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GENERALIZATION OF KÄHLER ANGLE AND INTEGRAL GEOMETRY IN COMPLEX PROJECTIVE SPACES

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1. Introduction

For a Riemannian homogeneous space G/K and submanifolds M and N in G/K, the function $g \mapsto \mathrm{vol}(gM \cap N)$ defined on G is measurable, so we can consider the integral

$$\int_G \mathrm{vol}(gM \cap N) d\mu_G(g).$$

The Poincaré formulas mean some equalities which represent the above integral by some geometric invariants of M and N. Many mathematicians have obtained Poincaré formulas of submanifolds in various homogeneous spaces.

Howard [5] showed a Poincaré formula in a general situation, which is stated in Theorem 7 of this paper. Before this, Santaló [8] showed a Poincaré formula of complex submanifolds in complex projective spaces. The above integral for complex submanifolds M and N in the complex projective space is equal to the product of the volumes of M and N multiplied by some constant. Kang and the author [6] formulated a Poincaré formula of real surfaces and complex hypersurfaces in the complex projective space by the use of Kähler angle. We also obtained a Poincaré formula of two real surfaces in the complex projective plane in [7].

In order to obtain Poincaré formulas of general submanifolds in complex projective spaces, we generalize the notion of Kähler angle and introduce *multiple Kähler angle*. The multiple Kähler angle characterizes the orbit of the action of the unitary group on the real Grassmann manifold. Using the multiple Kähler angles and the Poincaré formulas obtained by Howard, we can formulate Poincaré formulas of any real submanifolds in the complex projective spaces. In the cases of low dimensions, we describe the Poincaré formulas in more explicit way.

In Section 2 we define the multiple Kähler angle of a real vector subspace in a complex vector space and show that it is a complete invariant with respect to the natural action of the unitary group. We also explain its properties from the viewpoint of isometric transformation groups.

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In Section 3 we review the Poincaré formula obtained by Howard and formulate Poincaré formulas of any real submanifolds in the complex projective spaces by the use of the multiple Kähler angles.

The Poincaré formula obtained by Howard plays an important and fundamental role in this paper. We give another proof of it in Section 4.

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2. Multiple Kähler angle

In this section we define the multiple Kähler angle and show its properties. We consider the complex n-dimensional vecter space $\mathbf{C^n}$ with the standard real inner product $\langle \cdot, \cdot \rangle$ and orthogonal almost complex structure J. The natural action of the unitary group $U(n)$ on $\mathbf{C^n}$ induces its action on the Grassmann manifold $G_k^{\mathbf{R}}(\mathbf{C^n})$ consisting of real k-dimensional subspaces in $\mathbf{C^n}$. We define the multiple Kähler angle using the standard Kähler form ω on $\mathbf{C^n}$ defined by $\omega(u,v) = \langle Ju, v \rangle$ for u, v in $\mathbf{C^n}$.

Definition 1. Let $1 < k \le n$. For a real k-dimensional vector subspace V in $\mathbf{C^n}$, we take a canonical form of $\omega|_V$ as an alternating 2-form, that is, we can take an orthonormal basis $\alpha^1, \dots, \alpha^k$ of the dual space V^* which satisfies

$$\omega|_V = \sum_{i=1}^{[k/2]} \cos\theta_i \alpha^{2i-1} \wedge \alpha^{2i}, \qquad 0 \le \theta_1 \le \cdots \le \theta_{[k/2]} \le \pi/2.$$

We put $\theta(V) = (\theta_1, \dots, \theta_{[k/2]})$ and call it the *multiple Kähler angle* of V. If $n < k < 2n-1$ we define the multiple Kähler angle of real k-dimensional vector subspace V in $\mathbf{C^n}$ as that of its orthogonal complement $V^\perp$, that is, $\theta(V) = \theta(V^\perp)$.

We give some remarks on the multiple Kähler angle.

Remark 2. Let $k \le n$. For a real k-dimensional vector subspace V in $\mathbf{C^n}$ the following statements hold.

(1) The action of $U(n)$ preserves the multiple Kähler angle.
(2) $U(n)$ acts transitively on $G_1^{\mathbf{R}}(\mathbf{C^n})$. So it is not necessary to define the multiple Kähler angle for the case of $k = 1$.
(3) If $k = 2$, the multiple Kähler angle is nothing but the Kähler angle, which was used by Chern and Wolfson [1] in the theory of minimal surfaces.
(4) $\theta(V) = (0, \dots, 0)$, if and only if there is a complex $[k/2]$-dimensional subspace in V.
(5) $\theta(V) = (\pi/2, \dots, \pi/2)$, if and only if V and JV are perpendicular.

We denote by $\mathbf{e_1}, \dots, \mathbf{e_n}$ the standard unitary basis of $\mathbf{C^n}$.

Proposition 3. *Let $k \le n$. For $\theta = (\theta_1, \dots, \theta_{[k/2]})$ with $0 \le \theta_1 \le \cdots \le \theta_{[k/2]} \le \pi/2$ we put*

$$G_{k;\theta}^n = \{V \in G_k^{\mathbf{R}}(\mathbf{C^n}) \mid \theta(\mathbf{V}) = \theta\}.$$

Then $U(n)$ acts transitively on $G^n_{k;\theta}$. Moreover we put

$$V^k_\theta = \sum_{i=1}^{[k/2]} \mathrm{span}_{\mathbf{R}}\{\mathbf{e_{2i-1}}, \cos\theta_{\mathbf{i}}\sqrt{-1}\mathbf{e_{2i-1}} + \sin\theta_{\mathbf{i}}\mathbf{e_{2i}}\} \quad (+\mathbf{Re_k}),$$

where the last term is added only when k is odd. Then $G^n_{k;\theta} = U(n)\cdot V^k_\theta$ holds.

Proof. It is sufficient to show $G^n_{k;\theta} = U(n)\cdot V^k_\theta$. The action of $U(n)$ preserves the multiple Kähler angle, so we have $G^n_{k;\theta} \supset U(n)\cdot V^k_\theta$. We shall show $G^n_{k;\theta} \subset U(n)\cdot V^k_\theta$. Take $V \in G^n_{k;\theta}$ and an orthonormal basis $\alpha^1, \dots, \alpha^k$ of V^* which satisfies

$$\omega|_V = \sum_{i=1}^{[k/2]} \cos\theta_i \alpha^{2i-1} \wedge \alpha^{2i}.$$

We take the dual basis $e_1, \dots, e_k$ of $\alpha^1, \dots, \alpha^k$. We put

$$V_i^{\mathbf{C}} = \mathrm{span}_{\mathbf{C}}\{e_{2i-1}, e_{2i}\} \quad (1 \le i \le [k/2]), \qquad V_o^{\mathbf{C}} = \mathbf{Ce_k}.$$

By the canonical form of $\omega|_V$ we can see that

$$\sum_{i=1}^{[k/2]} V_i^{\mathbf{C}} \quad (+V_o^{\mathbf{C}})$$

is an orthogonal direct sum decomposition. So there exists $g \in U(n)$ which satisfies

$$gV_i^{\mathbf{C}} \subset \mathrm{span}_{\mathbf{C}}\{\mathbf{e_{2i-1}}, \mathbf{e_{2i}}\}, \quad \mathbf{gV_o^C} \subset \mathbf{Ce_k}.$$

The Kähler angle of $g(\mathrm{span}_{\mathbf{R}}\{e_{2i-1}, e_{2i}\})$ in $\mathrm{span}_{\mathbf{C}}\{\mathbf{e_{2i-1}}, \mathbf{e_{2i}}\}$ is equal to θ_i, thus by Lemma 2.2 in Kang-T.[6] we can transform $g(\mathrm{span}_{\mathbf{R}}\{e_{2i-1}, e_{2i}\})$ to

$$\mathrm{span}_{\mathbf{R}}\{\mathbf{e_{2i-1}}, \cos\theta_{\mathbf{i}}\sqrt{-1}\mathbf{e_{2i-1}} + \sin\theta_{\mathbf{i}}\mathbf{e_{2i}}\}$$

using the action of $U(2)$. Moreover we can transform ge_k to $\mathbf{e_k}$ using the action of $U(1)$ when k is odd. Therefore there exists $h \in U(n)$ which satisfies $hV = V^k_\theta$.

Proposition 4. *For any real vector subspace V in $\mathbf{C^n}$, the equation $\theta(V^\perp) = \theta(V)$ holds.*

Proof. If $\dim V$ is not equal to n, the equation $\theta(V^\perp) = \theta(V)$ holds by the definition. So it is sufficient to consider the case of $\dim V = n$. Moreover we can suppose $V = V^n_\theta$ which is defined in Proposition 3. The orthogonal complement of V^n_θ is given by

$$(V^n_\theta)^\perp = \sum_{i=1}^{[n/2]} \mathrm{span}_{\mathbf{R}}\{-\sin\theta_i\sqrt{-1}\mathbf{e_{2i-1}} + \cos\theta_{\mathbf{i}}\mathbf{e_{2i}}, \sqrt{-1}\mathbf{e_{2i}},\} \quad (+\mathbf{R}\sqrt{-1}\mathbf{e_n}),$$

where the last term is added only when n is odd. We take an element g in $U(n)$ which satisfies

$$g(\mathbf{e_{2i-1}}) = \sqrt{-1}\mathbf{e_{2i}}, \qquad \mathbf{g(e_{2i})} = \sqrt{-1}\mathbf{e_{2i-1}} \qquad (\mathbf{g(e_n)} = \sqrt{-1}\mathbf{e_n}).$$

Then we obtain $g((V_\theta^n)^\perp) = V_\theta^n$ and

$$\theta(V_\theta^n) = \theta(g((V_\theta^n)^\perp)) = \theta((V_\theta^n)^\perp).$$

Now we explain the meaning of the multiple Kähler angle from the viwpoint of isometric transformation groups. We can describe the Grassmann manifold

$$G_k^{\mathbf{R}}(\mathbf{C^n}) = \mathbf{O(2n)/O(k) \times O(2n-k)}$$

as a homogeneous space. This is a compact symmetric space and $O(2n)/U(n)$ is also a compact symmetric space. So the action of $U(n)$ on $G_k^{\mathbf{R}}(\mathbf{C^n})$ is a Hermann action. In general for a compact Lie group G and two symmetric pairs (G, K_1) and (G, K_2) the natural action of K_2 on the compact symmetric space G/K_1 is called a *Hermann action*. It is known that a Hermann action has a flat section, which meets perpendicularly all of the orbits. This is a result of Hermann [4]. Heintze, Palais, Terng and Thorbergsson [2] generalized the notion of Hermann action. See also this for Hermann action. We show that the set of all V_θ^k described in Proposition 3 is a concrete flat section of the action of the unitary group on the real Grassmann manifold.

Proposition 5. *Let $k \leq n$. The set $\{V_\theta^k \mid \theta \in \mathbf{R}^{[\mathbf{k/2}]}\}$ is a flat section of the action of $U(n)$ on $G_k^{\mathbf{R}}(\mathbf{C^n})$.*

Remark 6. The multiple Kähler angle can be regarded as a function defined on the Grassmann manifold $G_k^{\mathbf{R}}(\mathbf{C^n})$ and is invariant under the action of the unitary group $U(n)$. So the multiple Kähler angle is determined by its values on a section of the action of $U(n)$. The multiple Kähler angle on the section described in Proposition 5 is a simple coordinate of the flat torus.

Proof. We write

$$S = \{V_\theta^k \mid \theta \in \mathbf{R}^{[\mathbf{k/2}]}\}.$$

Proposition 3 shows that all of the orbits of the action of $U(n)$ on $G_k^{\mathbf{R}}(\mathbf{C^n})$ meet S. So we have to show that S is a flat submanifold in $G_k^{\mathbf{R}}(\mathbf{C^n})$ and that all of the orbits of the action of $U(n)$ perpendicularly meet S.

From some fundamental results of totally geodesic submanifolds and curvature tensors of symmetric spaces which we can see in Helgason [3], the definition of V_θ^k shows that S is a totally geodesic submanifold in $G_k^{\mathbf{R}}(\mathbf{C^n})$ which is flat with respect to the metric induced from that of $G_k^{\mathbf{R}}(\mathbf{C^n})$.

The property that all of the orbits of the action of $U(n)$ on $G_k^{\mathbf{R}}(\mathbf{C^n})$ perpendicularly meet S is equivalent to that all of the orbits of the action of $U(n)$ on the oriented Grassmann manifold $\tilde{G}_k^{\mathbf{R}}(\mathbf{C^n})$ consisting of oriented real k-dimensional subspaces in $\mathbf{C^n}$ perpendicularly meet $\tilde{S}$ coresponding to S. The oriented Grassmann manifold $\tilde{G}_k^{\mathbf{R}}(\mathbf{C^n})$ can be regarded as a submanifold in the real k-th exterior product $\wedge_k^{\mathbf{R}}(\mathbf{C^n})$. We take

$$\xi_0 = \left(\wedge_{i=1}^{[k/2]} \mathbf{e_{2i-1}} \wedge (\cos\theta_\mathbf{i}\sqrt{-1}\mathbf{e_{2i-1}} + \sin\theta_\mathbf{i}\mathbf{e_{2i}})\right) \wedge \mathbf{e_k} \in \mathbf{S},$$

where $\wedge \mathbf{e_k}$ is added only when k is odd and we use this rule in this proof. The tangent space $T_{\xi_0}\tilde{S}$ is given by

$$T_{\xi_0}S = \mathrm{span}_{\mathbf{R}} \left\{ \left(\wedge_{i=1}^{[k/2]}\xi_i^j\right) \wedge \mathbf{e_k} \;\middle|\; 1 \le j \le [k/2] \right\},$$

where

$$\xi_i^j = \begin{cases} \mathbf{e_{2i-1}} \wedge (\cos\theta_\mathbf{i}\sqrt{-1}\mathbf{e_{2i-1}} + \sin\theta_\mathbf{i}\mathbf{e_{2i}}) & (i \ne j), \\ \mathbf{e_{2i-1}} \wedge (-\sin\theta_\mathbf{i}\sqrt{-1}\mathbf{e_{2i-1}} + \cos\theta_\mathbf{i}\mathbf{e_{2i}}) & (i = j). \end{cases}$$

We define a basis

$$\{E_{ij} \mid 1 \le i < j \le n\} \cup \{F_{ij} \mid 1 \le i \le j \le n\}$$

of the Lie algebra of the unitary group $U(n)$ by

$$E_{ij}\mathbf{e_k} = \begin{cases} \mathbf{e_j} & (k = i), \\ -\mathbf{e_i} & (k = j), \\ 0 & (k \ne i, j), \end{cases} \qquad \mathbf{F_{ij}e_k} = \begin{cases} \sqrt{-1}\mathbf{e_j} & (k = i), \\ \sqrt{-1}\mathbf{e_i} & (k = j), \\ 0 & (k \ne i, j). \end{cases}$$

We can see

$$\begin{aligned} \left\langle \left(\wedge_{i=1}^{[k/2]}\xi_i^j\right) \wedge \mathbf{e_k}, \left.\frac{\mathbf{d}}{\mathbf{dt}}\right|_{\mathbf{t=0}} \exp \mathbf{tE_{pq}}\xi_\mathbf{0} \right\rangle &= \\ \left\langle \left(\wedge_{i=1}^{[k/2]}\xi_i^j\right) \wedge \mathbf{e_k}, \left.\frac{\mathbf{d}}{\mathbf{dt}}\right|_{\mathbf{t=0}} \exp \mathbf{tF_{rs}}\xi_\mathbf{0} \right\rangle &= 0, \end{aligned}$$

which shows that all of the orbits of the action of $U(n)$ perpendicularly meet $\tilde{S}$. This completes the proof of the proposition.

If $n < k \le 2n$ we put

$$V_\theta^k = (V_\theta^{2n-k})^\perp.$$

3. Integral geometry in complex projective spaces

In this section we discuss Poincaré formula in complex projective spaces. Before this we have to recall the generalized Poincaré formula in Riemmannian homogeneous spaces obtained by Howard [5].

We assume that E is a real vector space with an inner product. For two vector subspaces V and W we define $\sigma(V, W)$ by

$$\sigma(V, W) = |v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q|,$$

where $\{v_i\}$ and $\{w_j\}$ are orthonormal bases of V and W respectively. This definition is independent of the choice of v_i and w_j. We assume that G/K is a Riemannian homogeneous space. We denote by o the origin of G/K. For any x and y in G/K and vector subspaces V and W in $T_x(G/K)$ and $T_y(G/K)$, we define $\sigma_K(V, W)$ by

$$\sigma_K(V, W) = \int_K \sigma((dg_x)_o^{-1}V, dk_o^{-1}(dg_y)_o^{-1}W)d\mu_K(k),$$

where g_x and g_y are elements of G such that $g_x o = x$ and $g_y o = y$. This definition is independent of the choice of g_x and g_y in G such that $g_x o = x$ and $g_y o = y$. The σ_K satisfies the following equations: For any $g \in G$

$$\sigma_K(V, W) = \sigma_K(dgV, W) = \sigma_K(V, dgW) = \sigma_K(W, V).$$

This σ_K is invariant under the action of G. So it is sufficient to consider σ_K only in the tangent space $T_o(G/K)$ at the origin o. Using σ_K we can state the generalized Poincaré formula obtained by Howard.

Theorem 7 (Howard[5]). *Assume that G is unimodular. Let M and N be submanifolds of G/K such that $\dim M + \dim N \geq \dim(G/K)$. Then the following equation holds:*

$$\int_G \mathrm{vol}(gM \cap N) d\mu_G(g) = \int_{M \times N} \sigma_K(T_x^\perp M, T_y^\perp N) d\mu_{M\times N}(x, y).$$

Although this formula holds in a general situation, the integrand σ_K of the integral in the right hand side is explicitly described only in a few cases. For example in the case that the homogeneous space G/K is a space of constant curvature, the isotropy group K acts transitively on the Grassmann manifolds consisting of subspaces in $T_o(G/K)$, so σ_K is constant, that is, $\sigma_K(V, W)$ is not dependent on V and W. In this case we can explicitly describe σ_K as follows:

$$\sigma_{SO(n)}(V^p, V^q) = \frac{\mathrm{vol}(S^{p+q-n})\mathrm{vol}(SO(n+1))}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)}.$$

In the case that G/K is a two point homogeneous space of dimension n, σ_K is given by

$$\sigma_K(V^p, V^{n-1}) = \frac{\mathrm{vol}(K)\mathrm{vol}(S^{p-1})\mathrm{vol}(S^n)}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})},$$

which is a result of Howard [5](p.21). In general, the actions of K on the Grassmann manifolds are not transitive. The function σ_K is defined on the product of the Grassmann manifolds consisting of subspaces in the tangent space at the origin. By the invariance of σ_K under the action of K, we can regard the function σ_K is defined on the product of the orbit spaces of the actions of K on the Grassmann manifolds. We can apply this and the argument on the multiple Kähler angle to the complex projective space, and obtain the following theorem.

Theorem 8. *For any integers p and q which satisfy $p \leq 2n \leq p + q$ and $q \leq 2n \leq p + q$, we define*

$$\begin{aligned} \sigma^n_{p,q}(\theta^{(p)}, \theta^{(q)}) &= \int_{U(1)\times U(n)} \sigma(V^{2n-p}_{\theta^{(p)}}, k^{-1} \cdot V^{2n-q}_{\theta^{(q)}}) d\mu_{U(1)\times U(n)}(k) \\ & \qquad (\theta^{(p)} \in \mathbf{R}^{[\min\{\mathbf{p},\mathbf{2n-p}\}/\mathbf{2}]},\ \theta^{(\mathbf{q})} \in \mathbf{R}^{[\min\{\mathbf{q},\mathbf{2n-q}\}/\mathbf{2}]}). \end{aligned}$$

If q is equal to $2n-1$, we can define $\sigma^n_{p,q}$ by the use of any subspace of dimension $2n-1$. Then for any real p-dimensional submanifold M and any real q-dimensional *submanifold N in $\mathbf{CP^n}$, we have*

$$\int_{U(n+1)} \mathrm{vol}(gM \cap N) d\mu_{U(n+1)}(g) = \int_{M\times N} \sigma^n_{p,q}(\theta(T_xM), \theta(T_yN)) d\mu_{M\times N}(x,y).$$

Remark 9. (1) The function $\sigma^n_{2n-1,q}$ is defined by

$$\sigma^n_{2n-1,q}(\theta^{(q)}) = \int_{U(1)\times U(n)} \sigma(V, k^{-1}\cdot V^{2n-q}_{\theta^{(q)}}) d\mu_{U(1)\times U(n)}(k),$$

where V is any subspace of dimension $2n-1$. This definition is independent of the choice of V, because $U(n)$ acts transitively on the Grassmann manifold $G^{\mathbf{R}}_{2n-1}(\mathbf{C^n})$. Since the complex projective space is a two point homogeneous space, the result of Howard mentioned above says that $\sigma^n_{2n-1,q}(\theta^{(q)})$ is not dependent on $\theta^{(q)}$, that is,

$$\sigma^n_{2n-1,q} = \sigma^n_{q,2n-1} = \frac{\mathrm{vol}(U(1)\times U(n))\mathrm{vol}(S^{q-1})\mathrm{vol}(S^{2n})}{\mathrm{vol}(S^q)\mathrm{vol}(S^{2n-1})},$$

(2) The Poincaré formulas of complex submanifolds have been already obtained by Santaló [8] and reformulated by Howard [5]. By their results we can write

$$\sigma^n_{2p,2q}(0,0) = \frac{\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbf{CP^p})\mathrm{vol}(\mathbf{CP^q})}$$

for p and q which satisfy $p \le n \le p+q$ and $q \le n \le p+q$.

By the transfer principle mentioned in the paragraph 3.5 in [5], we get the following corollary of Theorem 8.

Corollary 10. *Let G/K be a complex space form with isotropy subgroup $U(1)\times U(n)$. Then the formula in Theorem 8 holds for submanifolds M and N in G/K.*

So the next problem is to calculate the integral of $\sigma^n_{p,q}$. Some calculations of integrals show the following:

$$\sigma^n_{2,2n-2}(\theta,0) = \sigma^n_{2n-2,2}(\theta,0) = \frac{\mathrm{vol}(U(n+1))}{2\mathrm{vol}(\mathbf{CP^1})\mathrm{vol}(\mathbf{CP^{n-1}})}(1+\cos^2\theta), \tag{1}$$

$$\sigma^2_{2,2}(\theta,\tau) = \frac{\mathrm{vol}(U(3))}{\mathrm{vol}(\mathbf{CP^1})^2}\left(\frac{1}{4}(1+\cos^2\theta)(1+\cos^2\tau) + \frac{1}{2}\sin^2\theta\sin^2\tau\right) \tag{2}$$

The equation (1) is obtained by Kang-T.[6]. The equation (2) is obtained by the formula:

$$\sigma^2_{2,2}(\theta,\tau) = \frac{\mathrm{vol}(U(3))}{\mathrm{vol}(\mathbf{RP^2})^2}(2 + 2\cos^2\theta\cos^2\tau + \sin^2\theta\sin^2\tau),$$

which is proved in Kang-T.[7] and $\mathrm{vol}(\mathbf{RP^2}) = \mathbf{2}\mathrm{vol}(\mathbf{CP^1})$. The author [9] recently obtained

$$\begin{aligned}
&\sigma^n_{2,2n-2}(\theta,\tau)\\
= \ &\frac{\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbf{CP^1})\mathrm{vol}(\mathbf{CP^{n-1}})}\\
&\times\left(\frac{1}{4}(1+\cos^2\theta)(1+\cos^2\tau)+\frac{n}{4(n-1)}\sin^2\theta\sin^2\tau\right)
\end{aligned}$$

by the use of (1) and (2). This equation implies the following theorem.

Theorem 11 (T.[9]). *For any real 2-dimensional submanifold M and real $(2n-2)$-dimensional submanifold N in the complex projective space $\mathbf{CP^n}$ of dimension n, we have*

$$\begin{aligned}
&\int_{U(n+1)} \#(gM\cap N)d\mu_{U(n+1)}(g)\\
= \ &\frac{\mathrm{vol}(U(n+1))\mathrm{vol}(N)}{\mathrm{vol}(\mathbf{CP^1})\mathrm{vol}(\mathbf{CP^{n-1}})}\\
&\times\int_{M\times N}\left(\frac{1}{4}(1+\cos^2\theta_x)(1+\cos^2\tau_y)+\frac{n}{4(n-1)}\sin^2\theta_x\sin^2\tau_y\right)\\
&\cdot d\mu_{M\times N}(x,y),
\end{aligned}$$

where θ_x is the Kähler angle of T_xM and τ_y is the Kähler angle of $T_y^\perp N$.

Remark 12. In order to formulate Poincaré formulas in a Riemannian symmetric space G/K we have to investigate the natural action of K on the Grassmann manifold $G_k^{\mathbf{R}}(T_o(G/K))$ consisting of k-dimensional subspaces in $T_o(G/K)$. As we have shown above, in the case of complex projective spaces all of these actions have flat sections. Due to this we can characterize each orbit of this action and formulate Poincaré formulas in complex projective spaces by the coordinate of the flat section which we call the multiple Kähler angle. It is known that the action of K on the Grassmann manifold $G_k^{\mathbf{R}}(T_o(G/K))$ does not have a flat section in general. In general cases the author does not know whether the action of K on $G_k^{\mathbf{R}}(T_o(G/K))$ has a section or not. It is important to study Poincaré formulas in other Riemannian symmetric spaces.

4. Another proof of the Poincaré formula

In his paper [5] Howard first proved the Poincaré formula in Lie groups and using this he proved the Poincaré formula for general Riemannian homogeneous spaces. In this section we give a direct proof of Theorem 7.

We define a subset $I(G\times(G/K)^2)$ in $G\times(G/K)^2$ by

$$I(G\times(G/K)^2)=\{(g,x,y)\in G\times(G/K)^2 \mid gx=y\}.$$

We first show that $I(G \times (G/K)^2)$ is a regular submanifold in $G \times (G/K)^2$. We define a submersion

$$p : G \times (G/K)^2 \to (G/K)^2 \ ; \ (g, x, y) \mapsto (gx, y).$$

Let

$$D(G/K) = \{(x, x) \in (G/K)^2 \mid x \in G/K\}.$$

Then $D(G/K)$ is a regular submanifold in $(G/K)^2$. So its inverse image $p^{-1}(D(G/K))$ under the submersion p is a regular submanifold in $G \times (G/K)^2$. Since

$$I(G \times (G/K)^2) = p^{-1}(D(G/K))$$

holds, $I(G \times (G/K)^2)$ is a regular submanifold in $G \times (G/K)^2$. We note that

$$\begin{aligned} \dim I(G \times (G/K)^2) &= \dim G + 2\dim(G/K) - \dim(G/K) \\ &= \dim G + \dim(G/K). \end{aligned}$$

Next we define a mapping q by

$$q : I(G \times (G/K)^2) \to (G/K)^2 \ ; \ (g, x, y) \mapsto (x, y).$$

We show that this is a fiber bundle with fiber K. For each $(x, y) \in (G/K)^2$ we choose $g_x, g_y \in G$ which satisfy $g_x o = x$, $g_y o = y$. Then we can see

$$q^{-1}(x, y) \supset (g_y K g_x^{-1}) \times \{(x, y)\}.$$

Conversely for $(g, x, y) \in q^{-1}(x, y)$ we have $gx = y$ and

$$o = g_y^{-1} y = g_y^{-1} g x = g_y^{-1} g g_x o.$$

Thus $g_y^{-1} g g_x \in K$ and $g \in g_y K g_x^{-1}$. From these we obtain

$$q^{-1}(x, y) = (g_y K g_x^{-1}) \times \{(x, y)\},$$

which is diffeomorphic to K. Since G/K is a Riemannian homogeneous space G has a left invariant Riemannian metric which is biinvariant on K. This metric induces an inner product on the Lie algebra $\mathfrak{g}$ of G which is invariant under the action of $\mathrm{Ad}_G(K)$. We denote by $\mathfrak{k}$ the Lie algebra of K. Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be an orthogonal direct sum decomposition of $\mathfrak{g}$. Since $\mathrm{Ad}_G(K)\mathfrak{k} \subset \mathfrak{k}$, we have $\mathrm{Ad}_G(K)\mathfrak{p} \subset \mathfrak{p}$. There exist an open neighborhood U of the origin 0 in $\mathfrak{p}$ and an open neighborhood V of the origin o in G/K such that

$$U \to V \ ; \ u \mapsto \exp u(o)$$

is a diffeomorphism. Hence the mappings

$$\begin{aligned} U \to g_x V \ &; \ u \mapsto g_x \exp u(o), \\ U \to g_y V \ &; \ u \mapsto g_y \exp u(o) \end{aligned}$$

are also diffeomorphisms. We define a mapping

$$\varphi : K \times g_x V \times g_y V \to q^{-1}(g_x V \times g_y V)$$

by

$$\begin{aligned}
& \varphi(k, g_x \exp u(o), g_y \exp v(o)) \\
= \quad & (g_y \exp v k \exp(-u) g_x^{-1}, g_x \exp u(o), g_y \exp v(o)).
\end{aligned}$$

Then φ is of class C^∞. The inverse mapping

$$\varphi^{-1} : q^{-1}(g_x V \times g_y V) \to K \times g_x V \times g_y V$$

is given by

$$\begin{aligned}
& \varphi^{-1}(g, g_x \exp u(o), g_y \exp v(o)) \\
= \quad & (\exp(-v) g_y^{-1} g g_x \exp u, g_x \exp u(o), g_y \exp v(o))
\end{aligned}$$

and this is also of class C^∞. Thus φ is a diffeomorphism. Therefore φ gives a local triviality of $q : I(G \times (G/K)^2) \to (G/K)^2$ and $q : I(G \times (G/K)^2) \to (G/K)^2$ is a fiber bundle with fiber K.

We need the differential of φ at (k, x, y) later, so we calculate it now. For $T \in \mathfrak{k}$ we have

$$\begin{aligned}
d\varphi_{(k,x,y)}(dL_k T, 0, 0) &= \left.\frac{d}{dt}\right|_{t=0} (g_y k \exp(tT) g_x^{-1}, x, y) \\
&= (dL_{\varphi(k,x,y)} \mathrm{Ad}_G(g_x) T, 0, 0).
\end{aligned}$$

For $X \in \mathfrak{p}$ we have

$$\begin{aligned}
d\varphi_{(k,x,y)}(0, dg_x X, 0) &= \left.\frac{d}{dt}\right|_{t=0} (g_y k \exp(-tX) g_x^{-1}, g_x \exp(tX)(o), y) \\
&= (-dL_{\varphi(k,x,y)} \mathrm{Ad}_G(g_x) X, dg_x X, 0).
\end{aligned}$$

For $Y \in \mathfrak{p}$ we have

$$\begin{aligned}
d\varphi_{(k,x,y)}(0, 0, dg_y Y) &= \left.\frac{d}{dt}\right|_{t=0} (g_y \exp(tY) k g_x^{-1}, x, g_y \exp(tY)(o)) \\
&= (dL_{\varphi(k,x,y)} \mathrm{Ad}_G(g_x) \mathrm{Ad}_G(k^{-1}) Y, 0, dg_y Y).
\end{aligned}$$

Thus for $T \in \mathfrak{k}$, $X, Y \in \mathfrak{p}$ we have

$$\begin{aligned}
& d\varphi_{(k,x,y)}(dL_k T, dg_x X, dg_y Y) \\
= \quad & (dL_{\varphi(k,x,y)} \mathrm{Ad}_G(g_x)(T - X + \mathrm{Ad}_G(k^{-1})Y), dg_x X, dg_y Y).
\end{aligned}$$

Since $q : I(G \times (G/K)^2) \to (G/K)^2$ is a submersion,

$$I(M, N) = q^{-1}(M \times N)$$

is a submanifold in $I(G \times (G/K)^2)$. We note that

$$\begin{aligned}
\dim I(M, N) &= (\dim G + \dim(G/K)) - \mathrm{codim}(M \times N) \\
&= (\dim G + \dim(G/K)) - (2\dim(G/K) - \dim M - \dim N) \\
&\geq \dim G.
\end{aligned}$$

We define a mapping f of class C^∞ by

$$f : I(M,N) \to G \;\; ; \; (g,x,y) \mapsto g.$$

By the coarea formula we have

$$\int_{I(M,N)} Jf d\mu_{I(M,N)} = \int_G \mathrm{vol}(f^{-1}(g)) d\mu_G(g).$$

Since

$$\begin{aligned} f^{-1}(g) &= I(M,N) \cap (\{g\} \times (G/K)^2) \\ &= \{(g,x,gx) \mid gx \in gM \cap N\}, \end{aligned}$$

the mapping ψ defined by

$$\psi : gM \cap N \to f^{-1}(g); \; gx \mapsto (g,x,gx)$$

is bijective. If g is a regular value of f, then $f^{-1}(g)$ is a submanifold in $I(M,N)$. $gM \cap N$ is also a submanifold in G/K. Moreover ψ is a diffeomorphism. For a tangent vector X of $g^{-1}(gM \cap N)$ we have

$$d\psi(dgX) = (0, X, dgX).$$

Since dg is a linear isometric mapping,

$$\langle d\psi(dgX), d\psi(dgY)\rangle = \langle X, Y\rangle + \langle dgX, dgY\rangle = 2\langle X, Y\rangle.$$

We put

$$r = \dim(f^{-1}(g)) = \dim(gM \cap N) = \dim M + \dim N - \dim(G/K).$$

Then the following equation holds:

$$\mathrm{vol}(f^{-1}(g)) = 2^{r/2} \mathrm{vol}(gM \cap N).$$

Hence we obtain that

$$\int_{I(M,N)} Jf d\mu_{I(M,N)} = 2^{r/2} \int_G \mathrm{vol}(gM \cap N) d\mu_G(g).$$

We calculate the left hand side of this equation. For this, we take orthonormal bases $\{T_a\}$, $\{X_b\}$, $\{Y_c\}$ of $\mathfrak{k}$, $dg_x^{-1}T_xM$, $dg_y^{-1}T_yN$ respectively. By the above result for the differential of φ we see that

$$\begin{aligned} d\varphi_{(k,x,y)}(dL_kT_a, 0, 0) &= (dL_{\varphi(k,x,y)}\mathrm{Ad}_G(g_x)T_a, 0, 0) \\ d\varphi_{(k,x,y)}(0, dg_xX_b, 0) &= (-dL_{\varphi(k,x,y)}\mathrm{Ad}_G(g_x)X_b, dg_xX_b, 0) \\ d\varphi_{(k,x,y)}(0, 0, dg_yY_c) &= (dL_{\varphi(k,x,y)}\mathrm{Ad}_G(g_x)\mathrm{Ad}_G(k^{-1})Y_c, 0, dg_yY) \end{aligned}$$

is a basis of $T_{\varphi(k,x,y)}I(M \times N)$. Note that

$$\begin{aligned} & df_{\varphi(k,x,y)} d\varphi_{(k,x,y)}(dL_kT_a, dg_xX_b, dg_yY_c) \\ = \;& dL_{\varphi(k,x,y)}\mathrm{Ad}_G(g_x)(T_a - X_b + \mathrm{Ad}_G(k^{-1})Y_c). \end{aligned}$$

We extend $Y_d = \mathrm{Ad}_G(k)X_d$ $(1 \leq d \leq r)$ to an orthonormal basis $\{Y_c\}$ of $dg_y^{-1}T_yN$. Then

$$d\varphi_{(k,x,y)}(0, dg_xX_d, dg_yY_d) = (0, dg_xX_d, dg_yY_d) \qquad (1 \leq d \leq r)$$

is a basis of $\ker df_{\varphi(k,x,y)}$. Since

$$\begin{array}{rcl} \bar{T}_a &=& (dL_kT_a, 0, 0) \\ \bar{X}_b &=& (0, dg_xX_b, 0) \qquad (r+1 \leq b) \\ \bar{Y}_c &=& (0, 0, dg_yY_c) \qquad (r+1 \leq c) \\ \bar{Z}_d &=& \dfrac{1}{\sqrt{2}}(0, -dg_xX_d, dg_yY_d) \qquad (1 \leq d \leq r) \end{array}$$

are orthonormal,

$$d\varphi_{(k,x,y)}(\bar{T}_a), \quad d\varphi_{(k,x,y)}(\bar{X}_b), \quad d\varphi_{(k,x,y)}(\bar{Y}_c), \quad d\varphi_{(k,x,y)}(\bar{Z}_d)$$

is a basis of $(\ker df_{\varphi(k,x,y)})^{\perp}$. Moreover we get

$$\begin{array}{rcl} df_{\varphi(k,x,y)}d\varphi_{(k,x,y)}(\bar{T}_a) &=& dL_{\varphi(k,x,y)}\mathrm{Ad}_G(g_x)(T_a) \\ df_{\varphi(k,x,y)}d\varphi_{(k,x,y)}(\bar{X}_b) &=& dL_{\varphi(k,x,y)}\mathrm{Ad}_G(g_x)(-X_b) \\ df_{\varphi(k,x,y)}d\varphi_{(k,x,y)}(\bar{Y}_c) &=& dL_{\varphi(k,x,y)}\mathrm{Ad}_G(g_x)(\mathrm{Ad}_G(k^{-1})Y_c) \\ df_{\varphi(k,x,y)}d\varphi_{(k,x,y)}(\bar{Z}_d) &=& dL_{\varphi(k,x,y)}\mathrm{Ad}_G(g_x)(\sqrt{2}X_d). \end{array}$$

Hence

$$Jf = \frac{|dL_{\varphi(k,x,y)}\mathrm{Ad}_G(g_x)(\wedge_aT_a \wedge \wedge_b(-X_b) \wedge \wedge_c\mathrm{Ad}_G(k^{-1})Y_c) \wedge \wedge_d\sqrt{2}X_d|}{|d\varphi_{(k,x,y)}(\wedge_a\bar{T}_a \wedge \wedge_b\bar{X}_b \wedge \wedge_c\bar{Y}_c \wedge \wedge_d\bar{Z}_d)|}.$$

Its numerator is

$$\begin{array}{rl} & |dL_{\varphi(k,x,y)}\mathrm{Ad}_G(g_x)(\wedge_aT_a \wedge \wedge_b(-X_b) \wedge \wedge_c\mathrm{Ad}_G(k^{-1})Y_c) \wedge \wedge_d\sqrt{2}X_d| \\ = & 2^{r/2}|\mathrm{Ad}_G(g_x)(\wedge_aT_a \wedge \wedge_b(-X_b) \wedge \wedge_c\mathrm{Ad}_G(k^{-1})Y_c) \wedge \wedge_dX_d| \\ = & 2^{r/2}|\det \mathrm{Ad}_G(g_x)||\wedge_a T_a \wedge \wedge_bX_b \wedge \wedge_c\mathrm{Ad}_G(k^{-1})Y_c \wedge \wedge_dX_d| \\ & \text{(since } G \text{ is unimodular, we have } |\det \mathrm{Ad}_G(g_x)| = 1) \\ = & 2^{r/2}|\wedge_a T_a \wedge \wedge_bX_b \wedge \wedge_c\mathrm{Ad}_G(k^{-1})Y_c \wedge \wedge_dX_d| \\ & \text{(because } \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \text{ is an orthogonal direct sum and } \mathrm{Ad}_{\mathfrak{G}}(\mathfrak{k}^{-1})\mathfrak{Y}_{\mathfrak{c}} \in \mathfrak{p}) \\ = & 2^{r/2}|\wedge_b X_b \wedge \wedge_c\mathrm{Ad}_G(k^{-1})Y_c \wedge \wedge_dX_d| \\ = & 2^{r/2}\sigma(dg_x^{-1}T_x^{\perp}M, \mathrm{Ad}_G(k^{-1})dg_y^{-1}T_y^{\perp}N) \\ = & 2^{r/2}\sigma(dg_x^{-1}T_x^{\perp}M, dk_o^{-1}dg_y^{-1}T_y^{\perp}N). \end{array}$$

Its integral on K is equal to $2^{r/2}\sigma_K(T_x^{\perp}M, T_y^{\perp}N)$, so we obtain

$$\int_{I(M,N)} Jf d\mu_{I(M,N)} = 2^{r/2}\int_{M\times N}\sigma_K(T_x^{\perp}M, T_y^{\perp}N)d\mu_{M\times N}(x,y)$$

and thus

$$\int_G \mathrm{vol}(gM \cap N) d\mu_G(g) = \int_{M\times N} \sigma_K(T_x^\perp M, T_y^\perp N) d\mu_{M\times N}(x, y).$$

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