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On a problem of Osserman concerning the Jacobi operator

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Abstract:

We report on some recent results on the study of semi-Riemannian manifolds whose Jacobi operators have constant eigenvalues. Special attention is paid to the indefinite case, mainly to metrics of signature $(++--)$.

1 Introduction

The study of the Jacobi operator is a central topic in semi–Riemannian geometry. The geodesic deviation is measured by this part of the curvature tensor and, because of its basic role in the Jacobi equation, many geometric properties can be derived from the knowledge of the Jacobi operators. Since for each vector x, the Jacobi operator $R_x = R(\cdot\,, x)x$ is a self–adjoint operator, the study of its eigenvalues is of special interest. (Note that, in the Riemannian case, these eigenvalues indicate the extreme values of the sectional curvature and, in the Lorentzian case, they play a fundamental role in the construction of mathematical models in General Relativity [16]). It is worthwile to emphasize at this point that the eigenvalues λ_i of the Jacobi operator $R_x(p) : T_pM \to T_pM$ depend both on the point $p \in M$ and the direction $x \in T_pM$, i.e., $\lambda_i(p, x)$.

A Riemannian manifold (M, g) is said to be an *Osserman space* if the eigenvalues of the Jacobi operators are constant on the unit sphere bundle SM of M. In the indefinite setting, a semi–Riemannian manifold (M, g) is said to be *spacelike* (resp., *timelike*) *Osserman* if the (possibly complex) eigenvalues of the Jacobi operators are constant on the unit pseudo–sphere bundle S^+M (resp., pseudo–hyperbolic bundle S^-M) of M.

The purpose of this note is to present a brief exposition of some recent progresses on the study of the Osserman problem, with special attention to the indefinite cases, mainly the Lorentzian and $(++--)$–metrics.

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2 Osserman Riemannian spaces

The eigenvalues of the Jacobi operators on a two–point homogeneous space are known to be constant on the unit sphere bundle of the manifold, i.e., they are Osserman spaces. Osserman conjectured that the converse also holds [17]. This was proved by Chi in many cases, as shown in the following

Theorem 1 [5] *Let (M^n, g) be an n–dimensional Riemannian manifold, with $n \neq 4k$ $(k > 1)$. If M is Osserman, then it is locally isometric to a space of constant sectional curvature or to a Kähler manifold of constant holomorphic sectional curvature.*

The simplest case is the one corresponding to odd–dimensional manifolds. Roughly speaking, the proof by Chi can be described as follows. If λ denotes an eigenvalue of the Jacobi operator R_x, it can be used to define a smooth field of planes on the sphere S^{2k} as follows

$$\mathcal{D}_\lambda : x \in S^{2k} \mapsto \mathcal{D}_\lambda(x) \subset T_x S^{2k},$$

where $\mathcal{D}_\lambda(x) = \mathrm{Ker}(R_x - \lambda Id)$. Therefore, $\mathcal{D}_\lambda$ is a field of planes on S^{2k} whose dimension equals the multiplicity of λ. There are well–known topological restrictions to the existence of fields of planes on spheres. In particular, since no even–dimensional spheres admit nonsingular fields of planes, the multiplicity of λ must be zero or $2k = \dim M - 1$. This shows that the Jacobi operators have only one eigenvalue and thus M is a space of constant sectional curvature as a consequence of the Schur lemma.

Motivated by the ideas in [5], a possible two-step solution to the Osserman problem is suggested in [12]. The first step should consist in the characterization of the possible curvature tensors which are Osserman. The second one, based on the use of the second Bianchi identity, should allow to show the local symmetry of the space. In understanding the first step, there exists a clear relation between Osserman curvature tensors and Clifford algebras. This was originally pointed out by Gilkey [11] who constructed Riemannian manifolds whose curvature tensor is Osserman at a point but not corresponding to any rank–one symmetric space.

A Riemannian manifold is called a *pointwise Osserman space* if the eigenvalues of the Jacobi operators are independent of the unit vector x at each point $m \in M$, but they may change from point to point. It is worthwhile to note the existence of pointwise Osserman spaces which are not Osserman. We refer to [12] for some information about the pointwise Osserman condition and its relation with the global one.

3 Osserman Lorentzian spaces

Motivated by the special significance of the Jacobi operator in the mathematical formulation of General Relativity, in [8] and [9] the study of the Osserman problem was initiated in the Lorentzian case. A Lorentzian manifold is said to be *timelike Osserman at a point* $p \in M$ if the eigenvalues of the Jacobi operators are independent of the timelike unit vector $x \in T_pM$.

Now, if $\lambda_1, \ldots, \lambda_k$ denote the eigenvalues of the Jacobi operators corresponding to timelike unit vectors $x \in T_pM$, one easily recognizes that the sectional curvature

$$K(\pi = \langle\{x, y\}\rangle) = \frac{R(y, x, x, y)}{g(x, x)g(y, y) - g(x, y)^2}$$

is bounded from above and from below on nondegenerate planes with induced metric of signature $(+-)$. Then, it follows from a result of Kulkarni [14] (see also O'Neill [16]) that the sectional curvature is constant at the point $p \in M$. This shows that a Lorentzian manifold is timelike Osserman if and only if it is a space of constant sectional curvature.

The spacelike case was analyzed in [9] for $(n \leq 4)$–dimensional manifolds, and completely solved by Blažić, Bokan and Gilkey in [1]. Summarizing the results in [9] and [1], we have the following

Theorem 2 *Let (M, g) be a Lorentzian manifold of dim $M \geq 3$. The following conditions are equivalent to each other:*

(i) (M, g) is timelike Osserman at a point $p \in M$.

(ii) (M, g) is spacelike Osserman at a point $p \in M$.

(iii) The sectional curvature is constant at $p \in M$.

It also follows from the previous theorem that the pointwise and the global Osserman conditions are equivalent in the Lorentzian setting.

With respect to the study of the Jacobi operators corresponding to null directions, one has to introduce the following. Let $u \in T_pM$ be a null vector and define the *nondegenerate normal space* $\overline{u}^\perp$ by $\overline{u}^\perp = u^\perp / \, span\{u\}$, where $u^\perp$ is the orthogonal space to $span\{u\}$. Then the Jacobi operator R_u associated to u defines an endomorphism $\overline{R}_u : \overline{u}^\perp \mapsto \overline{u}^\perp$, where $\overline{R}_u \overline{z} = \pi(R(z, u)u)$. Here $\pi : u^\perp \mapsto \overline{u}^\perp$ denotes the canonical projection and $z \in u^\perp$ with $\pi(z) = \overline{z}$.

With the previous notation, we call a null vector $u \in T_pM$ *isotropic* if $\overline{R}_u = c_u \overline{Id}$, where $c_u \in \mathbf{R}$. Moreover, M is said to be *null isotropic* at $p \in M$ if every null vector $u \in T_pM$ is isotropic.

Let M be a Lorentzian manifold and let $z_p \in T_pM$ be an unit timelike vector. The *null congruence* $N(z_p)$ determined by z_p is defined by

$$N(z_p) = \{u \in T_pM / \langle u, u\rangle = 0 \text{ and } \langle u, z_p\rangle = -1\}.$$

We will say that M is *null Osserman with respect to* z_p if the characteristic polynomial of $\overline{R}_u$ is independent of $u \in N(z_p)$. (Note that this is equivalent to that the eigenvalues of $\overline{R}_u$ are independent of $u \in N(z_p)$ (counting multiplicities), since $\overline{R}_u$ is diagonalizable). M is said to be *pointwise null Osserman* if M is null Osserman with respect to an unit timelike vector z_p at each $p \in M$. Also, a pointwise null Osserman M is called *null Osserman* if the characteristic polynomial of $\overline{R}_u$ is independent of points $p \in M$.

The celestial sphere $S(z_p)$ of z_p is defined by

$$S(z_p) = \{x \in z_p^\perp / \langle x, x\rangle = 1\}.$$

Now, note that $N(z_p)$ is canonically diffeomorphic to $S(z_p)$ by the map

$$u \mapsto u - z_p = x.$$

Thus, by this identification, for each $u \in N(z_p)$, $\overline{u}^\perp$ can be identified with $T_x S(z_p)$. Hence, $\overline{R}_u : \overline{u}^\perp \mapsto \overline{u}^\perp$ can be used to define a linear function

$$\mathcal{R}_x : T_x S(z_p) \mapsto T_x S(z_p)$$

via the above identification. $\mathcal{R}_x$ can also be extended to a bundle homomorphism $\mathcal{R} : TS(z_p) \mapsto TS(z_p)$ by defining $\mathcal{R}(y) = \mathcal{R}_x(y)$, where $y \in T_x S(z_p)$. Note that $\overline{R}_u$ and $\mathcal{R}_x$ have the same characteristic polynomial and hence the same eigenvalues (counting multiplicities). Using this fact, the following result is obtained in [8].

Theorem 3 *Let M be a 4–dimensional globally null Osserman manifold. Then, the Jacobi operator satisfies $\overline{R}_u = c\overline{Id}$ for all null vectors $u \in N(z_p)$. Moreover,*

- (a) *If $c = 0$ then M is a Lorentzian space form.*
- (b) *If $c \neq 0$ then M is locally a Lorentzian warped product $I \times_{f^2} N$, where $I \subset \mathbf{R}$ and N is a real space form.*

4 Osserman manifolds with metric of signature $(++--)$

Semi–Riemannian metrics of signature $(++--)$ have been productively studied in recent years in relation to several problems, such as self–duality and anti–self–duality. Also in mathematical physics this type of indefinite metrics have appeared in various contexts, like in the procurement of the two–dimensional nonlinear principal sigma models (harmonic maps). (See [13] and the references therein). However, in comparison with Riemannian and Lorentzian metrics, the study of $(++--)$–metrics has not been developed so far. (See [15] for a study on the existence of such metrics on compact 4–manifolds). Therefore, in this section we will analyze the Osserman condition for such metrics. First of all, we will point out some significant differences with respect to both the Riemannian and Lorentzian Osserman spaces, by exhibiting a family of nonsymmetric Osserman metrics. Secondly we will mention about some recent results by Blažić, Bokan and Rakić on the classification of such manifolds.

4.1 Nonsymmetric examples

Next we will show some examples of Osserman semi–Riemannian manifolds with metric of signature $(++--)$ which are not locally symmetric. (Also their generalization to higher dimensions will be considered).

Note that since the metric is indefinite, the Jacobi operators can be nondiagonalizable. Therefore, we will pay special attention to those examples of Osserman manifolds with metric of signature $(++--)$ whose Jacobi operators are nondiagonalizable. At this point it is interesting to note that the Jacobi operators will be determined not only by the knowledge of their characteristic polynomial but also by their minimal polynomial. Moreover, note that although the Osserman condition ensures the constancy of the roots of the characteristic polynomial, the minimal polynomial may have nonconstant roots.

Let $M = \mathbf{R}^4$ be the 4–dimensional real Euclidean space, with usual coordinates (x_1, x_2, x_3, x_4). On this space we define the family of metrics $g_{(f_1,f_2)}$ given by

$$\begin{aligned} g_{(f_1,f_2)} &= x_3 f_1(x_1,x_2) dx_1 \otimes dx_1 + x_4 f_2(x_1,x_2) dx_2 \otimes dx_2 \\ &\quad + a\left\{ dx_1 \otimes dx_2 + dx_2 \otimes dx_1 \right\} \\ &\quad + b\left\{ dx_1 \otimes dx_3 + dx_3 \otimes dx_1 + dx_2 \otimes dx_4 + dx_4 \otimes dx_2 \right\}, \end{aligned} \tag{4.1}$$

where a and b are constant real numbers, $b \neq 0$, and f_1, f_2 are smooth real valued functions defined on $\mathbf{R}^2$ such that

$$\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} = 0. \tag{4.2}$$

After long calculations, we get that the Jacobi operator associated to a tangent vector $x = \sum_{i=1}^{4} \alpha_i \frac{\partial}{\partial x_i}$ has the form

$$R_x = \begin{pmatrix} A & 0 \\ * & {}^t A \end{pmatrix}$$

with respect to the basis $\left\{ \frac{\partial}{\partial x_i};\ i = 1, 2, 3, 4 \right\}$, where A is the 2×2–matrix

$$A = \frac{1}{2b} \frac{\partial f_1}{\partial x_2} \begin{pmatrix} \alpha_1 \alpha_2 & -\alpha_1^2 \\ \alpha_2^2 & -\alpha_1 \alpha_2 \end{pmatrix}.$$

We also determine the box $*$, although due to the complexity of its elements and since it is not the purpose of this note to give all the details, we do not include it here. However, it is interesting to note that, in many cases, that box is nonnull and, therefore, *the Jacobi operator is not diagonalizable* in those cases.

The previous expression for the Jacobi operators leads us to prove the following

Theorem 4 [10] *$(\mathbf{R}^4, g_{(f_1,f_2)})$ is an Osserman semi–Riemannian manifold, with metric of signature $(++--)$. Moreover, $(\mathbf{R}^4, g_{(f_1,f_2)})$ is a locally symmetric space if and only if, in addition to the condition (4.2), the functions f_1 and f_2 are solutions of the following system of partial differential equations:*

(i) $\displaystyle \frac{\partial^2 f_1}{\partial x_1 \partial x_2} + \frac{1}{2b} f_1 \frac{\partial f_1}{\partial x_2} = 0$

(ii) $\displaystyle \frac{\partial^2 f_1}{\partial x_2^2} + \frac{1}{2b} f_2 \frac{\partial f_1}{\partial x_2} = 0$

(iii) $\displaystyle \frac{1}{4b} a \left\{ 3 f_1 \frac{\partial f_1}{\partial x_2} - f_2 \frac{\partial f_1}{\partial x_1} - \frac{1}{b} f_1^2 f_2 \right\} - x_3 \left(\frac{\partial f_1}{\partial x_2} \right)^2 = 0$

(iv) $\displaystyle \frac{1}{4b} a \left\{ 3 f_2 \frac{\partial f_1}{\partial x_2} + f_1 \frac{\partial f_2}{\partial x_2} + \frac{1}{b} f_1 f_2^2 \right\} + x_4 \left(\frac{\partial f_1}{\partial x_2} \right)^2 = 0$

Remark 5 Although we impose the condition (4.2) from the beginning to simplify the calculations, it can be shown that such condition is, in fact, equivalent to that the manifold $(\mathbf{R}^4, g_{(f_1,f_2)})$ is Osserman.

Note that the previous process allows us to get examples of Osserman semi–Riemannian manifolds whose Jacobi operators are not diagonalizable. On the other hand, it is easy to choose functions f_1 and f_2 satisfying (4.2) which are not solutions of the system of differential equations in the previous theorem. Thus, we have many examples of Osserman semi–Riemannian manifolds with metric of signature $(++--)$ which are not locally symmetric.

Since the Jacobi operators are not necessarily diagonalizable, we study in [10] the behaviour of the minimal polynomial as follows

Theorem 6 [10] *The characteristic polynomial of the Jacobi operators is always $p_\lambda(R_x) = \lambda^4$, while the minimal polynomial $m_\lambda(R_x)$ satisfies:*

(i) $m_\lambda(R_x) = \lambda^3$ *at any point where* $\dfrac{\partial f_1}{\partial x_2} \neq 0$.

(ii) *At any point where* $\dfrac{\partial f_1}{\partial x_2} = 0$ *the function*

$$F(x_1, x_2, x_3, x_4) = 2b^2 \frac{\partial}{\partial x_2}\left(x_3 \frac{\partial f_1}{\partial x_2} - x_4 \frac{\partial f_1}{\partial x_1} \right) - a f_1 f_2$$

determines $m_\lambda(R_x)$ as follows:

(ii.a) $(\mathbf{R}^4, g_{(f_1,f_2)})$ *is of constant zero curvature ($m_\lambda(R_x) = \lambda$) at any point where F and* $\dfrac{\partial f_1}{\partial x_2}$ *vanish, and*

(ii.b) *the minimal polynomial $m_\lambda(R_x) = \lambda^2$ at those points where F is different from zero and* $\dfrac{\partial f_1}{\partial x_2} = 0$.

Remark 7 The above result allows us to construct examples of nonsymmetric Osserman manifolds with metric of signature $(++--)$ covering all the possible cases for the minimal polynomial (λ, λ^2, λ^3 and their combinations). We summarize them in the following table:

$\mathbf{M}$	$\mathbf{f_1(x_1, x_2)}$	$\mathbf{f_2(x_1, x_2)}$	$\mathbf{m_\lambda(R_x)}$
$\mathbf{R}^4$	x_2	$-x_1$	λ^3
$\mathbf{R}^4$	k_1	k_2	λ^2
$\mathbf{R}^4$	$x_1 x_2$	$-\frac{1}{2}x_1^2$	λ^3 if $x_1 \neq 0$ λ^2 if $x_1 = 0,\ x_4 \neq 0$ λ if $x_1 = x_4 = 0$
$\{p \in \mathbf{R}^4 / x_4(p) \neq 0\}$	$x_1 x_2$	$-\frac{1}{2}x_1^2$	λ^3 if $x_1 \neq 0$ λ^2 if $x_1 = 0$
$\mathbf{R}^4$	$x_1^3 x_2^3$	$-\frac{3}{4}x_1^4 x_2^2$	λ^3 if $x_1 x_2 \neq 0$ λ if $x_1 x_2 = 0$
$\mathbf{R}^4$	x_1	k	λ^2 if $x_1 \neq 0$ λ if $x_1 = 0$

Note that although the roots of the minimal polynomial may change from point to point, they are independent of the direction at each point of the manifold. This is due to the fact that the conditions in Theorem 6 do not depend on any particular direction.

Remark 8

1. It is still an open problem whether Riemannian Osserman spaces are locally homogeneous or not. In the Lorentzian case, since Osserman spaces have constant curvature they are locally symmetric, and thus, locally homogeneous. However, in the general setting, a semi–Riemannian Osserman space is not necessarily locally homogeneous, since they are even not curvature homogeneous as pointed out in the previous Remark.
2. Note that the product of the manifolds $(\mathbf{R}^4, g_{(f_1,f_2)})$ and the Euclidean spaces $\mathbf{R}^n_\nu$ allows us to get similar examples in higher dimensions and with metric of arbitrary signature. In particular, we get examples of Osserman semi–Riemannian manifolds of any signature (p, q), for $p, q \geq 2$, which are not locally symmetric (in fact, not even locally homogeneous).

4.2 A partial classification

We close this note with two results by Blažić, Bokan and Rakić which provide a solution to the Osserman problem for $(+ + - -)$–metrics under the natural assumption that the Jacobi operators are diagonalizable.

As a first step, in [2] it is obtained the possible expressions of the Jacobi operators. Indeed, for each unit (spacelike or timelike) vector x, there exists an orthonormal basis of the normal space $\langle x\rangle^\perp$ such that the expression of the Jacobi operator R_x has one of the following forms:

$$\textit{Type Ia}: \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

$$\textit{Type Ib}: \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \beta \neq 0$$

$$\textit{Type II}: \begin{pmatrix} \alpha - \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \alpha + \frac{1}{2} & 0 \\ 0 & 0 & \beta \end{pmatrix}, \quad \textit{or} \quad \begin{pmatrix} -\alpha + \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\alpha - \frac{1}{2} & 0 \\ 0 & 0 & \beta \end{pmatrix},$$

$$\textit{Type III}: \begin{pmatrix} \alpha & 0 & \frac{\sqrt{2}}{2} \\ 0 & \alpha & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \alpha \end{pmatrix},$$

corresponding to the cases when the Jacobi operator is diagonalizable ($Type\ Ia$), the characteristic polynomial of the Jacobi operator has a complex root ($Type\,Ib$), the minimal polynomial of the Jacobi operator has a double root ($Type\ II$) and the minimal polynomial of the Jacobi operator has a triple root ($Type\ III$).

Note that, in view of the examples in the Remark 7, different types of the Jacobi operator can be realized at different points of the manifold. Therefore, it seems natural to consider some further assumptions on the Jacobi operator in order to obtain a characterization of those Osserman $(++--)$–metrics.

Theorem 9 [2] *Let (M,g) be a connected 4–dimensional manifold equipped with a metric of signature $(++--)$. If (M,g) is spacelike or timelike Osserman and, moreover, all the Jacobi operators are assumed to be diagonalizable, then*

(i) (M,g) *is a space of constant curvature, or*

(ii) (M,g) *is an indefinite Kähler manifold of constant holomorphic sectional curvature, or*

(iii) (M,g) *is a para–Kähler manifold of constant paraholomorphic sectional curvature.*

Moreover, even since an Osserman space with metric of signature $(++--)$ is not necessarily locally symmetric, it is shown in [3] that the covariant derivative of the Jacobi curvature tensor is isotropic under the natural assumption of having constant roots for the minimal polynomial of the Jacobi operator.

Note that since both the characteristic and the minimal polynomials of the Jacobi operators are assumed to have constant roots, the form of the Jacobi operators should correspond to one of those listed at the beginning of this subsection. Then, the following result is obtained, by using the second Bianchi identity, after a long calculation:

Theorem 10 [3]
Let (M,g) be a connected 4–dimensional manifold equipped with a metric of signature $(++--)$. If (M,g) is Osserman and the minimal polynomial of the Jacobi operator has constant roots, then $\|\nabla R\| = 0$.

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