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Variation of the Total Normal Twist of Closed Curves in Euclidean Spaces

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Abstract:

The existence of parallel mates to a given closed space curves in the sense of H.R. Farran and S.A. Robertson [J. London Math. Soc. (2) **35** (1987), 527–538] depends on its total normal twist, i.e. the angle of rotation of a normal vector subject to parallel transfer in the normal bundle after one period. Parallel mates exist iff this quantity is an integer multiple of 2π. The aim of this note is to investigate the change of the total normal twist under isotopic deformations of closed curves. This enables us to characterize isotopies of space curves where the total normal twist remains invariant. Main result is a description of its change in terms of the oriented surface area covered by the unit tangents to all curves of the deformation family.

1. Introduction

The notion of *exterior parallelism* of two smooth immersions f_1, f_2 of the manifold M into Euclidean n-space E^n has been introduced by H.R. Farran and S.A. Robertson in [2] in the following sense: For every point $p \in M$ the affine spaces normal to $f_1(M)$ at $f_1(p)$ and $f_2(M)$ at $f_2(p)$ coincide. This condition has been shown to be equivalent to the condition that both immersions are connected by a parallel section of their normal bundles (see [3]). In the case of two regular smooth curves $c_1, c_2 : S^1 \longrightarrow E^n$ this means that there is a smooth normal vector field e_1 along c_1 such that

$$(1) \qquad c_2(t) = c_1(t) + \lambda e_1(t) \quad \text{and} \quad prn(\nabla_{\dot{c}_1(t)} e_1) = 0$$

for all $t \in S^1$, where $\dot{c}_1(t)$ denotes the tangent vector field of c_1 as usual and prn denotes the orthogonal projection to the corresponding normal (vector) space of

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c_1. Previous investigations of this notion for curves could be found in the paper [1] by F. J. Craveiro de Carvalho and S. A. Robertson and in the papers [4] and [5] by the second author of this note.

Let $c : S^1 \longrightarrow E^3$ be a regular closed smooth curve (at least of class C^3) for the subsequent considerations. Generally, global parallel normal vector fields along c do not exist. Parallel transfer of the normal plane along one period of c with respect to the normal connection leads to a rotation of the normal plane which is characterized (up to integer multiples of 2π) by an oriented angle $\alpha(c)$ which we call the *total normal twist* of c. For Frenet curves this quantity is given by their total torsion up to integer multiples of 2π. Looking at general orthonormal frame fields $\{T, e_1, e_2\}$ along c where T denotes the unit tangent field of c, and setting

$$\omega_{12} = < \nabla_T e_1, e_2 > = - < \nabla_T e_2, e_1 > = -\omega_{21}, \tag{2}$$

we get for the total normal twist of c (up to integer multiples of 2π)

$$\alpha(c) = \int_{S^1} \omega_{21}(t)dt \tag{3}$$

(see [4]) where there is no need to parametrize c with arclength.

Similarities of E^3 can be seen to preserve parallel mates (see [2] or [3]). The existence of parallel submanifolds to a given one even remains true under conformal transformations (see [6]). Moreover in the case studied in this paper we have the invariance of the total normal twist under similarities. The proof is quite easy:

Proposition 1: *Let the curve $\hat{c} : S^1 \longrightarrow E^3$ be obtained from c by $\hat{c} = h_a \circ c$, h_a denoting the homothety by $a \neq 0$. Then $\alpha(\hat{c}) = \alpha(c)$.*

Example 1: Consider the following torus knots on a torus in 3-space having radii R and r, possibly admitting self-intersections and surface singularities:

$$c_{m,k}(t) = (R + r\cos kt)(\cos mt, \sin mt, 0) + r(0, 0, \sin kt) \tag{4}$$

with integers (m, k) being relatively prime and $t \in [0, m \cdot k \cdot 2\pi]$. After simple computation we get for $\alpha_{m,k} := \alpha(c_{m,k})$

$$\alpha_{m,k} = \int_0^{k \cdot m \cdot 2\pi} \frac{m \cdot k}{[(r/R)^2 k^2 + m^2(1 + (r/R)\cos kt)^2]^{\frac{1}{2}}} dt. \tag{5}$$

Adjusting the parameter r/R appropriately we get a lot of examples for torus knots admitting parallel mates.

2. The first variation of the total normal twist

The aim of this section is to compute the instantaneous change of the total normal twist of c under a deformation. Doing this in the general framework turns out to be quite tedious. Hence we carry out our considerations in a more special framework, hoping that the results easily can be transfered to the general case again. The latter will be shown in the next section.

Let c be a regular curve as above. A *homotopy* of c is given by a differentiable map (at least C^3)

$$H : S^1 \times (-\varepsilon, \varepsilon) \longrightarrow E^3 \tag{6}$$

such that $H(t,0) = c(t)$, $t \in S^1$. A homotopy of c leads to a family c_τ of closed curves $c_\tau(t) = H(t,\tau)$ depending differentiably on $\tau \in (-\varepsilon, \varepsilon)$ such that $c_0(t) = c(t)$. The vector field V along c defined by $V(t) = \frac{\partial H}{\partial \tau}|_{(t,0)}$, $t \in S^1$, is a differentiable vector field along c, which is called the variational vector field of H.

Denote the length of c_τ by $l(\tau)$. Then $l(0)$ is the length of the original curve c. As usual let s denote the arclength parameter of c. The first simplification we may assume for our homotopy of c is given by

Normalization 1: Replace H by $\hat{H}$,

$$\hat{H}(s,\tau) = \frac{l(0)}{l(\tau)} \cdot H(s,\tau), \tag{7}$$

i.e. all curves of the family $\hat{H}(.,\tau) = \hat{c}_\tau$ have the same length $l(0)$. We may assume without loss of generality that all curves are regular. From Proposition 1 we get immediately that $\alpha(\hat{c}_\tau) = \alpha(c_\tau)$ for all $\tau \in (-\varepsilon, \varepsilon)$, i.e. there is no difference in the development of the total normal twist if we consider $\hat{H}$ instead of H. The change of the variational vector field is given by the following formula

$$\hat{V}(s) = V(s) + \{(1/l(0)) \int_0^{l(0)} < V(s), \nabla_T T(s) > ds\} c(s). \tag{8}$$

Hence, for the future considerations we may assume that all curves of the homotopy have the same arclength which then leads to a restriction for the choice of the original variational vector field as follows:

$$\int_0^{l(0)} < V(s), \nabla_T T(s) > ds = 0. \tag{9}$$

At present s may differ from the arclength parameter of the c_τ for $\tau \neq 0$. For further computations it will be useful to rearrange this. Hence we get after having adjusted H according to Normalization 1

Normalization 2: Replace H by $\bar{H}$,

$$\bar{H}(\tilde{s}(s,\tau),\tau) = H(s,\tau), \tag{10}$$

where $\tilde{s}(.,\tau)$ is the arclength parametrization of $H(.,\tau)$. Clearly $\alpha(\bar{c}_\tau) = \alpha(c_\tau)$ for all $\tau \in (-\varepsilon, \varepsilon)$. We get the corresponding variational vector field $\bar{V}$ obtained under this procedure as follows: Hence

$$\begin{aligned}\bar{V}(s) &= V(s) - < V(s), T(s) > T(s) + \\ &\quad \{< V(0), T(0) > + \int_0^s < V(r), (\nabla_T T)(r) > dr\}\ T(s),\end{aligned} \tag{11}$$

which can be rewritten in the form:

$$\bar{V}(s) = V^\perp(s) + \{< V(0), T(0) > + \int_0^s < V(r), (\nabla_T T)(r) > dr\} T(s). \tag{12}$$

Hence, for the future considerations we may assume that all curves of the homotopy have the same arclength and that they are parametrized by arclength. This again leads to a restriction for the choice of the original variational vector field given by:

$$\int_0^s < V(r), \nabla_T T(r) > dr = < V(r), T(r) > |_0^s, \text{or} \tag{13}$$

$$< \nabla_T V, T >= 0 \tag{14}$$

Now we are ready to compute the derivative of $\alpha(\tau) := \alpha(c_\tau)$ at $\tau = 0$ assuming both normalizations for the homotopy described above.

Let $\{T(s,\tau), e_1(s,\tau), e_2(s,\tau)\}$ be an positively oriented orthonormal frame field along H where the unit tangent field along the curves of this family is given by $T(s,\tau) = \frac{\partial H}{\partial s}|_{(s,\tau)}$ according to our general assumptions. Hence

$$e_2(s,\tau) = (T \overset{\rightarrow}{\times} e_1)|_{(s,\tau)} = (\frac{\partial H}{\partial s} \overset{\rightarrow}{\times} e_1)|_{(s,\tau)}.$$

This implies

$$\frac{d\alpha}{d\tau} = \frac{d}{d\tau}\int_0^l < \frac{\partial H}{\partial s} \overset{\rightarrow}{\times} \frac{\partial e_1}{\partial s}, e_1 > |_{(s,\tau)} ds. \tag{15}$$

Simple computation leads to

$$\begin{aligned}\frac{d\alpha}{d\tau} &= -\textstyle\int_0^l \quad \{< \frac{\partial H}{\partial \tau} \overset{\rightarrow}{\times} \frac{\partial^2 e_1}{\partial s^2}, e_1 > |_{(s,\tau)} + < \frac{\partial^2 H}{\partial s^2} \overset{\rightarrow}{\times} \frac{\partial e_1}{\partial \tau}, e_1 > |_{(s,\tau)} + \\ &\qquad 2 < \frac{\partial H}{\partial s} \overset{\rightarrow}{\times} \frac{\partial e_1}{\partial \tau}, \frac{\partial e_1}{\partial s} > |_{(s,\tau)}\} ds\end{aligned} \tag{16}$$

which gives at $\tau = 0$

$$
\begin{aligned}
\frac{d\alpha}{d\tau}|_{\tau=0} \quad &= \quad -\int_0^l \{< V \vec{\times} \nabla_T \nabla_T e_1, e_1 > + \\
&< \nabla_T T \vec{\times} \nabla_V e_1, e_1 > + 2 < T \vec{\times} \nabla_V e_1, \nabla_T e_1 >\} ds \qquad (17)
\end{aligned}
$$

e_1 is a unit normal vector field of c; $T, \nabla_V e_1, \nabla_T e_1$ are orthogonal to e_1. Hence we get that $\{T, \nabla_V e_1, \nabla_T e_1\}$ are linearly dependent, i. e. $< T \vec{\times} \nabla_V e_1, \nabla_T e_1 >= 0$. Furthermore by the same argument, $\{e_1, \nabla_T T, (\nabla_V e_1)^{\perp}\}$ are linearly dependent. Hence we can simplify the last expression for the derivative of α to

$$
\begin{aligned}
&\frac{d\alpha}{d\tau}|_{\tau=0} = \\
&-\int_0^l \{< V, \nabla_T \nabla_T e_1 \vec{\times} e_1 > - < \nabla_T V, e_1 >< \nabla_T T, T \vec{\times} e_1 >\} ds \qquad (18)
\end{aligned}
$$

In particular we get that $\frac{d\alpha}{d\tau}|_{\tau=0}$ only depends on V and not on how H extends globally.

Substituting equation (13) or (14) in this expression we get

$$
\begin{aligned}
&\frac{d\alpha}{d\tau}|_{\tau=0} = \\
&\int_0^l [- < \nabla_T T, e_1 >< \nabla_T V, e_2 > + < \nabla_T T, e_2 >< \nabla_T V, e_1 >] ds \qquad (19)
\end{aligned}
$$

which can be reinterpreted using the formula

$$
\begin{aligned}
&< \nabla_T V \vec{\times} \nabla_T T, T >= \\
&< \nabla_T V, e_1 >< \nabla_T T, e_2 > - < \nabla_T V, e_2 >< \nabla_T T, e_1 > \qquad (20)
\end{aligned}
$$

as an expression where e_1 and e_2 do not appear any more, i.e. the derivative of α only is given in terms of the original curve c_0 and the variational vector field V which is assumed to satisfy our normalization conditions derived above:

Proposition 2: *Let V be the variational vector field of a homotopy of a smooth regular closed curve c such that the normalization conditions (13) or (14) are satisfied, then the derivative of the total normal twist of c as a function of the deformation parameter is given at c by the following expression*

$$
\frac{d\alpha}{d\tau}|_{\tau=0} = \int_0^l < \nabla_T V \vec{\times} \nabla_T T, T > ds \qquad (21)
$$

which is equivalent to

$$\frac{d\alpha}{d\tau}|_{\tau=0} = \int_0^l < V \overset{\rightarrow}{\times} T, \nabla_T \nabla_T T > ds. \tag{22}$$

3. The change of the total normal twist under homotopies

The aim of this section is to remove the restriction made for V and to extend Proposition 2 to general homotopies. The modifications of the variational vector fields in the normalizations above are linear procedures. Their kernels easily can be described as follows:

$$\hat{V} = 0. \Leftrightarrow V = M \cdot c \tag{23}$$

for a real constant M, and for vector fields V satisfying the requirements of Normalization 2 we get

$$\bar{V} = 0. \Leftrightarrow V = \lambda \cdot T \tag{24}$$

for some smooth function λ of s satisfying $\lambda(0) = \lambda(l) = 0$. Applying both projections to a general variational vector field V in order to get the uniquely determined companion V_n (within our procedures) satisfying the requirements made for the validity of Proposition 2 we get

$$V_n(s) = V(s) + M_V \cdot c(s) + \lambda_V(s) \cdot T(s) \tag{25}$$

where

$$M_V := (1/l(0)) \int_0^{l(0)} < V(s), \nabla_T T(s) > ds \tag{26}$$

and

$$\begin{aligned} \lambda_V(s) &= \int_0^s < V(r), \nabla_T T(r) > dr - < V(r), T(r) > |_0^s - M_V \cdot s \\ &= -[\int_0^s < T(r), \nabla_T V(r) > dr + M_V \cdot s]. \end{aligned} \tag{27}$$

Substituting this expression in the formulas (21) and (22) from Proposition 2, we see that these formulas remain valid in the general case which gives

Theorem 1: *Let V be the variational vector field of a homotopy of a smooth regular closed curve c. The derivative of the total normal twist of c as a function of the deformation parameter is given at c by the following expression*

$$\frac{d\alpha}{d\tau}|_{\tau=0} = \int_0^l < \nabla_T V \overset{\rightarrow}{\times} \nabla_T T, T > ds$$

$$= \int_0^l < V \overset{\rightarrow}{\times} T, \nabla_T \nabla_T T > ds \tag{28}$$

where T denotes the unit tangent vector field of c.

Going back from the arclength parametrization in this formula to a general parametrization and from the unit tangent vector to the tangent belonging to this parametrization, we see that all problematic terms cancel and that we get an expression which is valid at all deformation parameters for our homotopy. Hence we have computed the derivative of α in terms of the homotopy as follows:

Theorem 2: *Let $H(t, \tau)$ be an isotopy of smooth regular closed curves c_τ such that all curves are parametrized over the segment $[0, 1]$. Then the derivative of the total normal twist $\alpha(\tau)$ of c_τ as a function of the deformation parameter τ is given by the following expression*

$$\frac{d\alpha}{d\tau} = \int_0^1 \frac{1}{\|\frac{\partial H}{\partial t}\|^3} det\ (\frac{\partial^2 H}{\partial t \partial \tau}, \frac{\partial^2 H}{\partial t^2}, \frac{\partial H}{\partial t})\ dt. \tag{29}$$

Here we may distribute the third power of $1/\|\frac{\partial H}{\partial t}\|$ to the entries in the determinant again. Problematic terms will cancel again, and we get a formula for the global change of the total normal twist during a deformation which can be interpreted geometrically:

Theorem 3: *Let $H(t, \tau)$ be an isotopy of smooth regular closed curves c_τ such that all curves are parametrized over the segment $[0, 1]$. Let c_{τ_1} and c_{τ_2} be the boundary curves of this isotopy. Then the difference between the total normal twist $\alpha(\tau_2)$ of c_{τ_2} and that of c_{τ_1} is given by the following expression*

$$\alpha(\tau_2) - \alpha(\tau_1) = \int_{\tau_1}^{\tau_2} \int_0^1 det\ (\frac{\partial T}{\partial \tau}, \frac{\partial T}{\partial t}, T)\ dt d\tau, \tag{30}$$

where $T := \frac{\partial H}{\partial t}/\|\frac{\partial H}{\partial t}\|$ is the field along H given by the unit tangents to the curves of the isotopy. Calling T the spherical tangent map of the isotopy, we see from this equation that the difference between the total normal twists is given by the oriented surface area of the spherical tangent map of the connecting isotopy.

The results above may be used to detect isotopies leaving the total normal twist invariant, and in particular to provide examples for deformations of curves remaining in the class of curves of global parallel rank 2. These considerations have to be left to other work.

4. Some remarks about curves in 4-space

Let $c : S^1 \longrightarrow E^4$ be a regular closed smooth curve in E^4 parametrized by arclength s. Let $\{T, e_1, e_2, e_3\}$ be an orthonormal frame field along c, Tdenoting the field of unit tangents. We may choose e_1 as a parallel section of the normal bundle of c, as has been shown in [4]. Hence starting normal parallel transfer of $e_i(0)$, $i = 1, \ldots, 3$, in $c(0)$ will leave $e_1(0)$ invariant after one period, and it will rotate $e_2(0)$ and $e_3(0)$ by an angle $\alpha(c)$ in the $(e_2(0), e_3(0))$-plane. This may be taken as the total normal twist of c in E^4. (This construction will fail for curves in 5-space, because there the existence of global parallel sections of the normal bundle cannot be guaranteed.)

Under these assumptions the calculation of the total normal twist in terms of the given frame field can be done like in 3-space. Setting $e_0 := T$ we start with the structure equations

$$\frac{de_i}{ds} = \sum_{j=0}^{3} \omega_{ij} e_j, \tag{31}$$

and because e_1 is parallel we get an ordinary differential equation for the rotation angle of the parallel extension of $e_2(0)$ and $e_3(0)$ in the (e_2, e_3)-frame which can be integrated, giving the following formula for the total normal twist of c:

$$\alpha(c) = \int_0^l \omega_{32}(s) ds \tag{32}$$

This quantity is invariant under similarities. Hence, considering homotopies of c we can apply Normalizations 1 and 2 again and compute $\frac{d\alpha}{d\tau}$:

$$\frac{d\alpha}{d\tau}|_{\tau=0} = \int_0^l [< \nabla_T T, e_3 >< \nabla_T V, e_2 > - < \nabla_T T, e_2 >< \nabla_T V, e_3 >] ds. \tag{33}$$

This equation can be rewritten in a form where e_2 and e_3 have disappeared, leading to

Proposition 3: *For a normalized homotopy of a smooth closed curve c in Euclidean 4-space the derivative of the the total normal twist at c is given by*

$$\frac{d\alpha}{d\tau}|_{\tau=0} = \int_0^l det(\nabla_T V, \nabla_T T, T, e_1) ds. \tag{34}$$

where e_1 is a parallel unit normal section of c.

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