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HAMILTONIAN FIELD THEORY REVISITED: A GEOMETRIC APPROACH TO REGULARITY

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ABSTRACT. The aim of the paper is to announce some recent results concerning Hamiltonian theory for higher order variational problems on fibered manifolds. A reformulation, generalization and extension of basic concepts such as Hamiltonian system, Hamilton equations, regularity, and Legendre transformation, is presented. The theory is based on the concept of *Lepagean* $(n+1)$*-form* (where n is the dimension of the base manifold). Contrary to the classical approach, where Hamiltonian theory is related to a single Lagrangian, within the present setting a Hamiltonian system is associated with an Euler–Lagrange form, i.e., with the *class of all equivalent Lagrangians*. Hamilton equations are introduced to be equations for integral sections of an exterior differential system, defined by a Lepagean $(n+1)$-form. Relations between extremals and solutions of Hamilton equations are studied in detail. A revision of the concepts of *regularity* and *Legendre transformation* is proposed, reflecting geometric properties of the related exterior differential system. The new look is shown to lead to new regularity conditions and Legendre transformation formulas, and provides a procedure of *regularization* of variational problems. Relations to standard Hamilton–De Donder theory, as well as to multisymplectic geometry are studied. Examples of physically interesting Lagrangian systems which are traditionally singular, but regular in this revised sense, are discussed.

1. INTRODUCTION

Hamiltonian theory belongs to the most important parts of the calculus of variations. The idea goes back to the first half of the 19th century and is due to Sir William Rowan Hamilton and Carl Gustav Jacob Jacobi who, for the case of *classical mechanics*, developed a method to pass from the *Euler–Lagrange equations* to another set of differential equations, now called *Hamilton equations*, which are "better adapted" to integration. This celebrated procedure, however, is applicable

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only to a certain class of variational problems, called *regular*. Later the method was formally generalized to higer order mechanics, and both first and higher order field theory, and became one of the constituent parts of the classical variational theory (cf. [8], [4]). In spite of this fact, it has been clear that this generalization of Hamiltonian theory suffers from a principal defect: allmost all physically interesting field Lagrangians (gravity, Dirac field, electromagnetic field, etc.) are *non-regular*, hence they cannot be treated within this approach.

Since the second half of the 20'th century, together with an increasing interest to bring the more or less heuristic classical variational theory to a modern framework of differential geometry, an urgent need to understand the geometric meaning of the Hamiltonian theory has been felt, in order to develop its proper generalizations as well as global aspects. There appeared many papers dealing with this task in different ways, with results which are in no means complete: from the most important ones let us mention here at least Goldschmidt and Sternberg [14], Aldaya and Azcárraga [1], Dedecker [5], [7], Shadwick [37], Krupka [21]–[23], Ferraris and Francaviglia [9], Krupka and Štěpánková [26], Gotay [16], Garcia and Muñoz [10], [11], together with a rather pessimistic Dedecker's paper [6] summarizing main problems and predicting that a way-out should possibly lead through some new understanding of such fundamental concepts as *regularity*, *Legendre transformation*, or even the Hamiltonian theory as such.

The purpose of this paper is to announce some very recent results, partially presented in [30]–[32] and [38], which, in our opinion, open a new way for understanding the Hamiltonian field theory. We work within the framework of *Krupka's theory of Lagrange structures on fibered manifolds* where the so called *Lepagean form* is a central concept ([18], [19], [21], [24], [25]). Inspired by fresh ideas and interesting, but, unfortunately, not very wide-spread "nonclassical" results of Dedecker [5] and Krupka and Štěpánková [26], the present geometric setting means a direct "field generalization" of the corresponding approach to *higher order Hamiltonian mechanics* as developed in [27] and [28] (see [29] for review). The key point is the concept of a *Hamiltonian system*, which, contrary to the usual approach, is not related with a single Lagrangian, but rather with an Euler–Lagrange form (i.e., with the *class of equivalent Lagrangians*), as well as of *regularity*, which is understood to be a *geometric property of Hamilton equations*. It turns out that "classical" results are incorporated as a special case in this scheme. Moreover, for many variational systems which appear singular within the standard approach, one obtains here a *regular* Hamiltonian counterpart (*Hamiltonian*, independent *momenta* which can be considered a part of certain *Legendre coordinates*, *Hamilton equations equivalent with the Euler–Lagrange equations*). This concerns, among others, such important physical systems as, eg., gravity, electromagnetism or the Dirac field, mentioned above.

2. Notations and preliminaries

All manifolds and mappings throughout the paper are smooth. We use standard notations as, eg., T for the tangent functor, J^r for the r-jet prolongation functor, d for the exterior derivative of differential forms, i_ξ for the contraction by a vector field ξ, and $*$ for the pull-back.

We consider a fibered manifold (i.e., surjective submersion) $\pi : Y \to X$, $\dim X = n$, $\dim Y = m + n$, its r-jet prolongation $\pi_r : J^rY \to X$, $r \geq 1$, and canonical jet projections $\pi_{r,k} : J^rY \to J^kY$, $0 \leq k < r$ (with an obvious notation $J^0Y = Y$). A fibered chart on Y is denoted by (V, ψ), $\psi = (x^i, y^\sigma)$, the associated chart on J^rY by (V_r, ψ_r), $\psi_r = (x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 \ldots j_r})$.

A vector field ξ on J^rY is called π_r-*vertical* (respectively, $\pi_{r,k}$-*vertical*) if it projects onto the zero vector field on X (respectively, on J^kY). We denote by $V\pi_r$ the distribution on J^rY spanned by the π_r-vertical vector fields.

A q-form ρ on J^rY is called $\pi_{r,k}$-*projectable* if there is a q-form ρ_0 on J^kY such that $\pi^*_{r,k}\rho_0 = \rho$. A q-form ρ on J^rY is called π_r-*horizontal* (respectively, $\pi_{r,k}$-*horizontal*) if $i_\xi \rho = 0$ for every π_r-vertical (respectively, $\pi_{r,k}$-vertical) vector field ξ on J^rY.

The fibered structure of Y induces a morphism, h, of exterior algebras, defined by the condition $J^r\gamma^*\rho = J^{r+1}\gamma^*h\rho$ for every section γ of π, and called the *horizontalization*. Apparently, horizontalization applied to a function, f, and to the elements of the canonical basis of 1-forms, $(dx^i, dy^\sigma, dy^\sigma_{j_1}, \ldots, dy^\sigma_{j_1 \ldots j_r})$, on J^rY gives

$$hf = f \circ \pi_{r+1,r}, \quad hdx^i = dx^i, \quad hdy^\sigma = y^\sigma_l dx^l, \quad \ldots, \quad hdy^\sigma_{j_1 \ldots j_r} = y^\sigma_{j_1 \ldots j_r l} dx^l.$$

A q-form ρ on J^rY is called *contact* if $h\rho = 0$. On J^rY, behind the canonical basis of 1-forms, we have also the basis $(dx^i, \omega^\sigma, \omega^\sigma_{j_1}, \ldots, \omega^\sigma_{j_1 \ldots j_{r-1}}, dy^\sigma_{j_1 \ldots j_r})$ adapted to the contact structure, where in place of the dy's one has the contact 1-forms

$$\omega^\sigma = dy^\sigma - y^\sigma_l dx^l, \quad \ldots, \quad \omega^\sigma_{j_1 \ldots j_{r-1}} = dy^\sigma_{j_1 \ldots j_{r-1}} - y^\sigma_{j_1 \ldots j_{r-1} l}\, dx^l.$$

Sections of J^rY which are *integral sections of the contact ideal* are called *holonomic*. Apparently, a section $\delta : U \to J^rY$ is holonomic if and only if $\delta = J^r\gamma$ where $\gamma : U \to Y$ is a section of π.

Notice that every p-form on J^rY, $p > n$, is contact. Let $q > 1$. A *contact* q-form ρ on J^rY is called 1-*contact* if for every π_r-vertical vector field ξ on J^rY the $(q-1)$-form $i_\xi\rho$ is horizontal. Recurrently, ρ is called i-*contact*, $2 \leq i \leq q$, if $i_\xi\rho$ is $(i-1)$-contact. Every q-form on J^rY admits a *unique decomposition*

$$\pi^*_{r+1,r}\rho = h\rho + p_1\rho + p_2\rho + \cdots + p_q\rho,$$

where $p_i\rho$, $1 \leq i \leq q$, is an i-contact form on $J^{r+1}Y$, called the i-*contact part* of ρ. It is helpful to notice that the chart expression of $p_i\rho$ in any fibered chart contains *exactly* i exterior factors $\omega^\sigma_{j_1 \ldots j_l}$ where l is admitted to run from 0 to r. For more details on jet prolongations of fibered manifolds, and the calculus of horizontal and contact forms the reader can consult eg. [18], [19], [24], [25], [29], [33], [34].

Finally, throughout the paper the following notation is used:

$$\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n, \quad \omega_i = i_{\partial/\partial x^i}\,\omega_0, \quad \omega_{ij} = i_{\partial/\partial x^j}\,\omega_i, \quad \text{etc.}$$

3. Hamiltonian systems

In this section we discuss the concept of a Hamiltonian system and of a Lagrangian system as introduced in [30], and the relation between Hamiltonian and Lagrangian systems.

Let $s \geq 0$, and put $n = \dim X$. A *closed* $(n+1)$-form α on J^sY is called a *Lepagean* $(n+1)$*-form* if $p_1\alpha$ is $\pi_{s+1,0}$-horizontal. If α is a Lepagean $(n+1)$-form and $E = p_1\alpha$ we also say that α is a *Lepagean equivalent of* E. By definition, in every fiber chart (V,ψ), $\psi = (x^i, y^\sigma)$, on Y,

$$E = E_\sigma \omega^\sigma \wedge \omega_0,$$

where E_σ are functions on $V_{s+1} \subset J^{s+1}Y$. A Lepagean $(n+1)$-form α on J^sY will be also called a *Hamiltonian system of order* s. A section δ of the fibered manifold π_s will be called a *Hamilton extremal of* α if

$$\delta^* i_\xi \alpha = 0 \quad \text{for every } \pi_s\text{-vertical vector field } \xi \text{ on } J^sY. \tag{3.1}$$

The equations (3.1) will be then called *Hamilton equations.*

Hamiltonian systems are closely related with Lagrangians and Euler–Lagrange forms. The relation follows from the properties of Lepagean n-forms (see eg. [21], [24], [25] for review). Recall that an n-form ρ on J^sY is said to be a *Lepagean* n*-form* if $hi_\xi d\rho = 0$ for every $\pi_{s,0}$-vertical vector field ξ on J^sY [18], [21]. Thus, every Lepagean $(n+1)$-form locally equals to $d\rho$ where ρ is a Lepagean n-form. Consequently, if α is a Lepagean $(n+1)$-form then its 1-contact part E is a *locally variational form.* In other words, there exists an open covering of $J^{s+1}Y$ such that, on each set of this covering, E coincides with the Euler–Lagrange form of a Lagrangian of order $r \leq s$, i.e.,

$$E = \Big(\frac{\partial L}{\partial y^\sigma} - \sum_{l=1}^{r} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial L}{\partial y^\sigma_{p_1 p_2 \dots p_l}}\Big)\omega^\sigma \wedge \omega_0.$$

This suggests the following definition of a *Lagrangian system*: Lepagean $(n+1)$-forms (possibly of different orders) are said to be *equivalent* if their one-contact parts coincide (up to a possible projection). In what follows, we denote the equivalence class of a Lepagean $(n+1)$-form α by $[\alpha]$, and call it a *Lagrangian system.* The minimum of the set of orders of the elements in the class $[\alpha]$ will then be called the *(dynamical) order* of the Lagrangian system $[\alpha]$.

Every Lagrangian system is locally characterized by Lagrangians of all orders starting from a certain minimal one, denoted by r_0, and called the *minimal order* for $[\alpha]$.

The *Euler–Lagrange equations* corresponding to a Lagrangian system $[\alpha]$ of order s now read

$$J^s\gamma^* i_{J^s\xi}\alpha = 0 \quad \text{for every } \pi\text{-vertical vector field } \xi \text{ on } Y, \tag{3.2}$$

where α is any representative of order s of the class $[\alpha]$. Notice that (3.2) are PDE of order $s+1$ for sections γ of the fibered manifold π, their solutions are *extremals* of $E = p_1\alpha$.

Let us stop for a moment to discuss relations between Hamiltonian systems and Lagrangian systems.

Every Hamiltonian system α of order s has a *unique* associated Lagrangian system $[\alpha]$; its order is $r \leq s$, and it is represented by the locally variational form $E = p_1\alpha$ (or, alternatively, by the family of all, generally only local, Lagrangians whose Euler–Lagrange forms locally coincide with E). Comparing the Hamilton equations (3.1) with the Euler–Lagrange equations (3.2) one can see immediately that *the sets of extremals and Hamilton extremals generally are not in one-to-one correspondence*, in other words, *Hamilton equations need not be equivalent with the Euler–Lagrange equations.* However, *the s-jet prolongation of every extremal is a Hamilton extremal*; in this sense there is an *inclusion* of the set of extremals into the set of Hamilton extremals. More precisely, there is a *bijection* between the set of extremals and the set of *holonomic* Hamilton extremals.

On the other hand, a Lagrangian system $[\alpha]$ of order r has *many* associated Hamiltonian systems, each of an order $s \geq r$. Consequently, to a given set of Euler–Lagrange equations one has many sets of Hamilton equations. Behind the given Euler–Lagrange expressions (respectively, a Lagrangian), Hamilton equations depend also upon "free functions" which come from the at least 2-contact part of $\pi^*_{s+1,s}\alpha$. Since α is locally the exterior derivative of a Lepagean n-form, it is convenient to discuss different possibilities on the level of Lepagean equivalents of a corresponding Lagrangian. To this purpose, let us first recall an important result, due to Krupka [25]: ρ is a Lepagean n-form of order s iff in any fibered chart (V,ψ), $\psi = (x^i, y^\sigma)$, on Y,

$$\pi^*_{s+1,s}\,\rho = \theta_\lambda + d\nu + \mu, \tag{3.3}$$

where

$$\theta_\lambda = L\omega_0 + \sum_{k=0}^{r-1}\Big(\sum_{l=0}^{r-k-1}(-1)^l d_{p_1}d_{p_2}\dots d_{p_l}\,\frac{\partial L}{\partial y^\sigma_{j_1\dots j_k p_1\dots p_l i}}\Big)\omega^\sigma_{j_1\dots j_k}\wedge\omega_i, \tag{3.4}$$

ν is an arbitrary contact $(n-1)$-form, and μ is an arbitrary at least 2-contact n-form; in the formula (3.4), r denotes the order of $h\rho$. It should be stressed that the decomposition (3.3) is generally not invariant with respect to transformations of fibered coordinates. θ_λ is called the *local Poincaré–Cartan equivalent* of the Lagrangian $\lambda = h\rho$, each of the (invariant) forms $\Theta = \theta_\lambda + p_1 d\nu$ is then called the *Poincaré–Cartan form.* Now, let α be a Hamiltonian system. If locally $\alpha = d\rho$ where the Lepagean n-form ρ is *at most i-contact* $(1 \leq i \leq n)$ we speak about *Hamilton p_i-theory* and call the corresponding Hamilton equations (3.1) *Hamilton p_i-equations* [30].

In particular, *Hamilton p_1-equations* are based upon the Poincaré–Cartan form Θ. In an usual approach to Hamiltonian field theory only these equations are considered (cf. [1], [3], [7]–[17], [21]–[23], [26], [33], [35]–[37] and references therein); they are often called *Hamilton–De Donder equations.* Obviously, behind a Lagrangian, they depend upon ν. (With the exception of the case of first order Lagrangians when $\nu = 0$, i.e., θ_λ is invariant, and the Hamilton–De Donder equations are unique and completely determined by the Lagrangian).

Hamilton p_2-equations are based upon a Lepagean form $\rho = \Theta + \nu$ where ν is 2-contact; for first order Lagrangians they have been studied in [31], [32], a second order generalization is due to [38]. Hamilton p_n-equations are based upon a general Lepagean n-form; the first oder case was considered by Dedecker [5].

Each of these Hamiltonian systems can be viewed as a *different extension* of the original variational problem. In this way, in any concrete situation, one can utilize a possibility to apply additional (geometric or physical) requirements to choose from many alternative Hamiltonian systems the "best one". A deeper insight into this question is subject of the next sections.

Comments 1. Let us mention main differences between the presented approach and the usual one.

(i) Hamiltonian systems, Hamilton equations. Roughly speaking, there are two main geometric ways approaching Hamiltonian field theory. One, close to the classical calculus of variations, declares the philosophy to assign a *unique* Hamiltonian system to a single *Lagrangian.* This task is represented by the *Hamilton–De Donder theory*, based upon the Poincaré–Cartan form of a (global) Lagrangian λ, which gives "good" results for first order Lagrangians, and is considered problematic in the higher order case (cf. eg. [6]). The second approach is more or less axiomatic, and is based upon the so called *multisymplectic form* (eg., [3] and references therein). (Recall that an $(n+1)$-form Ω on a manifold M is called *multisymplectic* if it is closed, and the "musical map" $\xi \to i_\xi\Omega$, mapping vector fields on M to n-forms, is injective.)

Our approach is close to both of them, but different. This can be seen immediately if the definitions of the multisymplectic and Lepagean $(n+1)$-form are compared. For the "zero order" case one gets that *every multisymplectic form on Y is a Lepagean $(n+1)$-form, however, a Lepagean $(n+1)$-form need not be multisymplectic.* For higher order the difference becomes even sharper, since *a multisymplectic form on J^sY, $s \geq 1$, need not be Lepagean.* Apparently, our motivation was to define a Hamiltonian system to be, contrary to the multisymplectic definition, *sufficiently general* on the one hand (covering *all* Lagrangians without any a priori restriction), and, on the other hand, *directly related* with a variational problem (defined by a *locally variational form*). Among others, this also means that our *Hamiltonian system is assigned not to a particular Lagrangian* (as always done), *but to the whole class of all equivalent Lagrangians* corresponding to a given *Euler–Lagrange form.* Other differences are connected with the concepts of *regularity* and *Legendre transformation*, and will be discussed in the next sections.

(ii) Lagrangian systems. One should notice that also in the definition of a *Lagrangian system* and of the *order* of a Lagrangian system we differ from other authors. While usually by a "Lagrangian system of order r" one means a *global Lagrangian on J^rY*, by our definition a Lagrangian system is the *equivalence class of Lagrangians* giving rise to an Euler–Lagrange form; the order of a Lagrangian system is then, as a property of the whole equivalence class, determined via the order of the Euler–Lagrange form. In this way, only properties directly connected with dynamics, hence common to all the equivalent Lagrangians, enter in this definition, while distinct properties of particular Lagrangians which are not essential for the dynamics are eliminated (the latter are namely connected with the fact that the family of equivalent Lagrangians contains Lagrangians of all orders starting from a minimal one, which, as functions, may look completely different from each other, and whose domains usually are open subsets of the corresponding jet prolongations of the underlying fibered manifold; a global Lagrangian often even does not exist at all, obstructions lie in the topology of Y). In the sense of our definition, for example, the *Dirac field* is a Lagrangian system of order *zero* (and not of order one); indeed, in this case the class $[\alpha]$ is represented by the global form $d\theta_\lambda$ projectable onto Y, since the corresponding Lagrangian λ is a global first order Lagrangian affine in the first order derivatives. Similarly, the *Einstein gravitational field*, which is usually defined by the *scalar curvature Lagrangian* (global second order Lagrangian), is a Lagrangian system of order *one*, since the corresponding Poincaré–Cartan form is projectable onto J^1Y.

4. Regular Hamiltonian systems

A section δ of π_s is called a *Dedecker's section* [30] if $\delta^*\mu = 0$ for every at least 2-contact form μ on J^sY.

Consider a Hamiltonian system α on J^sY. Denote $E = p_1\alpha$, $F = p_2\alpha$, $G = \pi^*_{s+1,s}\alpha - E - F$ (i.e., G is the at least 3-contact part of $\pi^*_{s+1,s}\alpha$), and

$$\hat{\alpha} = E + F. \tag{4.1}$$

We shall call the form $\hat{\alpha}$ the *principal part of α*.

A Dedecker's section which is a Hamilton extremal of $\pi^*_{s+1,s}\alpha$ will be called *Dedecker–Hamilton extremal* of α.

It is easy to obtain the following relation between the sets of extremals, Hamilton extremals, and Dedecker–Hamilton extremals of a Hamiltonian system α on J^sY [30]:

If γ is an extremal of $E = p_1\alpha$ then for every Lepagean equivalent α of E, α defined on J^sY ($s \geq 0$), the section $\delta = J^s\gamma$ is a Hamilton extremal of α, and $\hat{\delta} = J^{s+1}\gamma$ is its Dedecker–Hamilton extremal.

For every $\alpha \in [\alpha]$, defined on J^sY ($s \geq 0$), and for every its Dedecker–Hamilton extremal $\hat{\delta}$, the section $\delta = \pi_{s+1,s} \circ \hat{\delta}$ is a Hamilton extremal of α.

Denote by $\mathcal{D}^s_\alpha$ and $\mathcal{D}^{s+1}_{\hat{\alpha}}$ the family of n-forms $i_\xi\alpha$ and $i_\xi\hat{\alpha}$ respectively, where ξ runs over all π_s-vertical vector fields on J^sY, respectively, over all π_{s+1}-vertical vector fields on $J^{s+1}Y$. Notice that the rank of $\mathcal{D}^{s+1}_{\hat\alpha}$ is never maximal, since $i_{\partial/\partial y^\nu_P}\hat\alpha = 0$ for all multiindices P of the lenght $s+1$.

Apparently, Hamilton extremals and Dedecker–Hamilton extremals of α are integral sections of the ideal generated by the family $\mathcal{D}^s_\alpha$ and $\mathcal{D}^{s+1}_{\hat\alpha}$, respectively. Hence, "regularity" can be understood to be a *property of the ideal* $\mathcal{D}^{s+1}_{\hat\alpha}$ as follows [30]. For convenience, let us consider the cases $s=0$ and $s>0$ separately.

We call a Hamiltonian system α on Y *regular* if $\operatorname{rank}\mathcal{D}^1_{\hat\alpha}$ is constant and equal to $\operatorname{rank} V\pi = m$. Let $s \geq 1$ and $r \geq 1$. A Hamiltonian system α on J^sY will be called *regular of degree* r if the system of local generators of $\mathcal{D}^{s+1}_{\hat\alpha}$ contains all the n-forms

$$\omega^\sigma \wedge \omega_i, \quad \ldots, \quad \omega^\sigma_{j_1\ldots j_{r-1}} \wedge \omega_i, \tag{4.2}$$

and $\operatorname{rank}\mathcal{D}^{s+1}_{\hat\alpha} = n \operatorname{rank} V\pi_{r-1} + \operatorname{rank} V\pi_{s-r}$.

We refer to (4.2) as local *canonical* 1-*contact* n-*forms* of order r.

Roughly speaking, regularity of degree r means that the system $\mathcal{D}^{s+1}_{\hat\alpha}$ contains all the canonical contact n-forms on J^rY, and the rank of the "remaining" subsystem of $\mathcal{D}^{s+1}_{\hat\alpha}$ is the greatest possible one.

Notice that by this definition, every Dedecker–Hamilton extremal of a regular Hamiltonian system with degree of regularity r is *holonomic up to the order* r, i.e.,

$$\pi_{s+1,r} \circ \hat\delta = J^r(\pi_{s+1,0} \circ \hat\delta).$$

Moreover, we have the following main theorem (for the proof we refer to [30]).

Theorem 4.1. [30] *Let α be a Hamiltonian system on J^sY, r_0 the minimal order for $E = p_1\alpha$. Suppose that α is regular of degree r_0. Then every Dedecker–Hamilton extremal $\hat\delta$ of α is of the form*

$$\pi_{s+1,r_0} \circ \hat\delta = J^{r_0}\gamma$$

where γ is an extremal of E.

Taking into account this result, a Hamiltonian system of order $s \geq 1$ which is regular of degree r_0 will be called simply *regular*. Thus, *regular* Hamilton equations and the corresponding Euler–Lagrange equations are *almost equivalent* in the sense that *extremals are in bijective correspondence with classes of Dedecker–Hamilton extremals*, $\gamma \to [J^{s+1}\gamma]$, where $\hat\delta \in [J^{s+1}\gamma]$ iff $\pi_{s+1,r_0} \circ \hat\delta = J^{r_0}\gamma$.

We shall call a regular Hamiltonian system *strongly regular* if the Hamilton and Euler–Lagrange equations are *equivalent* in the sense that *extremals are in bijective correspondence with classes of Hamilton extremals*, $\gamma \to [J^s\gamma]$, where $\delta \in [J^s\gamma]$ iff $\pi_{s,r_0} \circ \hat\delta = J^{r_0}\gamma$. (Clearly, for $s=1$ this precisely means a bijective correspondence between extremals and Hamilton extremals).

The concept of regularity of a *Lagrangian system* is now at hand: regularity can be viewed as the property that *there exists* at least one associated Hamiltonian system which is regular; obviously, the order of this Hamiltonian system may differ from the order of the Lagrangian system. Hence, in accordance with [30], we call a Lagrangian system $[\alpha]$ *regular* if the family of associated Hamiltonian systems contains a regular Hamiltonian system. Similarly, we call a Lagrangian system *strongly regular* if the family of associated Hamiltonian systems contains a strongly regular Hamiltonian system.

5. Regularity conditions

The above geometric definition of regularity enables one to find explicit *regularity conditions*. Keeping notations introduced so far, we write $\hat{\alpha} = E + F$, where $E = E_\sigma \omega^\sigma \wedge \omega_0$, and

$$F = \sum_{|J|,|P|=0}^{s} F^{J,P,i}_{\sigma\nu}\, \omega^\sigma_J \wedge \omega^\nu_P \wedge \omega_i, \quad F^{J,P,i}_{\sigma\nu} = -F^{P,J,i}_{\nu\sigma}; \tag{5.1}$$

here J, P are multiindices of the lenght k and l, respectively, $J = (j_1 j_2 \dots j_k)$, $P = (p_1 p_2 \dots p_l)$, where $0 \le |J|, |P| \le s$, i.e., $0 \le k, l \le s$, and, as usual, $1 \le i \le n$, $1 \le \sigma, \nu \le m$. Since $d\alpha = 0$, E is an *Euler–Lagrange form* of order $s+1$, i.e., the functions E_σ satisfy the identities

$$\frac{\partial E_\sigma}{\partial y^\nu_{p_1 \dots p_l}} - \sum_{k=l}^{s+1} (-1)^k \binom{k}{l} d_{p_{l+1}} d_{p_{l+2}} \dots d_{p_k} \frac{\partial E_\nu}{\partial y^\sigma_{p_1 \dots p_l}} = 0, \quad 0 \le l \le s+1, \tag{5.2}$$

called *Anderson–Duchamp–Krupka conditions* for local variationality of E [2], [20]. The condition $d\alpha = 0$ means that, in particular,

$$p_2 d\alpha = p_2 dE + p_2 dF = 0. \tag{5.3}$$

In fibered coordinates this equation is equivalent with the following set of identities

$$\frac{\partial E_\sigma}{\partial y^\nu} - \frac{\partial E_\nu}{\partial y^\sigma} - d_i F^{\,,i}_{\sigma\nu} = 0, \tag{5.4}$$

$$\begin{aligned} (F^{0,S,i}_{\sigma\nu})_{\mathrm{sym}(Si)} &= \frac{1}{2} \frac{\partial E_\sigma}{\partial y^\nu_{Si}}, \\ (F^{0,P,i}_{\sigma\nu})_{\mathrm{sym}(Pi)} &= \frac{1}{2} \frac{\partial E_\sigma}{\partial y^\nu_{Pi}} - d_j F^{0,Pi,j}_{\sigma\nu}, \quad 0 \le |P| \le s-1, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} (F^{J,S,i}_{\sigma\nu})_{\mathrm{sym}(Si)} &= 0, \quad 1 \le |J| \le s, \\ (F^{Jj,P,p}_{\sigma\nu})_{\mathrm{sym}(Pp)} + (F^{J,Pp,j}_{\sigma\nu})_{\mathrm{sym}(Jj)} + d_i F^{Jj,Pp,i}_{\sigma\nu} &= 0, \quad 0 \le |J|, |P| \le s-1, \end{aligned} \tag{5.6}$$

where sym means symmetrization in the indicated indices, and $S = (p_1 p_2 \dots p_s)$. Denote

$$f^{J,P,i}_{\sigma\nu} = F^{J,P,i}_{\sigma\nu} - (F^{J,P,i}_{\sigma\nu})_{\mathrm{sym}(Pi)}. \tag{5.7}$$

Then from (5.5) we easily get

$$(F_{\sigma\nu}^{,p_1\dots p_{l-1},p_l})_{\mathrm{sym}(p_1\dots p_{l-1}p_l)} = \frac{1}{2}\sum_{k=0}^{s+1-l}(-1)^k d_{i_1}d_{i_2}\dots d_{i_k}\frac{\partial E_\sigma}{\partial y^\nu_{p_1\dots p_l i_1 i_2\dots i_k}} - d_i f_{\sigma\nu}^{,p_1\dots p_l,i}, \quad 1 \le l \le s+1, \tag{5.8}$$

and the recurrent formulas (5.6) give us the remaining F's expressed by means of the $(F_{\sigma\nu}^{0,P,i})_{\mathrm{sym}(Pi)}$'s and the above f's. As a result, one gets F (5.1) *determined by the Euler–Lagrange expressions E_σ and the (free) functions $f_{\sigma\nu}^{J,P,i}$*. Moreover, (5.4), and the antisymmetry conditions for the $F_{\sigma\nu}^{J,P,i}$'s, lead to the identities (5.2), as expected.

Now, we are prepared to find explicit regularity conditions for α. By definition, $\mathcal{D}_{\hat\alpha}^{s+1}$ is locally spanned by the following n-forms:

$$\begin{aligned} \eta_\nu^P &= -i_{\partial/\partial y^\nu_P}\hat\alpha = \sum_{|J|=0}^{s} 2F_{\sigma\nu}^{J,P,i}\,\omega_J^\sigma \wedge \omega_i, \quad 1 \le |P| \le s, \\ \eta_\nu &= -i_{\partial/\partial y^\nu}\hat\alpha = -E_\nu\omega_0 + \sum_{|J|=0}^{s} 2F_{\sigma\nu}^{J,0,i}\,\omega_J^\sigma \wedge \omega_i. \end{aligned} \tag{5.9}$$

One can see from (5.6) that *the functions $F_{\sigma\nu}^{J,P,i}$ where $|J|+|P| \ge s+1$, depend only upon the f's (5.7).* The (invariant) choice

$$f_{\sigma\nu}^{J,P,i} = 0, \quad |J|+|P| \ge s+1 \tag{5.10}$$

then leads to

$$F_{\sigma\nu}^{J,P,i} = 0, \quad |J|+|P| \ge s+1, \tag{5.11}$$

and we obtain

$$\begin{aligned} \eta_\nu^S &= 2F_{\sigma\nu}^{0,S,i}\,\omega^\sigma \wedge \omega_i, \\ \eta_\nu^P &= 2F_{\sigma\nu}^{0,P,i}\,\omega^\sigma \wedge \omega_i + 2F_{\sigma\nu}^{j_1,P,i}\,\omega_{j_1}^\sigma \wedge \omega_i, \quad |P| = s-1, \\ &\dots \\ \eta_\nu^P &= \sum_{|J|=0}^{r_0-2} 2F_{\sigma\nu}^{J,P,i}\,\omega_J^\sigma \wedge \omega_i + 2F_{\sigma\nu}^{j_1\dots j_{r_0-1},P,i}\,\omega_{j_1\dots j_{r_0-1}}^\sigma \wedge \omega_i, \quad |P| = s-r_0+1, \\ &\dots \\ \eta_\nu^{p_1} &= \sum_{|J|=0}^{s} 2F_{\sigma\nu}^{J,p_1,i}\,\omega_J^\sigma \wedge \omega_i, \\ \eta_\nu &= -E_\nu\omega_0 + \sum_{|J|=0}^{s} 2F_{\sigma\nu}^{J,0,i}\,\omega_J^\sigma \wedge \omega_i. \end{aligned} \tag{5.12}$$

where

$$
\begin{aligned}
F^{Jj,P,i}_{\sigma\nu} &= (-1)^{|J|+1}(F^{0,JjP,i}_{\sigma\nu})_{\mathrm{sym}(JjPi)} - (f^{J,Pi,j}_{\sigma\nu})_{\mathrm{sym}(Jj)} + f^{Jj,P,i}_{\sigma\nu},\\
&= (-1)^{|J|+1}\,\frac{1}{2}\,\frac{\partial E_\sigma}{\partial y^\nu_{JjPi}} - (f^{J,Pi,j}_{\sigma\nu})_{\mathrm{sym}(Jj)} + f^{Jj,P,i}_{\sigma\nu}, \quad |J|+|P| = s-1.
\end{aligned}
\tag{5.13}
$$

The results can be summarized as follows.

Theorem 5.1. [30] *Let α be Hamiltonian system of order s, let r_0 denote the minimal order of the corresponding Lagrangians. Suppose that*

$$f^{J,P,i}_{\sigma\nu} = 0, \quad |J|+|P| \geq s+1, \tag{5.14}$$

and

$$
\begin{aligned}
\operatorname{rank}(F^{p_1\dots p_s,0,i}_{\nu\sigma}) &= mn,\\
\operatorname{rank}(F^{p_1\dots p_{s-1},j_1,i}_{\nu\sigma}) &= mn^2,\\
&\dots\\
\operatorname{rank}(F^{p_1\dots p_{s-r_0+1},j_1\dots j_{r_0-1},i}_{\nu\sigma}) &= nm\binom{n+r_0-2}{r_0-1},\\
\operatorname{rank}\begin{pmatrix} 0 & F^{P,J,i}_{\nu\sigma}\\ -E_\nu & 2F^{0,J,i}_{\nu\sigma}\end{pmatrix} &= \mathit{maximal}, \quad 0\leq |J|\leq s, \quad 1\leq |P| \leq s-r_0,
\end{aligned}
\tag{5.15}
$$

where in the above matrices, the (ν, P) label rows, and the (σ, J, i) label columns. Then α is regular.

For the most frequent cases of second and first order locally variational forms this result is reduced to the following:

Corollary 5.1. [30] *Let $s = 1$. The following are necessary conditions for α be regular:*

$$r_0 = 1, \tag{5.16}$$

$$f^{j,p,i}_{\sigma\nu} = 0, \quad f^{i,0,j}_{\sigma\nu} = f^{j,0,i}_{\nu\sigma}, \tag{5.17}$$

$$\det\Big(\frac{\partial E_\sigma}{\partial y^\nu_{ij}} - 2f^{i,0,j}_{\sigma\nu}\Big) \neq 0, \tag{5.18}$$

where in the indicated $(mn\times mn)$-matrix, (ν, j) label the rows and (σ, i) the columns.
Then

$$
\begin{aligned}
\hat\alpha = E_\sigma\omega^\sigma\wedge\omega_0 &+ \Big(\frac{1}{4}\Big(\frac{\partial E_\sigma}{\partial y^\nu_i} - \frac{\partial E_\nu}{\partial y^\sigma_i}\Big) - d_j f^{i,0,j}_{\sigma\nu}\Big)\omega^\sigma\wedge\omega^\nu\wedge\omega_i\\
&+ \Big(\frac{\partial E_\sigma}{\partial y^\nu_{ij}} - 2f^{i,0,j}_{\sigma\nu}\Big)\omega^\sigma\wedge\omega^\nu_j\wedge\omega_i.
\end{aligned}
\tag{5.19}
$$

In terms of a first order Lagrangian $\lambda = L\omega_0$ the regularity conditions (5.17) and (5.18) read

$$\det\Big(\frac{\partial^2 L}{\partial y^\sigma_j \partial y^\nu_k} - g^{ij}_{\sigma\nu}\Big) \neq 0, \tag{5.20}$$

where

$$g^{ij}_{\sigma\nu} = -g^{ji}_{\sigma\nu} = -g^{ij}_{\nu\sigma}, \tag{5.21}$$

and (in the notations of (3.3)) it holds

$$\hat{\alpha} = d\theta_\lambda + p_2 d\mu, \quad p_2\mu = \frac{1}{4}\, g^{ij}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}. \tag{5.22}$$

If, in particular, E is projectable onto J^1Y, i.e.,

$$\frac{\partial E_\sigma}{\partial y^\nu_{ij}} = 0 \tag{5.23}$$

for all (σ, ν, i, j), we can consider for E either a first order Hamiltonian system (s=1), or a zero order Hamiltonian system (s=0) (the latter follows from the fact that variationality conditions imply that E_σ are polynomials of order n in the first derivatives). Taking into account the above corollary for $s = 1$, and the definition of regularity for $s = 0$, we obtain regularity conditions for *first order* locally variational forms as follows.

Corollary 5.2. [30] *Every locally variational form E on J^1Y is regular.*

(1) *Let $W \subset J^1Y$ be an open set, and $g^{ij}_{\sigma\nu}$ be functions on W such that*

$$-g^{ij}_{\sigma\nu} = g^{ji}_{\sigma\nu} = g^{ij}_{\nu\sigma}, \quad \det(g^{ij}_{\sigma\nu}) \neq 0. \tag{5.24}$$

Then every closed $(n+1)$-form α on W such that

$$\hat{\alpha} = E_\sigma\omega^\sigma \wedge \omega_0 + \frac{1}{2}\Big(\frac{\partial E_\sigma}{\partial y^\nu_i} + d_j g^{ji}_{\sigma\nu}\Big)\omega^\sigma \wedge \omega^\nu \wedge \omega_i + g^{ji}_{\sigma\nu}\omega^\sigma \wedge \omega^\nu_j \wedge \omega_i \tag{5.25}$$

is a regular first order Hamiltonian system related to E.

(2) *Suppose that*

$$\operatorname{rank}\Big(\frac{\partial E_\sigma}{\partial y^\nu_i}\Big) = \operatorname{rank}\Big(\frac{\partial^2 L}{\partial y^\sigma \partial y^\nu_i} - \frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu}\Big) = m, \tag{5.26}$$

where in the indicated $(m \times mn)$-matrices, (σ) label the rows and (ν, i) the columns, and $\lambda = L\omega_0$ is any (local) first order Lagrangian for E. Then there exists a unique regular zero order Hamiltonian system related to E; it is given by the $(n+1)$-form α on Y which in every fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$ is expressed as follows:

$$\begin{aligned} \pi^*_{1,0}\alpha &= E_\sigma\omega^\sigma \wedge \omega_0 \\ &+ \sum_{k=1}^{n} \frac{1}{(k+1)!}\, \frac{\partial^k E_\sigma}{\partial y^{\nu_1}_{i_1} \dots \partial y^{\nu_k}_{i_k}}\, \omega^\sigma \wedge \omega^{\nu_1} \wedge \dots \wedge \omega^{\nu_k} \wedge \omega_{i_1 \dots i_k}. \end{aligned} \tag{5.27}$$

6. Legendre transformation

Let α be a *regular* Hamiltonian system of order $s \geq 1$. Then all the canonical 1-contact n-forms

$$\omega^\sigma \wedge \omega_i, \quad \dots, \quad \omega^\sigma_{j_1 \dots j_{r_0-1}} \wedge \omega_i, \tag{6.1}$$

where r_0 is the minimal order for the locally variational form $E = p_1 \alpha$, belong to the exterior differential system generated by $\mathcal{D}^{s+1}_{\hat{\alpha}}$. However, the generators of $\mathcal{D}^{s+1}_{\hat{\alpha}}$ naturally associated with the fibered coordinates (i.e., (5.9)), are of the form of *linear combinations* of (6.1), and, moreover, for $s > 1$, *nonzero generators are not linearly independent.* In this sense the generators associated with fibered coordinates are *not canonical.* In what follows we shall discuss the existence of coordinates on J^sY, *canonically* adapted to the system $\mathcal{D}^{s+1}_{\hat{\alpha}}$, i.e., such that a part of the naturally associated generators *coincides* with the forms (6.1), and all *superfluous generators vanish.* More precisely, we shall be interested in the existence of *local charts*, (W, χ), $\chi = (x^i, y^\sigma_J, p^{J,i}_\sigma, z^L)$, on J^sY such that (x^i, y^σ_J) are local fibered coordinates on $J^{r_0-1}Y$, and the generators of $\mathcal{D}^{s+1}_{\hat{\alpha}}$ naturally associated with the coordinates $p^{J,i}_\sigma$ coincide with the n-forms (6.1), and those associated with z^L vanish. Hence,

$$\begin{aligned} &i_{\partial/\partial z^L}\, \hat{\alpha} = 0, \quad \forall L, \\ &i_{\partial/\partial p^{J,i}_\sigma}\, \hat{\alpha} = \omega^\sigma_J \wedge \omega_i, \quad 0 \leq |J| \leq r_0 - 1. \end{aligned} \tag{6.2}$$

Consequently, the *nonzero generators are linarly independent.* We shall call such "canonical" cordinates on J^sY *Legendre coordinates* associated with the regular Hamiltonian system α.

In the following theorem we adopt the notations of (3.3) and (3.4).

Theorem 6.1. [30] *Let α be a regular Hamiltonian system on J^sY, let $x \in J^sY$ be a point. Suppose that in a neighborhood W of x*

$$\alpha = d\rho, \qquad \rho = \theta_\lambda + d\nu + \mu, \tag{6.3}$$

where λ is a Lagrangian of the minimal order, r_0, for $E = p_1\alpha$, defined on $\pi_{s,r_0}(W)$, and μ is such that

$$p_2\mu = \sum_{|J|,|K|=0}^{r_0-1} g^{J,K,i_1 i_2}_{\sigma\nu}\, \omega^\sigma_J \wedge \omega^\nu_K \wedge \omega_{i_1 i_2}, \tag{6.4}$$

where $g^{J,K,i_1 i_2}_{\sigma\nu}$ are functions on $\pi_{s,r_0-1}(W)$, satisfying the antisymmetry conditions $g^{J,K,i_1 i_2}_{\sigma\nu} = -g^{J,K,i_2 i_1}_{\sigma\nu} = -g^{K,J,i_1 i_2}_{\nu\sigma}$. Put

$$p^{J,i}_\sigma = \sum_{l=0}^{r_0-|J|-1} (-1)^l d_{p_1} d_{p_2} \dots d_{p_l} \frac{\partial L}{\partial y^\sigma_{J p_1 \dots p_l i}} + 4 g^{J,K,il}_{\sigma\nu} y^\nu_{Kl}, \quad 0 \leq |J| \leq r_0 - 1. \tag{6.5}$$

Then for any suitable functions z^L on W, (W, χ), where $\chi = (x^i, y^\sigma_J, p^{J,i}_\sigma, z^L)$, is a Legendre chart for α.

Using (6.5) we can write

$$\rho = -H\omega_0 + \sum_{|J|=0}^{r_0-1} p_\sigma^{J,i} dy_J^\sigma \wedge \omega_i + \eta + d\nu + \mu_3, \tag{6.6}$$

where

$$\eta = \sum_{|J|,|K|=0}^{r_0-1} g_{\sigma\nu}^{J,K,i_1 i_2}\, dy_J^\sigma \wedge dy_K^\nu \wedge \omega_{i_1 i_2}, \tag{6.7}$$

μ_3 is at least 3-contact, and

$$H = -L + p_\sigma^{J,i} y_{Ji}^\sigma + 2g_{\sigma\nu}^{J,K,i_1 i_2} y_{Ji_1}^\sigma y_{Ki_2}^\nu. \tag{6.8}$$

We call the functions H (6.8) and $p_\sigma^{J,i}$ (6.5) a *Hamiltonian* and *momenta* of α. Now,

$$\hat{\alpha} = -dH \wedge \omega_0 + \sum_{|J|=0}^{r_0-1} dp_\sigma^{J,i} \wedge dy_J^\sigma \wedge \omega_i + d\eta - p_3 d\eta, \tag{6.9}$$

and computing $i_{\partial/\partial z^L}\,\hat{\alpha}$, $i_{\partial/\partial p_\sigma^{J,i}}\,\hat{\alpha}$, and $i_{\partial/\partial y_J^\sigma}\,\hat{\alpha}$ we get that $\mathcal{D}_{\hat{\alpha}}^{s+1}$ is spanned by the following nonzero n-forms:

$$\begin{aligned} &-\frac{\partial H}{\partial p_\sigma^{J,i}}\omega_0 + dy_J^\sigma \wedge \omega_i = \omega_J^\sigma \wedge \omega_i, \\ &\Big(\frac{\partial H}{\partial y_J^\sigma} - 4\frac{\partial g_{\sigma\nu}^{JK,i_1 i_2}}{\partial x^{i_2}} y_{Ki_1}^\nu - 2\Big(\frac{\partial g_{\nu\rho}^{KP,i_1 i_2}}{\partial y_J^\sigma} + \frac{\partial g_{\sigma\nu}^{JK,i_1 i_2}}{\partial y_P^\rho} + \frac{\partial g_{\rho\sigma}^{PJ,i_1 i_2}}{\partial y_K^\nu}\Big) y_{Ki_1}^\nu y_{Pi_2}^\rho\Big)\omega_0 \\ &+ dp_\sigma^{J,i} \wedge \omega_i, \end{aligned} \tag{6.10}$$

where $0 \le |J| \le r_0 - 1$. Notice that if $d\eta = 0$ we get $\mathcal{D}_{\hat{\alpha}}^{s+1}$ spanned by

$$\omega_J^\sigma \wedge \omega_i, \quad \frac{\partial H}{\partial y_J^\sigma}\,\omega_0 + dp_\sigma^{J,i} \wedge \omega_i, \quad 0 \le |J| \le r_0 - 1. \tag{6.11}$$

Hamilton equations in Legendre coordinates thus take the following form.

Theorem 6.2. [30]

(1) *A section $\delta : U \to W$ is a Dedecker–Hamilton extremal of α (6.3), (6.6) if, along δ,*

$$\begin{aligned} \frac{\partial y_J^\sigma}{\partial x^i} &= \frac{\partial H}{\partial p_\sigma^{J,i}}, \\ \frac{\partial p_\sigma^{J,i}}{\partial x^i} &= -\frac{\partial H}{\partial y_J^\sigma} + 4\frac{\partial g_{\sigma\nu}^{JK,i_1 i_2}}{\partial x^{i_2}} \frac{\partial H}{\partial p_\nu^{K,i_1}} \\ &\quad + 2\Big(\frac{\partial g_{\nu\rho}^{KP,i_1 i_2}}{\partial y_J^\sigma} + \frac{\partial g_{\sigma\nu}^{JK,i_1 i_2}}{\partial y_P^\rho} + \frac{\partial g_{\rho\sigma}^{PJ,i_1 i_2}}{\partial y_K^\nu}\Big) \frac{\partial H}{\partial p_\nu^{K,i_1}} \frac{\partial H}{\partial p_\rho^{P,i_2}}. \end{aligned} \tag{6.12}$$

If, in particular, $d\eta = 0$, (6.12) take the "classical" form

$$\frac{\partial y^\sigma_J}{\partial x^i} = \frac{\partial H}{\partial p^{J,i}_\sigma}, \quad \frac{\partial p^{J,i}_\sigma}{\partial x^i} = -\frac{\partial H}{\partial y^\sigma_J}, \quad 0 \leq |J| \leq r_0 - 1. \tag{6.13}$$

(2) *If μ_3 is π_{s,r_0-1}-projectable then (6.12) (resp. (6.13)) are equations for Hamilton extremals of α.*

As a consequence of (2) we obtain that *extremals are in bijective correspondence with classes of Hamilton extremals* (with the equivalence in the sense of Sec. 4, i.e., that δ_1 is equivalent with δ_2 iff $\pi_{s,r_0} \circ \delta_1 = \pi_{s,r_0} \circ \delta_2$). In other words,

Corollary 6.1. [30] Hamiltonian systems satisfying the condition (2) of Theorem 6.2 are strongly regular.

The above result shows another geometrical meaning of Legendre transformation: Hamiltonian systems which are *regular and admit Legendre transformation* according to Theorem 6.1 either are strongly regular, or can be easily brought to a stongly regular form (by modifying the term μ_3).

Comments 2. Let us compare our approach to regularity and Legendre transformation with other authors.

(i) Standard Hamilton–DeDonder theory. In the usual formulation of Hamiltonian field theory *Legendre transformation* is a map associated with a *Lagrangian, defined* by the following formulas:

$$p^i_\sigma = \frac{\partial L}{\partial y^\sigma_i} \quad \text{if} \quad L = L(x^i, y^\nu, y^\nu_j), \tag{6.14}$$

and

$$p^{j_1\ldots j_k i}_\sigma = \sum_{l=0}^{r-k-1} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \frac{\partial L}{\partial y^\sigma_{j_1\ldots j_k p_1 \ldots p_l i}}, \quad 0 \leq k \leq r-1, \tag{6.15}$$

for a Lagrangian of order $r \geq 2$ ([8], [7], [16], [33], [35], [37], etc.). These formulas have their origin in the (noninvariant) decomposition of the Poincaré–Cartan form θ_λ (3.4) in the canonical basis (dx^i, dy^σ_J), $0 \leq |J| \leq r-1$, i.e.,

$$\theta_\lambda = (L - \sum_{|J|=0}^{r-1} p^{Ji}_\sigma y^\sigma_{Ji})\omega_0 + \sum_{|J|=0}^{r-1} p^{Ji}_\sigma dy^\sigma_J \wedge \omega_i. \tag{6.16}$$

However, for global Lagrangians of order $r \geq 2$ the form (3.4) is neither unique nor globally defined. It is replaced by $\Theta = \theta_\lambda + p_1 d\nu$ (cf. notations of (3.3)), and, consequently, (6.15) are replaced by more general formulas

$$p^{j_1\ldots j_k i}_\sigma = \sum_{l=0}^{r-k-1} (-1)^l d_{p_1} d_{p_2} \ldots d_{p_l} \Big(\frac{\partial L}{\partial y^\sigma_{j_1\ldots j_k p_1 \ldots p_l i}} + c^{j_1\ldots j_k p_1 \ldots p_l i}_\sigma \Big), \tag{6.17}$$

$0 \leq k \leq r-1$, where $c^{j_1\ldots j_k p_1\ldots p_l i}_\sigma$ are auxiliary (free) functions (Krupka [23], Gotay [16]). In the Hamilton–De Donder theory, a *Lagrangian* is *called regular* if the Legendre map defined by (6.15), resp. (6.17) is regular. In the case of a first order Lagrangian this means that λ satisfies the condition

$$\det\Big(\frac{\partial^2 L}{\partial q^\sigma_j \partial q^\nu_k}\Big) \neq 0 \tag{6.18}$$

at each point of J^1Y. Since for higher order Lagrangians the Legendre transformation (6.17) depends upon $p_1 d\nu$, one could expect that the corresponding regularity condition conserves this property. Surprizingly enough, it has been proved in [23] and [16] that the regularity conditions *do not depend upon the functions* $c_\sigma^{j_1\dots j_k p_1 \dots p_l i}$, and are of the form [37], [23], [16]

$$\operatorname{rank}\left(\frac{1}{[j_1\dots j_{2r-s}(p_{r+1}\dots p_s]\,[p_1\dots p_r)]}\;\frac{\partial^2 L}{\partial y^\sigma_{j_1\dots j_{2r-s}(p_{r+1}\dots p_s}\partial y^\nu_{p_1\dots p_r)}}\right) = \max \tag{6.19}$$

where $[j_1\dots j_{2r-s}p_{r+1}\dots p_s]$ and $[p_1\dots p_r]$ denotes the number of all different sequences arising by permuting the sequence $j_1\dots j_{2r-s}p_{r+1}\dots p_s$ and $p_1\dots p_r$, respectively; as usual, $r \le k \le 2r-1$, and, in the indicated matrices, $\sigma, j_1 \le \dots \le j_{2r-k}$ label columns and $\nu, p_1 \le \dots \le p_k$ label rows, and the bracket denotes symmetrization in the corresponding indices. (Notice that within the present approach, by (6.3), the nondependence of regularity conditions upon the c's is trivial). As a result one obtains that if a Lagrangian satisfies (6.18) (respectively, (6.19)) then every solution δ of the *Hamilton–De Donder equations* $\delta^* i_\xi d\theta_\lambda = 0$ (respectively, $\delta^* i_\xi d\Theta = 0$), is of the form $\delta = J^1\gamma$ (respectively, $\pi_{2r-1,r}\circ\delta = J^r\gamma$) where γ is an extremal of λ. However, while in the Legendre coordinates defined by (6.15) the local Hamilton–De Donder equations, i.e., $\delta^* i_\xi d\theta_\lambda = 0$, take the familiar "canonical" form,

$$\frac{\partial y^\sigma_J}{\partial x^i} = \frac{\partial H}{\partial p^{J,i}_\sigma}, \quad \frac{\partial p^{J,i}_\sigma}{\partial x^i} = -\frac{\partial H}{\partial y^\sigma_J},$$

for the global Hamilton–De Donder equations, i.e., $\delta^* i_\xi d\Theta = 0$, using the "Legendre transformation" defined by (6.17), one does not generally obtain a similar "canonical" representation.

(ii) A generalization of the concepts of regularity and Legendre transformation within the Hamilton–De Donder theory. In the paper [26] second order Lagrangians, *affine in the second derivatives, and admitting first order Poincaré–Cartan forms* were studied. Notice that in the sense of the regularity condition (6.19), Lagrangians of this kind are apparently *singular*. In [26], the definition of a regular Lagrangian is extended in the following way: a Lagrangian is called *regular* if the solutions of the Euler–Lagrange and Hamilton–De Donder equations are *equivalent* (in the sense that the sets of solutions are in bijective correspondence). The following results were proved:

Theorem 6.3. [26] *Consider a Lagrangian of the form $\lambda = L\omega_0$ where, in fibered coordinates, L admits an (obviously invariant) decomposition*

$$L = L_0(x^i, y^\sigma, y^\sigma_j) + h^{pq}_\nu(x^i, y^\sigma)\, y^\nu_{pq}. \tag{6.20}$$

Then θ_λ is projectable onto J^1Y, and, consequently, Hamilton–De Donder equations are equations for sections $\delta : U \to J^1Y$. If the condition

$$\det\Big(\frac{\partial^2 L_0}{\partial y_i^\sigma \partial y_k^\rho} - \frac{\partial h_\sigma^{ik}}{\partial y^\rho} - \frac{\partial h_\rho^{ik}}{\partial y^\sigma}\Big) \neq 0 \tag{6.21}$$

is satisfied then λ is regular, i.e., the Euler–Lagrange and the Hamilton–De Donder equations of λ are equivalent, and the mapping

$$(x^i, y^\sigma, y_j^\sigma) \to (x^i, y^\sigma, p_\sigma^j), \quad p_\sigma^j = \frac{\partial L_0}{\partial y_j^\sigma} - \frac{\partial h_\sigma^{jk}}{\partial x^k} - \Big(\frac{\partial h_\sigma^{jk}}{\partial y^\nu} + \frac{\partial h_\nu^{jk}}{\partial y^\sigma}\Big) y_k^\nu \tag{6.22}$$

is a local coordinate transformation on J^1Y.

The formula (6.22) comes from the following noninvariant decomposition of the Poincaré–Cartan form

$$\theta_\lambda = -H\omega_0 + p_\sigma^j dy^\sigma \wedge \omega_j + d(h_\sigma^{ij} y_j^\sigma\, \omega_i), \tag{6.23}$$

where

$$H = -L_0 + \frac{\partial L_0}{\partial y_j^\sigma} - \frac{\partial h_\sigma^{jk}}{\partial y^\nu} y_j^\sigma y_k^\nu. \tag{6.24}$$

The functions H and p_σ^j were called in [26] the *Hamiltonian* and *momenta* of the Lagrangian (6.20), and (6.22) was called *Legendre transformation.* In the Legendre coordinates (6.22) the Hamilton–De Donder equations read

$$\frac{\partial H}{\partial y^\nu} = -\frac{\partial p_\nu^i}{\partial x^i}, \quad \frac{\partial H}{\partial p_\nu^k} = \frac{\partial y^\nu}{\partial x^k}. \tag{6.25}$$

As pointed out by Krupka and Štěpánková, the above results directly apply to the *Einstein–Hilbert Lagrangian* (scalar curvature) of the General Relativity Theory (for explicit computations see [26]). The above ideas were applied to study also some other kinds of higher order Lagrangians with projectable Poincaré–Cartan forms by [10] (cf. also comments in [11]).

(iii) Dedecker's approach to first order Hamiltonian field theory. In [5], Dedecker proposed a Hamilton theory for first order Lagrangians on contact elements. If transfered to fibered manifolds, it becomes a "nonstandard" Hamiltonian theory. The core is to consider Hamilton equations of the form

$$\delta^* i_\xi d\rho = 0, \tag{6.26}$$

where ρ is a Lepagean equivalent of a first order Lagrangian of the form

$$\rho = \theta_\lambda + \sum_{k=2}^{n} g_{\sigma_1 \ldots \sigma_k}^{i_1 \ldots i_k} \omega^{\sigma_1} \wedge \cdots \wedge \omega^{\sigma_k} \wedge \omega_{i_1 \ldots i_k}.$$

Dedecker showed that if the condition

$$\det\Big(\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu} - g_{\sigma\nu}^{ij}\Big) \neq 0 \tag{6.27}$$

is satisfied, where $g^{ij}_{\sigma\nu}$ are the components of the 2-contact part of ρ, then for every solution δ of the equations (6.26) which annihilates all the n-forms

$$\omega^{\sigma_1} \wedge \omega_{i_1}, \quad \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \omega_{i_1 i_2}, \quad \dots, \quad \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \dots \wedge \omega^{\sigma_n},$$

one gets $\delta = J^1\gamma$ where γ is a solution of the corresponding Euler–Lagrange equations. Accordingly, Dedecker called (6.27) a *regularity condition.* Also he pointed out that since (6.27) depends on a Lagrangian *and* the free functions $g^{ij}_{\sigma\nu}$, *one could consider the regularity problem for a Lagrangian as a problem on the existence of appropriate functions $g^{ij}_{\sigma\nu}$ such that (6.27) are satisfied.* He demonstrated such a *regularization procedure* on an example of a Lagrangian, singular within the traditional understanding (i.e., not satisfying the regularity condition (6.18)). He also introduced "momenta" as components at $dy^{\sigma_1} \wedge dy_{i_1}$, $dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \omega_{i_1 i_2}$, $\dots, dy^{\sigma_1} \wedge dy^{\sigma_2} \wedge \dots \wedge dy^{\sigma_n}$, and "Hamiltonian" as the component at ω_0 in the decomposition of ρ in the canonical basis. A "Legendre transformation" of this kind, however, becomes a map to a space which has no direct connection with the space where the dynamics of the problem takes place.

(iv) Regularity conditions (5.15) and their applications. It can be easily seen that all the regularity conditions mentioned in (i), (ii) and (iii) represent particular cases of the general regularity conditions (5.15). Our understanding of *regular variational problems* as those *admitting existence of a regular Hamiltonian theory* precisely develops Dedecker's idea published in [5]. Within this setting it is natural to investigate possible *regularizations* of Lagrangian systems. One can even ask for more, namely, if there is a *strongly regular* Hamiltonian theory admitting a *Legendre transformation* (in our sense). In such a case, apparently, one gets the original integration problem (to solve PDE of order $s + 1$) transfered to a *different* but *equivalent* integration problem to solve an *exterior differential system given in a "normal form".* This opens new possibilities to understand dynamics of variational problems and to search for effective integration methods, especially for those Lagrangian systems which traditionally are considered as singular.

These techniques recently have been applied to some concrete physically interesting Lagrangians, singular in the sense of (6.18) and (6.19). We have already mentioned [26] where regular Hamilton equations and Legendre coordinates for the *Einstein–Hilbert Lagrangian* of the General Relativity Theory have been obtained. In [31] and [32] regularizations of the *Dirac field* Lagrangian and of the *electromagnetic field* Lagrangian have been studied, and the corresponding momenta, Hamiltonian, and canonical Hamilton equations have been found. It should be stressed that in all these cases one obtains to Euler–Lagrange equations their strongly regular Hamiltonian counterparts which depend *only on the Lagrangian* of the problem.

(v) Relation of (5.15) with Saunders' concept of "Euler regularity". Taking in the regularity conditions (5.15) all the auxiliary terms $f^{J,P,i}_{\sigma\nu}$ equal to zero, one obtains formulas which depend only upon partial derivatives of the Euler–Lagrange

expressions E_σ. In particular, one gets the following regularity conditions:

$$\operatorname{rank}\Big(\frac{\partial E_\sigma}{\partial y^\nu_{p_1\dots p_{s-k}j_1\dots j_k i}}\Big) = \text{maximal}, \quad 0 \le k \le s,$$

where in the above matrices, the (ν, P) label rows, and the (σ, J, i) label columns. For $s = 1$ this means that

$$\det\Big(\frac{\partial E_\sigma}{\partial y^\nu_{pi}}\Big) \neq 0,$$

and for $s = 0$, from the definition of regularity of a zero order Hamiltonian system (see Sec. 4), we get

$$\operatorname{rank}\Big(\frac{\partial E_\sigma}{\partial y^\nu_i}\Big) = m.$$

Comparing this with the concept of regularity in *higher order mechanics* introduced in [27] (see [29] for a detailed exposition), we can see that these formulas mean "a direct generalization" to higher order field theory of the corresponding formulas obtained in mechanics. Accordingly, the Hamiltonian theory for higher order mechanics developed in [27] and [28] is a *particular case* of the Hamiltonian field theory in this paper and [30].

On the contrary, another generalization of the regularity conditions of [27] to field theory, proposed by Saunders in [36], and called by him *Euler regularity*, gives regularity conditions which are *different* from ours. Namely, for an Euler–Lagrange form of order $s+1$ the Saunders' Euler regularity condition reads

$$\operatorname{rank}\Big(\frac{\partial E_\sigma}{\partial y^\nu_{p_1\dots p_{s+1}}}\Big) = m,$$

where (σ) label rows and (ν, P) label columns (i.e., the functional equations $E_\sigma = 0$ can be solved for n derivatives of order $s+1$). As shown by Saunders on examples, "Euler regularity" unfortunately does not represent a meaningful alternative to the (not quite satisfactory) standard concept of regularity (6.18), (6.19).

(vi) Regularity and multisymplectic forms. A regular Lepagean $(n+1)$-form (Hamiltonian system) α on J^sY is a multisymplectic form. However, the converse generally is not true, i.e., multisymplectic forms (even those which are Lepagean) do not coincide with regular Hamiltonian systems. For details we refer to [30].

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