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ALMOST COMPLEX SUBMANIFOLDS OF QUATERNIONIC MANIFOLDS

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Abstract. It is a report on some recent results concerning the almost complex submanifolds of a quaternionic, in particular quaternionic Kähler, manifold. Some extensions of these results to submanifolds of a quaternionic Kähler manifold with torsion (QKT manifold) are considered.

1. Introduction

We report on some recent results concerning the submanifolds of special type, in particular the almost complex submanifolds, of a quaternionic Kähler manifold and we point out that some of this results extend to quaternionic manifolds. Also we report on the classification of Kähler manifolds with parallel cubic line bundle, which were defined in [5] and whose interest is related to the consideration of maximal Kähler submanifolds of a quaternionic Kähler manifold.

The classification of almost complex submanifolds of known quaternionic spaces is far to be completed. In fact, the almost complex submanifolds of a non symmetric Alekseevsky space, [8, 9], were not studied at all.

On the other hand the classification of the immersions of Kähler manifolds with parallel cubic line bundle into a quaternionic Kähler manifold different from the quaternionic projective space $\mathbb{H}P^n$ is still an open problem.

We take this opportunity to point out the interest to start to study the almost quaternionic and almost complex submanifolds of a quaternion Kähler manifold with torsion, [13]. We prove also some results which extend to such manifolds those valid for special submanifolds of quaternionic Kähler manifolds.

Finally, as a starting point to study complex submanifolds of a hypercomplex manifold $(M^{4n}, H = (J_\alpha))$, we prove that any integrable complex structure J compatible with H is parallel with respect to the Obata connection.

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2. Almost quaternionic and almost complex submanifolds of an almost quaternionic manifold $(\widetilde{M}^{4n}, Q)$

Let $\widetilde{M} \equiv \widetilde{M}^{4n}$ be a $4n$-dimensional manifold. We recall the basic notions of quaternionic geometry, see [3].

An *almost hypercomplex structure* $H = (J_\alpha), \alpha = 1, 2, 3$, on $\widetilde{M}$ is a triple of anticommuting almost complex structures with $J_1 J_2 = J_3$.

An **almost quaternionic structure** Q on $\widetilde{M}$ is a rank-3 subbundle $Q \to \widetilde{M}$ of the bundle $\mathrm{End} T\widetilde{M} \to \widetilde{M}$ which is locally generated by an almost hypercomplex structure $H = (J_1, J_2, J_3)$, that is locally $Q_x = \mathbb{R}J_{1|x} + \mathbb{R}J_{2|x} + \mathbb{R}J_{3|x}$. The pair $(\widetilde{M}, Q)$ is called an **almost quaternionic manifold** and $H = (J_1, J_2, J_3)$ is called a **local basis of** Q .

The **twistor bundle** $Z(\widetilde{M}) \to \widetilde{M}$ of the almost quaternionic manifold $(\widetilde{M}^{4n}, Q)$ is the S^2-bundle whose fiber $Z(\widetilde{M})_x$ at $x \in \widetilde{M}$ consists of all complex structures subordinated to the quaternionic structure Q_x, i.e.

$$Z(\widetilde{M})_x = \{J \in Q_x \quad | \quad J^2 = -Id\}$$

A (local) section $J : U \subset \widetilde{M} \to Q$ is called a **compatible almost complex structure** on $U \subset \widetilde{M}$.

Let $(\widetilde{M}^{4n}, Q)$ be an almost quaternionic manifold.

It is natural to consider the following classes of special submanifolds of $(\widetilde{M}^{4n}, Q)$.

Definition 2.1. A submanifold $M^{4k} \subset \widetilde{M}^{4n}$ is called an **almost quaternionic submanifold** if its tangent spaces are Q-invariant, that is

$$\forall x \in M^{4k}, \forall J \in Q_x \qquad \text{one has} \qquad J T_x M^{4k} = T_x M^{4k}$$

An almost quaternionic submanifold M^{4k} carries an almost quaternionic structure Q' induced by Q which consists of the restrictions to the fibers of TM of the endomorphisms of Q.

Definition 2.2. Let $M^{2m} \subset \widetilde{M}^{4n}$ be a submanifold and J an almost complex structure on M^{2m}. (M^{2m}, J) is called an **almost complex submanifold** of $(\widetilde{M}^{4n}, Q)$ if for every $x \in M^{2m}$, there exists $\widetilde{J} \in Q_x$ such that $\widetilde{J}_{|T_x M^{2m}} = J_x$, that is J is the restriction to TM of a compatible almost complex structure of $\widetilde{M}^{4n}$. If J is an (integrable) complex structure, then (M^{2m}, J) is called a **complex submanifold** of $\widetilde{M}^{4n}$.

3. Quaternionic submanifolds of a quaternionic manifold $(\widetilde{M}^{4n}, Q)$

Let Q be an almost quaternionic structure on $\widetilde{M} \equiv \widetilde{M}^{4n}$. Then Q is called a **quaternionic structure** and $(\widetilde{M}, Q)$ a **quaternionic manifold** if there exists a torsion-free connection $\widetilde{\nabla}$ on $T\widetilde{M}$ preserving the subbundle Q (such a $\widetilde{\nabla}$ is called a **quaternionic connection** and if it exists it is not unique).

The following result was proved in [2], see also [17].

Theorem 3.1. *Let M^{4k} be an almost quaternionic submanifold of the quaternionic manifold $(\widetilde{M}^{4n}, Q)$. Then $(M^{4k}, Q' = Q|TM)$ is a quaternionic manifold. Moreover it is totally geodesic with respect to any quaternionic connection $\widetilde{\nabla}$ of $(\widetilde{M}^{4n}, Q)$ and $\widetilde{\nabla}|TM$ is a quaternionic connection.*

Proof. It follows from the fact that the second fundamental form h of an almost quaternionic submanifold M with respect to any Q-invariant decomposition $T_x\widetilde{M} = T_xM + T_x^{\perp}M$ satisfies the identities

$$h(J_\alpha X, Y) = J_\alpha h(X, Y), \ \forall X, Y \in T_xM, \ \alpha = 1, 2, 3,$$

and hence vanishes. □

In the case of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$ the theorem was proved by A. Gray [11], D.V. Alekseevsky [1].

Due to this result, an almost quaternionic submanifold $(M, Q' = Q|TM)$ of a quaternionic manifold $(\widetilde{M}, Q)$ is called a **quaternionic submanifold**.

Remark 3.2. Locally any quaternionic submanifold can be considered as a complex manifold. In fact, locally a quaternionic manifold always admits a compatible (integrable) complex structure, see [18], pag. 125, and also [6].

Remarkable examples of quaternionic manifolds are the quaternionic Kähler manifolds. We recall that a **quaternionic Kähler manifold** $(\widetilde{M}^{4n}, Q, \widetilde{g})$ is a quaternionic manifold $(\widetilde{M}^{4n}, Q)$ with a given Riemannian metric $\widetilde{g}$ which is **Hermitian** with respect to Q, i.e. all endomorphisms of Q are skew-symmetric, and whose Levi-Civita connection $\widetilde{\nabla} = \nabla^{\widetilde{g}}$ preserves Q.

Main known examples of quaternionic Kähler manifolds are the so called *Wolf spaces*, which are symmetric with positive scalar curvature: with the exception of few exotic spaces they are

$$\mathbb{H}P^n = \frac{Sp(n+1) \cdot Sp(1)}{Sp(n) \cdot Sp(1)} \quad , \quad \mathrm{Gr}_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n) \times U(2))} \quad ,$$

$$\widetilde{\mathrm{Gr}}_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n) \times SO(4)} \,.$$

Other examples are the symmetric duals of Wolf spaces and, more generally, the homogeneous quaternionic Kähler manifolds with a transitive solvable group of motions, [9].

We recall that a classification of quaternionic submanifolds of symmetric quaternionic Kähler manifolds was obtained by Tasaki [20]. For $n = 2$ see also [14].

4. Almost complex submanifolds of a quaternionic manifold $(\widetilde{M}^{4n}, Q)$

Let $(\widetilde{M}^{4n}, Q)$ be a quaternionic manifold and $\widetilde{\nabla}$ a quaternionic connection on $(\widetilde{M}^{4n}, Q)$. For any local basis $H = (J_\alpha)$ of Q one has

$$\widetilde{\nabla} J_\alpha = \omega_\gamma \otimes J_\beta - \omega_\beta \otimes J_\gamma \tag{4.1}$$

where (α, β, γ) is a cyclic permutation of 1,2,3 and ω_α are local 1-forms.

In general, the locally defined almost complex structures J_α are not integrable. The Nijenhuis tensor of J_α is given by

$$4N_{J_\alpha} = J_\beta \partial(\psi_\alpha \otimes Id - (\psi_\alpha \circ J_\alpha) \otimes J_\alpha) \tag{4.2}$$

where $\psi_\alpha := (\omega_\gamma \circ J_\alpha - \omega_\beta)$, and ∂ indicates the Spencer operator of alternation, $\partial(\psi \otimes Id)(X,Y) = \psi(X)Y - \psi(Y)X$.

Let (M^{2m}, J) be an almost complex submanifold. *We will assume that on an open neighbourhood of M^{2m} in $\widetilde{M}^{4n}$ there is given a local basis $H = (J_1, J_2, J_3)$ such that $J = J_{1|TM}$ on M.* (Note that locally this assumption can always be made.) We will call H a **basis of Q adapted to** M. We denote by $\psi = \psi_1|TM = (\omega_3 \circ J_1 - \omega_2)|TM$ the restriction of 1-form ψ_1 to M.

For any $x \in M^{2m}$ we put $K^\psi_x = \mathrm{Ker}(\psi_\mathrm{x}) \cap \mathrm{Ker}(\psi_\mathrm{x} \circ \mathrm{J}) \subset \mathrm{T_xM}$ and we denote by $\overline{T}_x M$ the maximal Q_x-invariant subspace of T_xM, that is

$$\overline{T}_x M = T_x M \cap J_2 T_x M.$$

Theorem 4.1. *Let (M^{2m}, J) be an almost complex submanifold of a quaternionic manifold $(\widetilde{M}^{4n}, Q)$. Then*

(1) *for any $x \in M$*

$$4N_{J_x} = J_2 \partial(\psi_x \otimes Id - (\psi_x \circ J) \otimes J) \tag{4.3}$$

and hence $N_J|_x = 0$ if and only if $\psi_x = 0$. In particular, J is integrable if and only if $\psi \equiv 0$.

(2) *If $\psi_x \neq 0$, then either $\bar{T}_x M = K^\psi_x$ or $\bar{T}_x M = T_x M$. In the first case* $\dim M = 4k+2$ *and in the second case* $\dim M = 4k$.

Proof. Since the restriction $N_{J_1}|TM$ to M of the Nijenhuis tensor of the almost complex structure J_1 coincides with the Nijenhuis tensor N_J of $J = J_1|TM$, we have the following formula

$$4N_J = J_2 \partial(\psi \otimes Id - (\psi \circ J) \otimes J).$$

It implies (1) and shows that $N_J(X,Y) \in T_xM \cap J_2 T_x M = \bar{T}_x M$ for $X, Y \in T_xM$. Assume now that $\psi_x \neq 0$. Then we can choose a vector $Z \in T_xM$ such that $\psi(Z) = 1, \psi(JZ) = 0$. We have a direct sum decomposition $T_xM = K^\psi_x \oplus \mathrm{span}\{Z, JZ\}$. For $Y \in K^\psi_x$ we get

$$4N_J(Z,Y) = J_2(\psi(Z)Y - \psi(JZ)JY) = J_2 Y \in \bar{T}_x M$$

from which it follows $J_2 K^\psi_x \subset \bar{T}_x M$ and hence $K^\psi_x \subset \bar{T}_x M$. If $\bar{T}_x M$ is a proper subspace of T_xM, then $\bar{T}_x M = K^\psi_x$. □

Corollary 4.2. *Let (M^{2m}, J) be an almost complex, but not complex submanifold of the quaternionic manifold $(\widetilde{M}^{4n}, Q)$.*

i) *If m is even, then the open submanifold $M' = \{\psi \neq 0\}$ of M is a quaternionic (totally geodesic) submanifold of $\tilde{M}$. Moreover, M is a quaternionic submanifold of $\tilde{M}$ if M' is a dense subset of M.*

ii) *If m is odd, then at any point $x \in M'$ the maximal quaternionic subspace $\bar{T}_x M = K_x^{\psi}$ has codimension 2 in $T_x M$.*

Corollary 4.3. *Let (M^{2m}, J) be an almost complex submanifold of the quaternionic manifold $(\widetilde{M}^{4n}, Q)$. Then J is integrable if*

(a) *codim*$(\overline{T}_x M) > 2 \qquad \forall x \in M$

(b) (M, J) *is analytic and* $\exists x \in M \mid$ *codim*$(\overline{T}_x M) > 2$.

Recall that a quaternionic Kähler manifold is Einstein. Denote by $\nu = \frac{K}{4n(n+2)}$, where K is the scalar curvature, the **reduced scalar curvature** of such a manifold.

Theorem 4.4. *(see [4, 5]) Let $(\widetilde{M}^{4n}, Q, \widetilde{g})$ be a complete quaternionic Kähler manifold with $\nu > 0$. Then any analytic almost complex submanifold (M^{4k}, J) with complete induced metric g is a Hermitian submanifold i.e. the complex structure J is integrable.*

Proof. Assume that the almost complex structure J on M^{4k} is not integrable. Then by Corollary 4.2, M^{4k} is a totally geodesic complete quaternionic Kähler submanifold of positive scalar curvature with an almost complex structure . By a result from [7] , a complete quaternionic Kähler manifold (M^{4k}, Q', g) with positive scalar curvature admits no globally defined compatible almost complex structure. This contradiction proves the Theorem. $\square$

5. Kähler submanifolds of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$

Let now report on some results concerning Kähler submanifolds of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$.

We assume $\widetilde{\nabla} = \nabla^{\widetilde{g}}$. We recall that for any given local basis $H = (J_\alpha)$ of Q the 1-forms $\omega_\alpha, \alpha = 1, 2, 3$, defined by the 4.1 verify the identities

$$d\omega_\alpha + \omega_\beta \wedge \omega_\gamma = -\nu F_\alpha \tag{5.1}$$

where $F_\alpha = \widetilde{g} \circ J_\alpha, \alpha = 1, 2, 3$, are Kähler forms. Moreover

$$dF_\alpha - F_\beta \wedge \omega_\gamma + \omega_\beta \wedge F_\gamma = 0 \tag{5.2}$$

In the sequel the local basis $H = (J_\alpha)$ will be always assumed to be adapted to the considered almost complex submanifolds.

The following three theorems were proved in [4, 5].

Theorem 5.1. *([5]) Let $(M^{2m}, J, g = \widetilde{g}_{|TM})$ be an almost Kähler submanifold of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$. Then it is Kähler if $m \neq 3$.*

Proof. The proof bases on the identity

$$dF_1 - F_2 \wedge \omega_3 + \omega_2 \wedge F_3 = 0$$

valid on M, see 5.2, where $H = (J_\alpha)$ is a basis of Q adapted to M. $\square$

Problem. It could be interesting to state conditions under which this theorem is still valid by assuming that $(\widetilde{M}^{4n}, Q, \widetilde{g})$ is a quaternionic Hermitian, not necessarily Kähler, manifold.

Theorem 5.2. *([4, 5]) The almost Hermitian submanifold $(M^{2m}, J, g), m > 1$, of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$ with $\nu \neq 0$ is a Kähler submanifold if and only if one of the following equivalent conditions holds:*

k_1) $\omega_{2|T_xM} = \omega_{3|T_xM} = 0 \quad , \quad \forall x \in M^{2m}$

k_2) *(M^{2m}, J, g) is* **totally complex***, i.e.*

$$J_2 T_x M \perp T_x M \quad , \quad x \in M$$

Proof. Let (M^{2m}, J, g) be an almost Hermitian submanifold of $\widetilde{M}$. Using 4.1 we get

$$\begin{aligned}(\widetilde{\nabla}_X J_1)Y &= (\nabla_X J)Y + h(X, JY) - J_1 h(X, Y) \\ &= \omega_3(X) J_2 Y - \omega_2(X) J_3 Y \quad , \quad X, Y \in TM\end{aligned}$$

Taking the orthogonal projection on T_xM we conclude that

$$(\nabla_X J)Y = 0 \Longleftrightarrow \big[\omega_3(X) J_2 Y - \omega_2(X) J_3 Y\big]^T = 0$$

where $[\quad]^T$ means the tangent part. It is clear that any one of the conditions k_1) or k_2) implies $\nabla J_{|_x} = 0 \,\forall x \in M$, that is (M, J, g) is Kähler. To prove that the conditions k_1), k_2) are also necessary for (M, J) to be Kähler, we first show that at a point $x \in M$ where $\nabla J_{|_x} = 0$ at least one of them must hold: in fact, from the identities $(\nabla_X J)Y = \big[\omega_3(X) J_2 Y - \omega_2(X) J_3 Y\big]^T = 0$, $(\nabla_X J)(JY) = -\big[\omega_3(X) J_3 Y + \omega_2(X) J_2 Y\big]^T = 0 \quad \forall\, X, Y \in T_xM$ one gets

$$\big[\omega_2^2(X) + \omega_3^2(X)\big]\big[J_2 Y\big]^T = 0 \quad , \quad \forall\, X, Y \in T_xM$$

and the claim follows immediately. Now we assume that (M, J, g) is Kähler and prove that both k_1) and k_2) must hold on M.

1) Suppose that k_1) does not hold at $x \in M$. Then k_2) holds on an open neighbourhood U_x of x in M and the structure equation 5.2 for $\alpha = 2, 3$ gives $(\omega_3 \wedge F_1)_{T_xM} = (\omega_2 \wedge F_1)_{T_xM} = 0$ which imply (since $\dim T_xM > 2$) $\omega_{3|T_xM} = \omega_{2|T_xM} = 0$, by contradicting the assumption.

2) On the other hand, assume that k_2) does not hold at $x \in M$. Hence k_1) holds on an open neighbourhood V_x of x and the structure equation 5.1 for $\alpha = 2, 3$ gives $\nu F_{2|_{T_xM}} = \nu F_{3|_{T_xM}} = 0$. Since $\nu \neq 0$ these give a contradiction. □

K. Tsukada [21] proved that k_2) implies k_1), also for $\nu = 0$.

M. Takeuchi [19] classified the maximal totally geodesic Kähler submanifolds of a symmetric quaternionic Kähler manifold. For the aforenamed Wolf spaces of positive scalar curvature they reduce to the following ones (which can be easily described in classical terms):

$$\mathbb{C}P^n \hookrightarrow \mathbb{H}P^n \quad , \quad \mathbb{C}P^p \times \mathbb{C}P^q \hookrightarrow \mathrm{Gr}_2(\mathbb{C}^{p+q+2}) \quad , \quad \mathrm{Gr}_2(\mathbb{R}^{n+2}) \hookrightarrow \mathrm{Gr}_2(\mathbb{C}^{n+2})$$

$$\mathrm{Gr}_2(\mathbb{C}^{n+2}) \hookrightarrow \widetilde{\mathrm{Gr}}_4(\mathbb{C}^{n+4}) \quad , \quad (Q_p(\mathbb{C}) \times Q_q(\mathbb{C}))/\mathbb{Z}_2 \hookrightarrow \widetilde{\mathrm{Gr}}_4(\mathbb{R}^{p+q+4})$$

where $Q_p(\mathbb{C}) = \frac{SO(p+2)}{SO(p) \times SO(2)}$ is the complex hyperquadric of dimension p.

K. Tsukada [21] classified all parallel Kähler immersions in $\mathbb{H}P^n$.

We quote also the following theorem, see [5].

Theorem 5.3. *Let $(\widetilde{M}^{4n}, Q, \widetilde{g})$ be a quaternionic Kähler manifold with $\nu \neq 0$. Assume that for an almost complex submanifold (M^{2m}, J) of $(\widetilde{M}^{4n}, Q)$ the 2nd fundamental form h satisfies the identity*

$$h(X, JY) = h(JX, Y) = J_1 h(X, Y)$$

Then $(M^{2m}, J, g), g = \widetilde{g}_{|M}$, it is either a Kähler submanifold or a quaternionic submanifold.

The application of this result to the case $h = 0$, i.e. M^{2m} totally geodesic, is evident.

6. Maximal Kähler submanifolds (M^{2n}, J, g) of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$

A particularly interesting case is that one of a maximal Kähler submanifold (M^{2n}, J, g) of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$.

Let $H = (J_1, J_2, J_3)$ be an almost hypercomplex structure adapted to M. Then

$$\widetilde{\nabla}_X J_1 = 0 \quad , \quad \widetilde{\nabla}_X J_2 = \omega(X) J_3 \quad , \widetilde{\nabla}_X J_3 = -\omega(X) J_2 \qquad \forall X \in TM$$

and

$$d\omega = -\nu F \tag{6.1}$$

where $F = g \circ J$ is the Kähler form of (g, J) on M.

By means of J_2 we can identify the normal bundle $T^\perp M$ with the tangent bundle TM:

$$\varphi = J_{2|T^\perp M} : \underset{\xi}{T^\perp_x M} \underset{\mapsto}{\to} \underset{J_2\xi}{T_x M} \qquad \forall x \in M$$

Then the second fundamental form h of M is identified with the tensor field $C \in \Gamma(TM \otimes S^2 T^* M)$ on M given by

$$C = J_2 \circ h$$

We called C the **shape tensor** of the submanifold (M^{2n}, J). The following properties hold:

(1) $C_X \in \mathrm{End}^{sym}(TM) \quad , \qquad X \in TM$

(2) $\{C_X, J\} := C_X \circ J + J \circ C_X = 0$

(in particular, $\mathrm{Trace}\, C_\cdot := \sum_{2n} C_{E_i} E_i = 0$ for any orthonormal basis (E_i)).

(3) The tensors $gC, gC \circ J$ defined by

$$gC(X, Y, Z) = g(C_X Y, Z) \qquad , \qquad (gC \circ J)(X, Y, Z) = gC(JX, Y, Z)$$

are symmetric, i.e. $gC, gC \circ J \in S^3 T^* M$.

Moreover

(4) $\nabla_X C = J_2 \nabla' h + \omega(X) C \circ J$ where ∇' is the connection induced on the bundle $T^{\perp} M \otimes S^2 T^* M$.

From this last identity it is clear that M^{2m} *is parallel, i.e.* $\nabla' h = 0$*, if and only if the following identity holds*:

(5) $\quad \nabla_X C = \omega(X) C \circ J$

In this case, by taking into account the (6.1), one has the identity

(6) $\quad (R_{XY} \cdot C)_Z = -\nu F(X,Y) C_Z$

Let define by

$$S_J = \{A \in \mathrm{End} TM \, , \, \{A, J\} = 0 \, , \, g(AX, Y) = g(X, AY)\}$$

the bundle of symmetric endomorphisms of TM which anticommute with J and by

$$S_J^{(1)} = \{A \in \mathrm{Hom}(TM, S_J) = T^* M \otimes S_J \, , \, A_X Y = A_Y X\}$$

its first prolongation.

Then the validity of both (1),(2) can be expressed by saying that C is a section of $S_J^{(1)}$

Remark 6.1. In fact it must be taken into account that the tensor field C of a parallel Kähler submanifold depends on the choice of J_2, which could be not globally defined, and hence it is determined up to a transformation of the form $C' = \sin\theta C + \cos\theta J \circ C$.

7. Kähler manifolds with parallel cubic line bundle

From previous results it is clear that independently from any immersion, *it is interesting to take in consideration a Kähler manifold* (M^{2m}, J, g) *(locally) admitting a tensor field* $C \in \Gamma(TM \otimes S^2 T^* M)$ *for which the (1),(2),(3) and (6) hold.*

To define the precise notion of such a manifold it is convenient first to translate in the complexified context the basic results stated for a maximal parallel Kähler submanifold (M^{2n}, J, g) of a quaternionic Kähler manifold $(\widetilde{M}^{4n}, Q, \widetilde{g})$.

Consider the decompositions

$$T^{\mathbb{C}} M = T^{(10)} M + T^{(01)} M \qquad , \qquad T^{*\mathbb{C}} M = T^{*(10)} M + T^{*(01)} M$$

Denote by $S_J^{(1)\mathbb{C}}$ the complexification of the bundle $S_J^{(1)}$ and by $g \circ S_J^{(1)\mathbb{C}}$ the associated subbundle of the bundle $S^3(T^* M)^{\mathbb{C}}$ (the **bundle of cubic forms**).

It can be proved that

$$g \circ S_J^{(1)\mathbb{C}} = S^3 T^{*(10)} M + S^3 T^{*(01)} M$$

Hence the following result follows by using the decomposition

$$gC = q + \overline{q} \in S^3 T^{*(10)} M + S^3 T^{*(01)} M$$

Theorem 7.1. *Let (M^{2n}, J) be a parallel Kähler submanifold of a quaternionic Kähler manifold $\widetilde{M}^{4n}$ with $\nu \neq 0$. If it is not totally geodesic then on M there is a canonically defined parallel complex line bundle L of the bundle $S^3(T^{*(10)}M)$ of holomorphic cubic forms such that the curvature of the connection ∇^L induced by the Levi-Civita connection ∇ has the curvature form*

$$R^L = i\nu F \tag{7.1}$$

where $F = g \circ J$ is the Kähler form of M

Definition 7.2. A parallel subbundle $L \subset S^3(T^{*(10)}M)$ with the curvature form (7.1) on a Kähler manifold M is called a **parallel cubic line bundle of type** ν.

8. Characterization of Kähler manifolds with parallel cubic line bundle

Let M be a complete simply connected Kähler manifold with the de Rham decomposition

$$M_1 \times M_2 \times \cdots \times M_p$$

into product of the flat Kähler manifold M_0 and the irreducible Kähler manifolds $M_i, i = 1, \dots, p$.

Assume that M **admits a parallel cubic line bundle L of type** $\nu \neq 0$.

Then *there is no flat factor M_0 and $p \leq 3$.*

Moreover the following proposition holds.

Proposition 8.1. *Under the above hypothesis, if M is reducible either*

$$M = M^2_\nu \times M^2_\nu \times M^2_\nu$$

where $M^2_\nu (= \mathbb{C}P^1$ or $\mathbb{C}\mathcal{H}^1)$ is a 2-dimensional manifold of constant curvature ν, or

$$M = M_1 \times M^2_\nu$$

where M_1 is a complete simply connected reducible Kähler-Einstein manifold with

$$Ric_{M_1} = \nu \frac{m}{2} g_{(1)} \qquad (2m = dim M_1)$$

such that

$$(S^2V^*)^{\mathfrak{h}'_1} \neq 0$$

where $\mathfrak{h}'_1 = [\mathfrak{h}_1, \mathfrak{h}_1]$ is the commutator of the holonomy Lie algebra $\mathfrak{h}_1$ of M_1 at a point $x \in M_1$, $V = T^{(10)}_x M_1$ and $W^{\mathfrak{h}}$ denotes the subspace of $\mathfrak{h}$-invariant vectors of a vector space W.

Conversely, any manifold M of these types has a parallel cubic line bundle.

Concerning the irreducible case we have the following proposition.

Proposition 8.2. *A complete simply connected irreducible Kähler manifold M^{2n} with holonomy Lie algebra $\mathfrak{h}$ at a point x admits a parallel cubic line bundle of type ν if and only if it is Kähler-Einstein with*

$$Ric_M = \frac{\nu}{3} ng$$

and

$$(S^3 V^*)^{\mathfrak{h}'} \neq 0$$

where $V = T_x^{(10)} M$ is the holomorphic tangent space with the natural action of the Lie algebra $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$.

Previous propositions reduce the classification of Kähler manifolds with parallel cubic line bundle to the determination of the irreducible holonomy Lie algebras $\mathfrak{h}$ of a Kähler manifold such that the representation of $\mathfrak{h}_1' = [\mathfrak{h}_1, \mathfrak{h}_1]$ on the holomorphic tangent space $V = T_x^{(10)} M$ has a non trivial invariant quadratic or cubic form, i.e. such that

$$S^2(V^*)^{\mathfrak{h}'} \neq 0 \qquad \text{or} \qquad S^3(V^*)^{\mathfrak{h}'} \neq 0$$

Such a study, by help of tables in [16], led to the following classification, [5].

List of simply connected Kähler manifolds M^{2n}
with parallel cubic line bundle L of type $\nu > 0$

<u>Case of reducible M^{2n}</u>

$$M^{2n} = \frac{SO_{n+1}}{SO_2 \cdot SO_{n-1}} \times P \quad , \quad M^4 = P \times P' \quad , \quad M^6 = P \times P' \times P'' \quad ,$$

$$M^8 = \frac{Sp_2}{U_2} \times P \qquad \text{(where} \quad P, P', P'' \cong \mathbb{C}P^1)$$

<u>Case of irreducible M^{2n}</u>

$$M^2 = P \quad , \quad M^{12} = \frac{Sp_3}{U_3} \quad , \quad M^{18} = \frac{SU_6}{S(U_3 \times U_3)} \quad , \quad M^{30} = \frac{SO_{12}}{U_6} \quad ,$$

$$M^{54} = \frac{E_7}{T^1 \cdot E_6}$$

They are all <u>symmetric</u>.

For $\nu < 0$ the manifold M^{2n} is one of the dual symmetric spaces.

For $\nu > 0$ they were obtained by K. Tsukada as parallel submanifolds of $\mathbb{H}P^n$, [21].

The problem to find other immersions is still open.

9. QKT manifolds $(\widetilde{M}^{4n}, Q, \widetilde{g}, \widetilde{\nabla})$

Recently a certain interest on quaternionic Kähler manifolds with torsion arose mainly from the point of view of theoretical physics, see [12], [13].

We recall that a **quaternionic Kähler manifold with torsion** (briefly **QKT manifold**) $(\widetilde{M}^{4n}, Q, \widetilde{g}, \widetilde{\nabla})$ is a 4n-dimensional quaternionic Hermitian manifold endowed with a **QKT linear connection** $\widetilde{\nabla}$: this means that $\widetilde{\nabla}$ preserves the quaternionic structure Q, as well the Q-Hermitian metric $\widetilde{g}$, and moreover the covariant torsion tensor $\hat{T} = g \circ \widetilde{T}$ (which is totally skew-symmetric) has (1,2)+(2,1) type with respect to all almost complex structures in Q, that is

$$\widetilde{T}_X - \widetilde{T}_{JX} J + J\widetilde{T}_{JX} + J\widetilde{T}_X J = 0 \qquad \forall X \in TM \quad , \quad \forall J \in Q \tag{9.1}$$

where $\widetilde{T}_X := \widetilde{T}(X, \cdot)$.

Then, see [13],

$$\widetilde{\nabla} = \nabla^{\widetilde{g}} + \frac{1}{2}\widetilde{T} \tag{9.2}$$

Of course, $\widetilde{T} = 0$ if and only if $(\widetilde{M}^{4n}, Q, \widetilde{g})$ is a quaternionic Kähler manifold.

In any case $(\widetilde{M}^{4n}, Q)$ *is a quaternionic manifold*. For this result and the most basic properties of QKT manifolds we send to [13]. We only remark that the following identities hold.

Let $H = (J_\alpha)$ be a local basis of Q.

Then the condition that $\widetilde{\nabla}$ preserves Q gives

$$\widetilde{\nabla} J_\alpha = \omega_\gamma \otimes J_\beta - \omega_\beta \otimes J_\gamma \tag{9.3}$$

and hence

$$\nabla^{\widetilde{g}}_X J_\alpha = \omega_\gamma(X) J_\beta - \omega_\beta(X) J_\gamma - \frac{1}{2}[\widetilde{T}_X, J_\alpha]$$

from which

$$dF_\alpha = \omega_\gamma \wedge F_\beta - \omega_\beta \wedge F_\gamma + J_\alpha \hat{T} \tag{9.4}$$

where $F_\alpha = g(J_\alpha \cdot, \cdot)$ and $J_\alpha \hat{T} = -\hat{T}(J_\alpha \cdot, J_\alpha \cdot, J_\alpha \cdot)$, $\alpha = 1, 2, 3$. Moreover

$$d(J_\alpha \hat{T}) = \rho_\beta \wedge F_\gamma - \rho_\gamma \wedge F_\beta - \omega_\beta \wedge (J_\gamma \hat{T}) + \omega_\gamma \wedge (J_\beta \hat{T})$$

where

$$\rho_\alpha = d\omega_\alpha + \omega_\beta \wedge \omega_\gamma \, .$$

In the following we prove some results on special submanifolds of $\widetilde{M}^{4n}$.

10. Q-invariant submanifolds of a QKT manifold $(\widetilde{M}^{4n}, Q, \widetilde{g}, \widetilde{\nabla})$

Let $M^{4k} \subset \widetilde{M}^{4n}$ be an almost quaternionic submanifold. By Lemma and proof of proposition 8 at page 31 of [2], we know that it is a totally geodesic submanifold with respect to $\widetilde{\nabla}$,

$$\widetilde{\nabla}_X Y \in \Gamma(TM) \qquad \forall X, Y \in \Gamma(TM)$$

As a consequence (or also from the 9.4), we deduce that the restriction of $\widetilde{T}$ to TM^{4k} takes values in TM^{4k} and the following result holds true.

Theorem 10.1. *Let M^{4k} be an almost quaternionic submanifold of the QKT manifold $\widetilde{M}^{4n}$. Then*
1) $(M^{4k}, Q' = Q_{|TM}, g = \widetilde{g}_{|TM}, \nabla' = \widetilde{\nabla}_{|TM})$ is a QKT manifold;
2) (M^{4k}, g) is a totally geodesic submanifold of $(\widetilde{M}^{4n}, \widetilde{g})$.

11. Almost complex submanifolds of a QKT manifold

Let $(\widetilde{M}^{4k}, Q, \widetilde{g}, \widetilde{\nabla})$ be a QKT manifold.

Let (M^{2m}, J) be an almost complex submanifold of $(\widetilde{M}^{4n}, Q)$ and $H = (J_\alpha)$ a local basis of Q adapted to M^{2m}.

Remark 11.1. As in section 4, denote by ψ the restriction of the 1-form $\omega_3 \circ J_1 - \omega_2$ to TM. Then, even if $\widetilde{\nabla}$ is not a quaternionic connection, the statements (1), (2) of Theorem 4.1 and his Corollaries continue to hold. In fact one still has

$$4N_{J_x} = J_2\partial(\psi_x \otimes Id - (\psi_x \circ J) \otimes J)\,,$$

see 4.2, since 9.1 holds.

We now prove some generalizations of the results of section 5.

Theorem 11.2. *a) If the almost Hermitian submanifold $(M^{2m}, J, g = \widetilde{g}_{|TM})$ of the QKT manifold $(\widetilde{M}^{4n}, Q, \widetilde{g}, \widetilde{\nabla})$ is a* **totally complex submanifold** *, then*
1) *J is integrable*
2) $\nabla^g_X J - (\nabla^g_{JX} J)J = 0 \qquad \forall X \in TM;$
3) $h(X, JY) - Jh(X, Y) + h(JX, Y) + Jh(JX, JY) = 0 \qquad \forall X, Y \in TM\,.$
where h is the 2nd fundamental form of M^{2m}.

On the other hand:
b) if $(M^{2m}, J, g = \widetilde{g}_{|TM})$ is a **Kähler submanifold** *of the QKT manifold $(\widetilde{M}^{4n}, Q, \widetilde{g}, \widetilde{\nabla})$, then the following identity holds:*

$$h(X, JY) - Jh(X, Y) + h(JX, Y) + Jh(JX, JY) = 0 \qquad \forall X, Y \in TM\,.$$

Proof. On the almost Hermitian submanifold $(M^{2m}, J, g = \widetilde{g}_{|TM})$ of the QKT manifold $(\widetilde{M}^{4n}, Q, \widetilde{g}, \widetilde{\nabla})$, for any $X, Y \in T_xM$, by 9.2, one has

$$\begin{aligned}(\nabla^g_X J)Y + h(X, JY) - Jh(X, Y) = \ & \omega_3(X)J_2Y - \omega_2(X)J_3X \\ & -\tfrac{1}{2}[\widetilde{T}(X, JY) - J\widetilde{T}(X, Y)]\end{aligned}$$

By substracting from this identity that one obtained by substituting X, Y with JX, JY one obtains the identity

(11.1)
$$\begin{aligned}(\nabla^g_X J)Y - (\nabla^g_{JX} J)JY + \ & h(X, JY) - Jh(X, Y) + h(JX, Y) + Jh(JX, JY) = \\ & [\omega_3(X) + \omega_2(JX)]J_2Y - [\omega_2(X) - \omega_3(JX)]J_3Y\end{aligned}$$

since

$$\widetilde{T}(X,JY) - J\widetilde{T}(X,Y) + \widetilde{T}(JX,Y) + J\widetilde{T}(JX,JY) = 0\,.$$

By considering the tangent part and the part in the orthogonal $TM^{\perp}$ of the identity 11.1 we get the following consequences.

a) If the submanifold is totally complex the identity 11.1, by considering the tangential and the normal part respectively, is equivalent to the identities

$$(\nabla^g_X J)Y - (\nabla^g_{JX} J)JY = 0\,,$$

and

$$\begin{aligned} h(X,JY) - Jh(X,Y) + \quad & h(JX,Y) + Jh(JX,JY) = \\ & [\omega_3(X) + \omega_2(JX)]J_2Y - [\omega_2(X) - \omega_3(JX)]J_3Y \end{aligned}$$

But the first one implies that

$$4N_J(X,Y) = (\nabla^g_{JX} J)Y - (\nabla^g_{JY} J)X - (\nabla^g_Y J)(JX) + (\nabla^g_X J)(JY) = 0$$

that is J is integrable. Hence, by previous remark, $\omega_2 - \omega_3 \circ J = 0$ on M and we can conclude that the identity 3) holds.

On the other hand:

b) if we assume that the almost Hermitian submanifold is Kähler then $\nabla^g J = 0$, of course, and $(\omega_2 - \omega_3 \circ J)_{|TM} = (\omega_3 + \omega_2 \circ J)_{|TM} = 0$ since J is integrable. Hence 11.1 becomes

$$h(X,JY) - Jh(X,Y) + h(JX,Y) + Jh(JX,JY) = 0 \qquad\qquad \forall\, X,Y \in TM$$

and the statement b) is also proved. □

12. (Integrable) complex structures J which are compatible with a hypercomplex structure $H = (J_\alpha)$

Let $H = (J_1, J_2, J_3)$ be a **hypercomplex** structure on a manifold M^{4n}, that is $N_{J_\alpha} = 0, \quad \forall \alpha = 1,2,3.$

Let $J = \sum_\alpha a_\alpha J_\alpha, \quad \sum_\alpha a_\alpha^2 = 1$ be an almost complex structure compatible with H. Then the following identities hold:

$$\begin{aligned} 4N_J(X,Y) &= [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] \\ &= \sum_\alpha [a_\gamma(J_\gamma X\cdot a_\alpha) + a_\beta(J_\beta X\cdot a_\alpha) + a_\alpha(J_\alpha X\cdot a_\alpha) - a_\beta(X\cdot a_\gamma) + a_\gamma(X\cdot a_\beta)]J_\alpha Y \\ &- \sum_\alpha [a_\gamma(J_\gamma Y\cdot a_\alpha) + a_\beta(J_\beta Y\cdot a_\alpha) + a_\alpha(J_\alpha Y\cdot a_\alpha) - a_\beta(Y\cdot a_\gamma) + a_\gamma(Y\cdot a_\beta)]J_\alpha X \\ &+ \sum_{\alpha,\rho} a_\alpha a_\rho([J_\alpha X, J_\beta Y] - J_\alpha[X, J_\beta Y] - J_\alpha[J_\beta X, Y]) - [X,Y] \end{aligned}$$

that is

$$\begin{aligned}
4N_J(X,Y) &= [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] \\
&= \sum_\alpha [a_\gamma(J_\gamma X\cdot a_\alpha) + a_\beta(J_\beta X\cdot a_\alpha) + a_\alpha(J_\alpha X\cdot a_\alpha) - a_\beta(X\cdot a_\gamma) + a_\gamma(X\cdot a_\beta)]J_\alpha Y \\
&- \sum_\alpha [a_\gamma(J_\gamma Y\cdot a_\alpha) + a_\beta(J_\beta Y\cdot a_\alpha) + a_\alpha(J_\alpha Y\cdot a_\alpha) - a_\beta(Y\cdot a_\gamma) + a_\gamma(Y\cdot a_\beta)]J_\alpha X \\
&+ \sum_\alpha a_\alpha^2([J_\alpha X, J_\alpha Y] - J_\alpha[X,J_\alpha Y] - J_\alpha[J_\alpha X,Y] - [X,Y]) \\
&+ \sum_{\alpha\neq\rho} a_\alpha a_\rho([J_\alpha X, J_\rho X] - J_\alpha[X,J_\rho Y] - J_\alpha[J_\rho X,Y])
\end{aligned}$$

where (α,β,γ) is a circular permutation of $(1,2,3)$.

By assumption, the J_α are integrable, that is

$$4N_{J_\alpha}(X,Y) = [J_\alpha X, J_\alpha Y] - J_\alpha[X,J_\alpha Y] - J_\alpha[J_\alpha X,Y] - [X,Y] = 0 \quad \alpha = 1,2,3\,.$$

This implies (see [3], Lemma 3.2)

$$\begin{aligned}
&[J_\alpha X, J_\rho X] - J_\alpha[X,J_\rho Y] - J_\alpha[J_\rho X,Y] + \\
&\qquad + [J_\rho X, J_\alpha X] - J_\rho[X,J_\alpha Y] - J_\rho[J_\alpha X,Y] = 0 \qquad \forall\ \alpha,\rho\,.
\end{aligned}$$

Hence

$$\begin{aligned}
4N_J(X,Y) &= [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] \\
&= \sum_\alpha [a_\gamma(J_\gamma X\cdot a_\alpha) + a_\beta(J_\beta X\cdot a_\alpha) + a_\alpha(J_\alpha X\cdot a_\alpha) - a_\beta(X\cdot a_\gamma) + a_\gamma(X\cdot a_\beta)]J_\alpha Y \\
&- \sum_\alpha [a_\gamma(J_\gamma Y\cdot a_\alpha) + a_\beta(J_\beta Y\cdot a_\alpha) + a_\alpha(J_\alpha Y\cdot a_\alpha) - a_\beta(Y\cdot a_\gamma) + a_\gamma(Y\cdot a_\beta)]J_\alpha X \\
&= \sum_\alpha [(JX)\cdot a_\alpha - a_\beta(X\cdot a_\gamma) + a_\gamma(X\cdot a_\beta)]J_\alpha Y \\
&- \sum_\alpha [(JY)\cdot a_\alpha) - a_\beta(Y\cdot a_\gamma) + a_\gamma(Y\cdot a_\beta)]J_\alpha X\,.
\end{aligned}$$

It follows that J *is integrable if and only if*

$$(JX)\cdot a_\alpha = a_\beta(X\cdot a_\gamma) - a_\gamma(X\cdot a_\beta) \qquad \alpha = 1,2,3\,.$$

By combining this with the other identity

$$-X\cdot a_\alpha = a_\beta(JX\cdot a_\gamma) - a_\gamma(JX\cdot a_\beta) \qquad \alpha = 1,2,3$$

one finds

$$\begin{aligned}
-X\cdot a_\alpha &= a_\beta(JX\cdot a_\gamma) - a_\gamma(JX\cdot a_\beta) \\
&= a_\beta[a_\alpha(X\cdot a_\beta) - a_\beta(X\cdot a_\alpha)] - a_\gamma[a_\gamma(X\cdot a_\alpha) - a_\alpha(X\cdot a_\gamma)] \\
&= a_\beta a_\alpha(X\cdot a_\beta) - [a_\beta^2(X\cdot a_\alpha) + a_\gamma^2(X\cdot a_\alpha)] + a_\gamma a_\alpha(X\cdot a_\gamma) \\
&= -a_\alpha^2(X\cdot a_\alpha) + a_\alpha^2(X\cdot a_\alpha)
\end{aligned}$$

that is

$$X\cdot a_\alpha = 0 \quad , \quad \alpha = 1,2,3\,.$$

Hence

Theorem 12.1. *The only complex structures J which are compatible with a hypercomplex structure $H = (J_\alpha)$ on a manifold M^{4n} are linear combination of J_α with constant coefficients.*

In other words, a complex structure J is compatible with the hypercomplex structure H if and only if it is parallel with respect to the Obata connection ∇^H.

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