z)}{P_n^{(j_1)}(z)}\,\frac{P_n^{(j_1)}(z)}{nP_n^{(j_1-1)}(z)}\cdots\frac{P_n^{(j_2+2)}(z)}{nP_n^{(j_2+1)}(z)}\,\frac{P_n^{(j_2+1)}(z)}{nP_n^{(j_2)}(z)}\,. \tag{21}$$

Then we have (17) from (16), (20) and (21). ☐

Theorem 4. *With the above conditions, the following statements hold:*

- *(Strong Asymptotic) If $\mu'(x)$ satisfy the Szegő condition*

$$\int_{-1}^{1}\frac{\log\mu'(x)\,dx}{\sqrt{1-x^2}} > -\infty$$

then, for all $j \in \mathbb{Z}_+$,

$$\frac{P_n^{(j)}(z)}{n^j\left(\dfrac{\varphi(z)}{2}\right)^n} \underset{n}{\rightrightarrows} \frac{\left(\sqrt{z^2-1}\right)^{1-j}}{z-\zeta}\,\frac{\mathcal{D}\left(\mu'(\cos\theta)\,|\sin\theta|,0\right)}{\mathcal{D}\left(\mu'(\cos\theta)\,|\sin\theta|,\varphi^{-1}(z)\right)}\,, \tag{22}$$

uniformly on compact subsets of Ω*, where* $\mathcal{D}(h,z)$ *is the Szegő function of* h

$$\mathcal{D}(h,z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log h(\theta)\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right), \quad |z|<1\,.$$

- *(Ratio Asymptotic) If* $\mu'(x)>0$ *a.e. in* $[-1,1]$ *then, for all* j_1, j_2, $k \in \mathbb{Z}_+$,

$$\frac{n^{j_2}}{(n+k)^{j_1}}\,\frac{P_{n+k}^{(j_1)}(z)}{P_n^{(j_2)}(z)} \underset{n}{\rightrightarrows} \left(\sqrt{z^2-1}\right)^{j_2-j_1}\left(\frac{\varphi(z)}{2}\right)^k, \tag{23}$$

uniformly on compact subsets of Ω.

- *(n-th Root Asymptotic) If the measure* μ *is such that for all measurable set* $E\subset \operatorname{supp}(\mu)$ *with* $\mu(E)=\mu([-1,1])$ *it holds that* $\operatorname{Cap}(E)=\frac{1}{2}$*, then, for all* $j\in\mathbb{Z}_+$,

$$\sqrt[n]{|P_n^{(j)}(z)|} \underset{n}{\rightrightarrows} \frac{|\varphi(z)|}{2}, \tag{24}$$

uniformly on compact subsets of Ω*, where* $\Omega=\mathbb{C}\setminus\mathbf{D}$, $\varphi(z)=z+\sqrt{z^2-1}$ *and the branch of the square root is chosen so that* $|z+\sqrt{z^2-1}|>1$ *for* $z\in\mathbb{C}\setminus[-1,1]$.

Proof. The theorem is a direct consequence of (17) in theorem 3 and the well known strong asymptotic, ratio asymptotic and n-th root asymptotic behavior of standard orthogonal polynomials L_n. □

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