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Bi-Hamiltonian manifolds – an application to the relativistic Toda lattice and a generalization to the Jacobi manifolds[1]

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Abstract: An application of bi-Hamiltonian manifolds: using mastersymmetries, we generate a hierarchy of Poisson tensors that provides a multi-Hamiltonian formulation for the relativistic Toda lattice.

A generalization of bi-Hamiltonian manifolds: we extend the notion of compatible Poisson tensors to the Jacobi structures.

All manifolds, maps, vector and tensor fields are assumed to be differentiable of class C^∞.

I) Bi-Hamiltonian manifolds — an application

1 Recursion operators and Poisson-Nijenhuis manifolds

The bi-Hamiltonian manifolds were introduced by F. Magri [5] in 1978. In this section we review the basic facts concerning recursion operators and bi-Hamiltonian manifolds, namely Poisson-Nijenhuis manifolds [3, 6].

Definition 4 *Let X be a vector field on a manifold M. A tensor field R of type $(1,1)$ (one time contravariant and one time covariant) is called a* recursion operator *of X if $\mathcal{L}_X R = 0$.*

As it is usual, we shall consider a tensor field of type $(1,1)$ as a vector bundle map of the tangent bundle TM into itself and we shall denote the transpose map by ${}^tR : T^*M \to T^*M$.

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Definition 5 *Let R be a tensor field of type $(1,1)$ on a manifold M. The* Nijenhuis torsion *of R is a tensor field $\tau(R)$ of type $(1,2)$ (one time contravariant and two times covariant) given, for any pair (X,Y) of vector fields on M, by*

$$\tau(R)(X,Y) = [RX, RY] - R([RX,Y] + [X,RY] - R[X,Y]). \tag{1.1}$$

Definition 6 *A tensor field R of type $(1,1)$ on the manifold M is called a* Nijenhuis operator *if its Nijenhuis torsion $\tau(R)$ vanishes.*

If R is a Nijenhuis operator, then for any vector field X on M,

$$\mathcal{L}_{RX} R = R\mathcal{L}_X R. \tag{1.2}$$

Proposition 2 *Let R be a recursion operator of a vector field X on the manifold M. If R is a Nijenhuis operator, then R is recursion operator of each $X_k = R^k X, k \in \mathbb{N}$. Furthermore, $[X_k, X_l] = 0$, for all $k,l \in \mathbb{N}$.*

Definition 7 *A* bi-Hamiltonian manifold *is a triple $(M, \Lambda_0, \Lambda_1)$ where Λ_0 and Λ_1 are* compatible *Poisson tensors on the smooth manifold M. The compatibility condition of Λ_0 and Λ_1 can be expressed by $[\Lambda_0, \Lambda_1] = 0$.*

If Λ is a bivector on the manifold M, it defines a vector bundle map $\Lambda^{\#} : T^*M \to TM$ such that, for any $x \in M$ and $\alpha, \beta \in T^*_x M$,

$$< \beta, \Lambda^{\#}_x(\alpha) >= \Lambda_x(\alpha, \beta).$$

Definition 8 *A* Poisson-Nijenhuis manifold *is a triple (M, Λ_0, R) where (M, Λ_0) is a Poisson manifold and R is a tensor field on M of type $(1,1)$ such that, if Λ_1 is the bivector on M corresponding to the bundle map $\Lambda_1^{\#} = R\Lambda_0^{\#}$, then*

$$\Lambda_1^{\#} = \Lambda_0^{\#} \circ {}^tR, \quad [\Lambda_0, \Lambda_1] = 0 \quad \textit{and} \quad [\Lambda_1, \Lambda_1] = 0.$$

From last definition, we may conclude that a Poisson-Nijenhuis manifold is a bi-Hamiltonian manifold and also that if (M, Λ_0, R) is a Poisson-Nijenhuis manifold, the Nijenhuis torsion of R vanishes on $\Lambda_0^{\#}(T^*M) \times \Lambda_0^{\#}(T^*M)$.

Let (M, Λ_0, R) be a Poisson-Nijenhuis manifold and define the following sequence of bivectors $\Lambda_k = R^k \Lambda_0, k \in \mathbb{N}$, on M. Then, $(\Lambda_k)_{k \in \mathbb{N}_0}$ is a hierarchy of Poisson tensors on M, that are pairwaise compatible.

Let us now consider the case of a bi-Hamiltonian manifold $(M, \Lambda_0, \Lambda_1)$ where the Poisson tensor Λ_0 is *non-degenerate.* In such a case, we may define the tensor field $R = \Lambda_1^{\#} \circ (\Lambda_0^{\#})^{-1}$ and (M, Λ_0, R) is a Poisson- Nijenhuis manifold. The tensor field R is a Nijenhuis operator and is also a recursion operator for all bi-Hamiltonian vector fields on M (that is, vector fields that are Hamiltonian with respect to both Poisson structures on M).

2 Mastersymmetries and bi-Hamiltonian systems

Mastersymmetries were introduced by B. Fuchssteiner [1] and mastersymmetries for bi-Hamiltonian systems were studied by W. Oevel [9]. First of all, we recall the notion of mastersymmetry.

Definition 9 *Let Y be a bi-Hamiltonian vector field on a manifold M. The vector field X on M is a* symmetry *of the differential equation $\dot{y} = Y(y)$ if $[X, Y] = 0$ and X is called a* mastersymmetry *if*

$$[[X, Y], Y] = 0 \quad \textit{and} \quad [X, Y] \neq 0.$$

Let us take a bi-Hamiltonian manifold $(M, \Lambda_0, \Lambda_1)$, with Λ_0 non-degenerate. As we have already remarked, if we set $R = \Lambda_1^{\#} \circ (\Lambda_0^{\#})^{-1}$ then (M, Λ_0, R) is a Poisson-Nijenhuis manifold. Suppose that the first cohomology of M is trivial. Then, if $Y_1 = \Lambda_0^{\#}(dH_1) = \Lambda_1^{\#}(dH_0)$ is a bi-Hamiltonian vector field on M, there exists a sequence $(H_k)_{k\in\mathbb{N}}$ of first integrals of Y_1 that are pairwise in involution with respect to both Poisson brackets on M. The Hamiltonians H_k are defined (up to additive constants) by $dH_k = ({}^tR)^k(dH_0)$. Under these conditions we have the following theorem due to Oevel.

Theorem 10 (Oevel) *Let X_0 be a vector field on the Poisson-Nijenhuis manifold (M, Λ_0, R) such that*

$$\mathcal{L}(X_0)\Lambda_0 = \alpha\Lambda_0, \quad \mathcal{L}(X_0)\Lambda_1 = \beta\Lambda_1 \quad \textit{and} \quad \mathcal{L}(X_0)H_0 = \gamma H_0,$$

for all $\alpha, \beta, \gamma \in \mathbb{R}$.

Then, if $Y_k = \Lambda_k^{\#}(dH_0)$, the vector fields $X_k = R^k X_0$ satisfy, for all $k, l \in \mathbb{N}$,

(i) $[X_k, X_l] = (\beta - \alpha)(l - k)X_{k+l}$;

(ii) $[X_k, Y_l] = (\beta + \gamma + (\beta - \alpha)(l - 1))Y_{k+l}$;

(iii) $\mathcal{L}(X_k)\Lambda_l = (\beta + (\beta - \alpha)(l - k - 1))\Lambda_{k+l}$;

(iv) $\mathcal{L}(X_k)H_l = (\gamma + (\beta - \alpha)(l + k))H_{k+l}$.

Remark that for all $k, l \in \mathbb{N}$, $[X_k, Y_l] \neq 0$ and $[[X_k, Y_l], Y_l] = 0$; that is, $(X_k)_{k\in\mathbb{N}}$ is a sequence of mastersymmetries for this bi-Hamiltonian system.

3 The relativistic Toda lattice

Let us consider $\mathbb{R}^{2n}$ with coordinates $(q^1,\dots,q^n,p_1,\dots,p_n)$. The relativistic Toda lattice (RTL), introduced by S. Ruijsenaars [10], is a finite-dimensional completely integrable system that admits the following bi-Hamiltonian formulation [8]:

$$\begin{aligned}\Lambda_0 &= \sum_{i=1}^{n}\exp(-p_i)\frac{\partial}{\partial q^i}\wedge\left(\frac{\partial}{\partial p_i}+\sum_{j=i+1}^{n}\frac{\partial}{\partial q^j}\right)\\ &+\sum_{i=1}^{n-1}\exp(q^i-q^{i+1}-p_{i+1})\left(\left(\frac{\partial}{\partial p_i}+\frac{\partial}{\partial q^{i+1}}\right)\wedge\left(\frac{\partial}{\partial p_{i+1}}+\sum_{j=i+2}^{n}\frac{\partial}{\partial q^j}\right)\right.\\ &\left.-\frac{\partial}{\partial p_{i+1}}\wedge\sum_{j=i+2}^{n}\frac{\partial}{\partial q^j}\right) \end{aligned}\tag{3.1}$$

$$\Lambda_1=\sum_{i=1}^{n}\frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i};\tag{3.2}$$

with Hamiltonians

$$H_0=\sum_{i=1}^{n}\left(\exp(q^i-q^{i+1}+p_i)+\exp(p_i)\right)\tag{3.3}$$

and

$$\begin{aligned}H_1 &= \sum_{i=1}^{n}\left(\frac{1}{2}\left(\exp(q^i-q^{i+1}+p_i)+\exp(p_i)\right)^2\right. &(3.4)\\ &\left.+\exp(q^{i-1}-q^i+p_{i-1})\left(\exp(q^i-q^{i+1}+p_i)+\exp(p_i)\right)\right), &(3.5)\end{aligned}$$

where, by convention, $q^0=-\infty$ and $q^{n+1}=+\infty$.

Since Λ_0 is a non-degenerate Poisson tensor on $\mathbb{R}^{2n}$, the triple $(\mathbb{R}^{2n},\Lambda_0,R)$ is a Poisson-Nijenhuis manifold, with $R=\Lambda_1^{\#}\circ(\Lambda_0^{\#})^{-1}$. On the other hand, if we take

$$X_0=\sum_{i=1}^{n}\frac{\partial}{\partial p_i},\tag{3.6}$$

we compute

$$\mathcal{L}(X_0)\Lambda_0=-\Lambda_0,\quad \mathcal{L}(X_0)\Lambda_1=0\quad \mathcal{L}(X_0)H_0=H_0.\tag{3.7}$$

So, we may apply Oevel's theorem to generate a hierarchy $(X_k)_{k\in\mathbb{N}}$ of mastersymmetries, satisfying the following conditions, with $Y_k=\Lambda_k^{\#}(dH_0)$:

$$[X_k,X_l] = (l-k)X_{k+l};\tag{3.8}$$

$$[X_k,Y_l] = lY_{k+l};\tag{3.9}$$

$$\mathcal{L}(X_k)\Lambda_l = (l-k-1))\Lambda_{k+l}; \tag{3.10}$$
$$\mathcal{L}(X_k)H_l = (1+l+k))H_{k+l}. \tag{3.11}$$

Thus we obtain a multi-Hamiltonian formulation for the RTL.

Let us consider the following map,

$$\pi : (q^1, \ldots, q^n, p_1, \ldots, p_n) \in \mathbb{R}^{2n} \to (c_1, \ldots, c_{n-1}, d_1, \ldots, d_n) \in \mathbb{R}^{2n-1},$$

with $c_i = \exp(q^i - q^{i-1} + p_i)$ and $d_i = \exp(p_i)$.

The Poisson tensors Λ_0 and Λ_1 on $\mathbb{R}^{2n}$ reduce by π onto Poisson tensors $\bar{\Lambda}_0$ and $\bar{\Lambda}_1$ on $\mathbb{R}^{2n-1}$, that are compatible. The Hamiltonians H_0 and H_1 can also be reduced by π. However, we may not reduce the recursion operator R and so we have no recursion operator on $\mathbb{R}^{2n-1}$. In spite of that, the mastersymmetries $X_k = R^k X_0$ are projectable vector fields by π. In fact, relations (3.8)-(3.11) still hold for the reduced objects. This provides a way of generating multi-Hamiltonian formulations for the RTL on $\mathbb{R}^{2n-1}$.

II) Bi-Hamiltonian manifolds — a generalization

4 Compatible Jacobi manifolds

The notion of *Jacobi manifold* was introduced by A. Lichnerowicz [4]. Let us recall that a *Jacobi manifold* is a triple (M, C, E) where M is a smooth manifold, C is a bivector on M and E is a vector field on M such that

$$[E, C] = 0 \quad \text{and} \quad [C, C] = 2E \wedge C. \tag{4.1}$$

If (M, C, E) is a Jacobi manifold, $C^\infty(M, \mathbb{R})$ has the structure of a *local Lie algebra* [2] given by the *Jacobi bracket*:

$$\{f, g\} = C(df, dg) + f(E.g) - g(E.f), \quad f, g \in C^\infty(M, \mathbb{R}). \tag{4.2}$$

With each function $f \in C^\infty(M, \mathbb{R})$, we may associate a vector field on M,

$$X_f = C^\#(df) + fE, \tag{4.3}$$

which is called the *Hamiltonian vector field* associated with f.

Let M be a smooth manifold equipped with two Jacobi structures, (C_1, E_1) and (C_2, E_2).

Definition 11 *The Jacobi manifolds (M, C_1, E_1) and (M, C_2, E_2) are said to be* compatible *if $(C_1 + C_2, E_1 + E_2)$ is again a Jacobi structure on M.*

All the results that follow can be found in [7]. Next theorem states the necessary and sufficient conditions for two Jacobi structures, defined on the same manifold, to be compatible.

Theorem 12 *The Jacobi manifolds (M, C_1, E_1) and (M, C_2, E_2) are compatible if and only if*

$$\begin{aligned}
&\textbf{(i)} \quad [C_1, C_2] = E_1 \wedge C_2 + E_2 \wedge C_1;\\
&\textbf{(ii)} \quad [E_1, C_2] + [E_2, C_1] = 0.
\end{aligned}$$

The compatibility of two Jacobi structures defined on the same manifold can be stated using some equivalent properties.

Theorem 13 *Let (M, C_1, E_1) and (M, C_2, E_2) be two Jacobi manifolds, whose Jacobi brackets are denoted respectively by $\{,\}_1$ and $\{,\}_2$. Given a function $f \in C^\infty(M, \mathbb{R})$, we denote by X_f and Y_f the Hamiltonian vector fields on M associated with f, with respect to the Jacobi structures (C_1, E_1) and (C_2, E_2), respectively.*

The following properties 1, 2, 3 and 4 are equivalent:

(i) *(a)* $[C_1, C_2] = E_1 \wedge C_2 + E_2 \wedge C_1$,

(b) $[E_1, C_2] + [E_2, C_1] = 0$;

(ii) for all $f, g, h \in C^\infty(M, \mathbb{R})$,

$$\begin{aligned}
&\{f, \{g, h\}_1\}_2 + \{g, \{h, f\}_1\}_2 + \{h, \{f, g\}_1\}_2 +\\
&+\{f, \{g, h\}_2\}_1 + \{g, \{h, f\}_2\}_1 + \{h, \{f, g\}_2\}_1 = 0;
\end{aligned}$$

(iii) for all $f, g \in C^\infty(M, \mathbb{R})$,

$$[X_f, Y_g] + [Y_f, X_g] - X_{\{f,g\}_2} - Y_{\{f,g\}_1} = 0;$$

(iv) for all $f \in C^\infty(M, \mathbb{R})$,

(a) $\mathcal{L}(Y_f)C_1 + \mathcal{L}(X_f)C_2 + (E_2.f)C_1 + (E_1.f)C_2 = 0$,

(b) $[X_f, E_2] + [Y_f, E_1] + X_{(E_2.f)} + Y_{(E_1.f)} = 0$.

Let (M, C_1, E_1) and (M, C_2, E_2) be two Jacobi manifolds. Let us take the corresponding associated homogeneous Poisson manifolds, (P, Λ_1) and (P, Λ_2), with $P = \mathbb{R} \times M$ and

$$\Lambda_i = e^{-t}\left(C_i + \frac{\partial}{\partial t} \wedge E_i\right), \quad i = 1, 2. \tag{4.4}$$

Proposition 3 *The Jacobi manifolds* (M, C_1, E_1) *and* (M, C_2, E_2) *are compatible if and only if the Poisson tensors* Λ_1 *and* Λ_2 *on* P *are compatible.*

Let us recall that if (M, C, E) is a Jacobi manifold with Jacobi bracket denoted by $\{,\}$ and if $h \in C^\infty(M, \mathbb{R})$ is a smooth function that never vanishes on M , we may define a new Jacobi bracket $\{,\}_h$ on M:

$$\{f, g\}_h = \frac{1}{h}\{hf, hg\}, \quad f, g \in C^\infty(M, \mathbb{R}). \tag{4.5}$$

The new Jacobi structure (C_h, E_h) on M is given by

$$C_h = hC, \quad E_h = C^\#(dh) + hE. \tag{4.6}$$

In such a case, the two Jacobi structures (C, E) and (C_h, E_h) on M are said to be *conformally equivalent.*

The next two Propositions show the relationship between the compatibility of Jacobi manifolds and the conformal equivalence of their Jacobi structures.

Proposition 4 *Let* (M, C, E) *be a Jacobi manifold and let* $h, k \in C^\infty(M, \mathbb{R})$ *be two distinct functions that never vanish on* M. *If* (C_h, E_h) *and* (C_k, E_k) *are two Jacobi structures on* M, *that are conformally equivalent to* (C, E), *then* (C_h, E_h) *and* (C_k, E_k) *are compatible, where* $C_h = hC$, $C_k = kC$, $E_h = hE + [C, h]$ *and* $E_k = kE + [C, k]$.

In particular, any Jacobi structure on M *which is conformally equivalent to* (C, E), *is compatible with* (C, E).

Proposition 5 *Let* (M, C_1, E_1) *and* (M, C_2, E_2) *be two compatible Jacobi manifolds. Then, for any function* $h \in C^\infty(M, \mathbb{R})$ *that never vanishes on* M, *the Jacobi manifolds* (M, C_{1h}, E_{1h}) *and* (M, C_{2h}, E_{2h}), *with* $C_{ih} = hC_i$ *and* $E_{ih} = hE_i + [C_i, h]$, $i = 1, 2$, *are compatible.*

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