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Huge Singly Periodic Minimal Surfaces in $\mathbf{R}^3$ and Symmetries

Francisco Martin and Domingo Rodriguez

First of all, we are going to start by giving a brief outline of the development of SPMS [1]. During the last century or more the only known SPMS with infinite number of ends

were the Riemann examples.

At the begining of the 90's, a large quantity of new examples appeared thanks to the work of Callahan, Hoffman, Meeks, Wolgemuth, among others. A problem of classification then arose due to the appearance of so many of these new examples. In this line of work we have obtained a uniqueness theorem for a discrete family of SPMS constructed by Callahan, Hoffman and Meeks in 1990.

But firstly we need to briefly summarize the principal results for complete embedded SPMS.

Let $\widetilde{x} : \widetilde{M} \longrightarrow \mathbf{R}^3$ be a proper minimal embedding of a surface $\widetilde{M}$ in three-dimensional Euclidean space, invariant under a cyclic group of translations $\mathcal{T}$. Using isothermal parameters, $\widetilde{M}$ has a natural **complex structure** in which the stereographic projection of the Gauss map of $\widetilde{x}$, $\widetilde{g}$, is a meromorphic function.

There is a holomorphic one form, $\widetilde{\omega}$, associated to this function, and both of them determine the minimal immersion as follows:

$$\widetilde{x} = \mathrm{Real}\left(\int (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)\right)$$

where

$$\widetilde{\phi}_1 = \frac{1}{2}(1-\widetilde{g}^2)\widetilde{\omega}, \quad \widetilde{\phi}_2 = \frac{i}{2}(1+\widetilde{g}^2)\widetilde{\omega}, \quad \widetilde{\phi}_3 = \widetilde{g}\widetilde{\omega}, \tag{0.1}$$

are holomorphic one-forms on $\widetilde{M}$ satisfying

$$\sum_{i=1}^{3} |\widetilde{\phi}_j|^2 \neq 0.$$

[1] Singly Periodic Minimal Surfaces

We label the pair $(\widetilde{g}, \widetilde{\omega})$ as the **Weierstrass representation** of $\widetilde{M}$.

We can work with the quotient $M = \widetilde{M}/\mathcal{T}$, and so we have a minimal immersion

$$x : \widetilde{M}/\mathcal{T} \longrightarrow \mathbf{R}^3/\mathcal{T}.$$

Since both the Gauss map and the holomorphic one form are translation-invariant, they can be induced to the quotient. If we label this new Weierstrass data as (g, ω), we can then reobtain the immersion x as follows:

$$x = \text{Real}\left(\int (\phi_1, \phi_2, \phi_3)\right)$$

labeling

$$\phi_1 = \frac{1}{2}(1 - g^2)\omega, \quad \phi_2 = \frac{i}{2}(1 + g^2)\omega, \quad \phi_3 = g\omega. \tag{0.2}$$

In particular, the period group

$$\Lambda = \left\{\text{Real}\left(\int_\gamma (\phi_1, \phi_2, \phi_3)\right) \; : \; [\gamma] \in H_1(M, \mathbf{z})\right\}$$

coincides with $\mathcal{T}$. The **total curvature** of M is denoted by

$$\mathcal{C}(M) = \int_M K \, dA,$$

where K represents the Gauss curvature. Meeks and Rosenberg proved that a properly embedded minimal surface in a complete non-simply connected flat three-manifold has finite total curvature, if and only if it has finite topology. Then, M is conformally diffeomorphic to a compact Riemann surface punctured in a finite number of points,

$$M \equiv \overline{M} - \{P_1, \ldots, P_r\}.$$

These points are called **the ends** of M.

Moreover, the Weierstrass data (g, ω) extends meromorphically to $\overline{M}$. Therefore, the degree of g is well defined and

$$\mathcal{C}(M) = -4\,\pi \deg(g).$$

A surprising relationship exists between the topology of a properly embedded periodic minimal surface in $\mathbf{R}^3$ and the behaviour of its ends. This was studied by Callaham, Hoffman, and Meeks, who proved the following theorem:

Theorem 1 (Callahan-Hoffman-Meeks) *Suppose $\widetilde{M}$ is a properly embedded minimal surface in $\mathbf{R}^3$, with more than one end, whose symmetry group is infinite. Then either $\widetilde{M}$ is a catenoid, or:*

(i) $\widetilde{M}$ *has an infinite number of ends;*

(ii) $\widetilde{M}$ *is invariant under a screw motion* $\mathcal{S}$*;*

(iii) all annular ends of $\widetilde{M}$ *are flat ends;*

(iv) the total curvature of $M = \widetilde{M}/\mathcal{S}$ *is finite if and only if* M *has finite topology, in which case*

$$\mathcal{C}(M) = 2\pi\,(\chi(M) - r(M))$$

where $r(M)$ *is the number of ends of* M*.*

We write the isometry group of $\widetilde{M}$ as $Iso(\widetilde{M})$. The subgroup of $Iso(\widetilde{M})$, which consists of isometries that are the rectriction of a rigid motion in $\mathbf{R}^3$, leaving $\widetilde{x}(\widetilde{M})$ invariant, is denoted by $Sym(\widetilde{M})$. Choi, Meeks, and White [4] proved the following:

Theorem 2 (Choi-Meeks-White) *A properly embedded minimal surface* $\widetilde{M}$ *in* $\mathbf{R}^3$ *which has more than one end is minimally rigid.*

In particular $Sym(\widetilde{M}) = Iso(\widetilde{M})$*.*

The first examples of periodic minimal surfaces with more than one end were discovered by **Riemann** in 1867. He constructed a one-parametric family of properly embedded minimal surfaces $\{R_\lambda : \lambda > 0\}$ which are invariant under a translation, $\mathcal{T}$. Furthermore, the quotient $R_\lambda/\mathcal{T}$ has genus one, two ends, and total curvature -8π. Recently, Meeks, Pérez, and Ros have characterized Riemann's examples as the only embedded minimal tori in $\mathbf{R}^3/\mathcal{T}$.

These surfaces were generalized by **Callahan**, **Hoffman**, and **Meeks** in 1989. The way in which they carried out this generalization was by increasing the genus and the symmetry of the surfaces at the same time. So they found examples of complete embedded minimal surfaces, with an infinite number of annular ends. This discrete family consists of a surface M_k, for each integer $k > 0$, which is invariant under a cyclic group of vertical translations, $\mathcal{T}$. The quotient $M_k/\mathcal{T}$ has:

- genus $2k+1$;
- two flat ends;
- $8(k+1)$ symmetries.

We have studied the surfaces M_k in depth, obtaining the following classification theorem:

Theorem 3 *Let* $\widetilde{M}$ *be a properly embedded minimal surface in* $\mathbf{R}^3$ *satisfying the following:*

(i) $\widetilde{M}$ *has an infinite number of annular ends;*

(ii) $\widetilde{M}$ *is invariant under a cyclic group of translations* $\mathcal{T}$*;*

(iii) $\widetilde{M}/\mathcal{T}$ *has genus* ϱ, $\varrho \geq 2$, *and* 2 *ends;*

(iv) $Iso(\widetilde{M}/\mathcal{T})$ *has at least* $4(\varrho+1)$ *elements.*

Then ϱ *is odd and* $\widetilde{M}$ *is, up to homotheties and rigid motions, the Callahan-Hoffman-Meeks example of genus* ϱ.

The main idea in the proof of this theorem is to use the great number of symmetries of the surface in order to determine the underlying complex structure for the quotient $\widetilde{M}/\mathcal{T}$. Furthermore, this abundance of symmetries permits the reduction of the number of possibilities for the Weierstrass representation for these kinds of surfaces.

We have included an outline of the proof of Theorem 3.

Let $\widetilde{x} : \widetilde{M} \longrightarrow \mathbf{R}^3$ be a properly embedded minimal surface in $\mathbf{R}^3$ satisfying:

(i) $\widetilde{M}$ has an infinite number of annular ends;

(ii) $\widetilde{M}$ is invariant under a cyclic group of translations, $\mathcal{T}$;

(iii) $\widetilde{M}/\mathcal{T}$ has genus ϱ, $\varrho \geq 2$, and two ends;

(iv) $Iso\left(\widetilde{M}/\mathcal{T}\right)$ has at least $4(\varrho+1)$ elements.

From Theorem 1 we deduce that the ends of $\widetilde{M}$ are flats and the total curvature of $M = \widetilde{M}/\mathcal{T}$ is $-4\pi(\varrho+1)$.

As we have mentioned above, M has the conformal type of a compact Riemann surface, $\overline{M}$, punctured in two points, P_1 and P_2, which correspond to the ends. Meeks and Rosenberg [12] proved that the Weierstrass data (g,η) extends meromorphically to $\overline{M}$. Since the ends of M are parallel we can suppose that, up to a rigid motion of $\mathbf{R}^3$, $g(P_i) \in \{0,\infty\}$, $i = 1, 2$.

In the following, the generator of $\mathcal{T}$ is denoted by τ.

If $\widetilde{A} \in Sym(\widetilde{M})$, then a simple argument of Linear Algebra gives $\widetilde{A}\circ\tau\circ\widetilde{A}^{-1} = \tau^{\varepsilon}$, where $\varepsilon \in \{1,-1\}$. In particular $\mathcal{T}$ is a normal subgroup of $Sym(\widetilde{M})$.

If we write $\Delta = Sym(\widetilde{M})/\mathcal{T}$, then as a consequence of the rigidity of $\widetilde{M}$ we obtain $Iso(M) = \Delta$. A symmetry of M induces, in a natural way, a conformal automorphism of M which extends to $\overline{M}$, leaving the set $\{P_1, P_2\}$ invariant. Since the subgroup of holomorphic transformations has index one or two in Δ, then Hurwitz's Theorem (see [5]) implies that Δ is finite. For any $\widetilde{A} \in Sym(\widetilde{M})$ its class in Δ is denoted by A. In the same way, we write the class of $\widetilde{Q} \in \widetilde{M}$ in

M as Q. If we label $\mathcal{H}$ as the subgroup of the holomorphic transformations of Δ and $\mathcal{G} = \{A \in \mathcal{H} \ / \ A(P_1) = P_1\}$, it is obvious that

$$[\Delta : \mathcal{H}] \leq 2, \quad [\mathcal{H} : \mathcal{G}] \leq 2 \tag{0.3}$$

If we take D, a conformal disk centered at P_1, invariant under $\mathcal{G}$, then $\{A_{|D} \ / \ A \in \mathcal{G}\}$ is a finite group of biholomorphisms of the disk fixing the origin, and so $\mathcal{G}$ is cyclic. Let J be a generator of $\mathcal{G}$. As the normal vectors at the ends are vertical, then any $\widetilde{J}$, in class J, maps a vertical line into another vertical line.

Under these conditions, and without loss of generality, we can assume that there exists a $\widetilde{J}$ on J which preserves the x_3-axis and fixes the origin. Taking (0.3) and hypothesis (iv) into account, we have $ord(\widetilde{J}) \geq \varrho + 1$, $\varrho \geq 2$. Then we can assume that either $\widetilde{J}$ is a counterclockwise rotation around the x_3-axis by angle $2\pi/ord(\widetilde{J})$ or $ord(\widetilde{J})$ is even and $\widetilde{J}$ is a counterclockwise rotation around the x_3-axis by angle $2\pi/ord(\widetilde{J})$ followed by a symmetry with respect to the (x_1, x_2)-plane. Since the translation vector of τ, $\vec{b}$, is a eigenvector of $\widetilde{J}$, we deduce that $\vec{b}$ has the direction of the x_3-axis. Up to an homothety, we can assume that $\tau(x) = x + (0, 0, 2)$.

For a point $Q \in \overline{M}$ we define:

- the isotropy group of Q in $\mathcal{G}$ as $\Im(Q) = \{A \in \mathcal{G} \ / \ A(Q) = Q\}$,
- $\mu(Q) = |\Im(Q)|$,
- the orbit of Q associated to $\mathcal{G}$ as $orb(Q) = \{Q, J(Q), \ldots, J^{|\mathcal{G}|-1}(Q)\}$.

Note that $\sharp(orb(Q)) = \frac{|\mathcal{G}|}{\mu(Q)}$. Using the Riemann-Hurwitz formula for the branched covering $\zeta : \overline{M} \to \overline{M}/\langle J \rangle$, we obtain

$$\chi(\overline{M}) = ord(J) \cdot \chi(\overline{M}/\langle J \rangle) - \left(2(ord(J) - 1) + \sum_{Q \in M} (\mu(Q) - 1) \right) \tag{0.4}$$

Let $Q_1, \ldots, Q_s$ be the points of M whose orbits are non-trivial. If we write $m_i = \sharp(orb(Q_i))$, $i = 1, \ldots, s$, then equality (0.4) becomes:

$$2(ord(J) - \varrho) + \sum_{i=1}^{s} (ord(J) - m_i) = ord(J) \cdot \chi(\overline{M}/\langle J \rangle) \tag{0.5}$$

Taking into account that $ord(J) \geq \varrho + 1$, one obtains $\chi(\overline{M}/\langle J \rangle) > 0$, that is to say, $\overline{M}/\langle J \rangle$ is the Riemann sphere. Thus, substituting $\chi(\overline{M}/\langle J \rangle)$ for 2 in (0.5), we have

$$\sum_{i=1}^{s} (ord(J) - m_i) = 2\varrho \tag{0.6}$$

As $\widetilde{x}$ is an embedding, one has $m_i \leq 2$, $\forall i \in \{1, \ldots, s\}$.

We now consider the behaviour of the Gauss map at an end and at a point on the x_3-axis in the presence of a rotational symmetry $\widetilde{R}$ of $\widetilde{M}$. A catenoid end still produces a simple zero or a simple pole in g. However, a flat end must correspond to a zero or a pole of order $jm+1$, where j is a positive integer and m is the order of $\widetilde{R}$. On the other hand, if $\widetilde{Q} \in \widetilde{M}$ is a point of the x_3-axis, then the order of $\widetilde{Q}$ as zero or pole of g is $jm-1$, for a positive integer j (for details see [2]).

Using this fact and the equality $deg(g) = \varrho+1$ (remember $\mathcal{C}(M) = -4\pi(\varrho+1)$), we are able to obtain

(i) $\widetilde{J}$ is a rotation followed by a symmetry, and so $ord(J)$ is even.

(ii) The order of J is $\varrho+1$. In particular, ϱ is odd.

(iii) $s = 2$ and $m_1 = m_2 = 1$.

For details see [9].

In the following, we have written $\varrho = 2k+1$, where k is a positive integer.

Note that this new notation leads to $deg(g) = 2(k+1)$ and $ord(J) = 2(k+1)$.

Summarizing all the information and using the above ideas about the behaviour of the Gauss map in the presence of a rotational symmetry, it is possible to deduce that the divisor associated to g as a meromorphic function on $\overline{M}$ is, up to relabelings,

$$[g] = \frac{P_1^{k+2} \cdot Q_1^k}{P_2^{k+2} \cdot Q_2^k} \tag{0.7}$$

In the following we consider the fundamental piece of $\widetilde{M}$ in the slab

$$\{(x_1, x_2, x_3) \in \mathbf{R}^3 \;/\; -1 < x_3 \leq 1\}$$

Let $\widetilde{Q}_1$ and $\widetilde{Q}_2$ be the unique points in the fiber of Q_1 and Q_2, respectively, which belong to the above slab. As $\widetilde{J}$ is a rotation followed by a symmetry with respect to the plane $x_3 = 0$ and J fixes the ends P_1 and P_2, then the ends are asymptotic to the planes $x_3 = 0$ and $x_3 = 1$, respectively. In the same way, as $J(Q_i) = Q_i$, $i = 1, 2$, if we write $\widetilde{x}(\widetilde{Q}_i) = (0, 0, q_i)$, $q_i \in]-1, 1]$, $i = 1, 2$, then

$$(0,0,q_i) = \widetilde{J}(0,0,q_i) + (0,0,2n_i) = (0,0,2n_i - q_i), \quad n_i \in \mathbf{z}, \; i = 1, 2.$$

which implies $q_i = n_i \in \mathbf{z} \cap]-1, 1]$ and so

$$\{\widetilde{x}(\widetilde{Q}_1), \widetilde{x}(\widetilde{Q}_2)\} = \{(0,0,0), (0,0,1)\}.$$

This fact implies that any symmetry of $\overline{M}$ which fixes the ends must fix the points Q_1 and Q_2, and any symmetry of $\overline{M}$ which interchanges the ends must interchange the points Q_1 and Q_2.

Using similar techniques to those introduced by Hoffman and Meeks in [6], we obtain two possible conformal structures for $\overline{M}$:

(A) The surface $\overline{M}$ is conformally diffeomorphic to

$$\overline{M}_{k\,a} = \left\{ (z,w) \in (\mathbf{C} \cup \{\infty\})^2 \;:\; w^{2(k+1)} = \frac{z(z-a)}{az-1} \right\}$$

Furthermore, $P_1 = (0,0)$, $P_2 = (\infty,\infty)$, $Q_1 = (a,0)$, $Q_2 = (1/a,\infty)$ and $J(z,w) = (z, e^{\frac{\pi i}{k+1}} w)$. In this case the Weierstrass data are

$$M_{k\,a} = \overline{M}_{k\,a} - \{P_1, P_2\}, \quad g = A\frac{w^k}{z}, \quad g\eta = B\frac{dz}{(az-1)w^{k+1}}$$

where $A \in \mathbf{R} - \{0\}$ and $B \in \mathbf{C} - \{0\}$.

(B) The surface $\overline{M}$ is conformally diffeomorphic to

$$\overline{N}_{k\,a} = \left\{ (z,w) \in (\mathbf{C} \cup \{\infty\})^2 \;:\; w^{2(k+1)} = \frac{z(z-a)}{az+1} \right\}$$

Furthermore, $P_1 = (0,0)$, $P_2 = (\infty,\infty)$, $Q_1 = (a,0)$, $Q_2 = (-1/a,\infty)$ and $J(z,w) = (z, e^{\frac{\pi i}{k+1}} w)$. Now, the Weierstrass data are

$$N_{k\,a} = \overline{N}_{k\,a} - \{P_1, P_2\}, \quad g = A\frac{w^k}{z}, \quad g\eta = B\frac{dz}{(az+1)w^{k+1}}$$

where $A \in \mathbf{R} - \{0\}$ and $B \in \mathbf{C} - \{0\}$.

The first one corresponds to the Callahan-Hoffman-Meeks example and it is studied in [9, sec. 4], where we give a new approach to the solution of the period problem and we prove that this solution is unique. The second one does not give any example, because we have proved that the period problem has no solutions for this kind of surfaces (see [9, sec. 5]).

Finally, we would like to point out that the hypotheses of Theorem 3 are essential and cannot be changed, as there are examples (see [7] and [8]) which show this fact.

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Francisco Martin
Departamento de Geometria y Topologia
Facultad de Ciencias
Universidad de Granada
18071 Granada
Spain
Email: fmartin@goliat.ugr.es