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Mean Curvature One Surfaces in Hyperbolic Space

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It is well known that mean curvature one surfaces in hyperbolic space can be isometrically associated to minimal surfaces in euclidean space. Two associated surfaces have the same first fundamental form and their second fundamental forms differ by a diagonal matrix ([L], [ET] .)

In 1987 R. Bryant ([B]) studied the surfaces of mean curvature one in hyperbolic space as local projections of null curves in the space of the 2×2 Hermitian symmetric matrices with its Cartan-Killing metric. More recently, Umehara and Yamada ([UY-1], [UY-2], [RUY]) produced explicit technical tools to construct examples of these surfaces by means of the null curves.

In a joint work with Célia Góes ([GG]) we have a local Weierstrass type representation for mean curvature one surfaces in the hyperbolic space. This representation depends on the hyperbolic Gauss map first introduced by Bryant. Some known examples are described and we also have a new one, associated to the minimal Bonnet surface constructed explicitly by the author and Barbara Nelli ([GN]).

In the classical minimal surfaces theory we have a classification theorem for non-planar minimal surfaces with planar curvature lines ([B], [D], [N]). These surfaces are the catenoid, Enneper's surface and Bonnet's surface. To study the hyperbolic associated surfaces, we observe that the planar curvature lines will be corresponding, and that some of the symmetries are preserved. Then we have the well known associated hyperbolic families of surfaces called *catenoid cousins, Enneper cousins,* ([B], [RUY]). We also exhibit a new family of surfaces that called *Bonnet cousins* ([GN]) and we obtain the hyperbolic surfaces associated to non-planar minimal surfaces with planar curvature lines.

A minimal surface can be analitycally described by means of its Weierstrass representation. We have an integral formula for the immersion

$$x : M \longrightarrow \mathbb{R}^3$$

given by:

$$x(p) = \frac{1}{2}\int_{p_o}^{p} ((1-g^2), i\,(1+g^2), 2g)\, f dz,\ \ p_o,\, p \in M$$

where M is noncompact Riemann surface, f is holomorphic and g in a meromorphic function satisfying a regularity condition. The function g describes the normal classical Gauss map of the surface.

The induced metric is given by

$$ds^2 = \frac{|f|^2}{4}(1+|g|^2)^2\, |dz|^2$$

The universal recovering of the catenoid can be described by

$$g(z) = e^{kz}, \quad f(z) = e^{-kz},$$

the Enneper surface is given by

$$g(z) = z, \quad f(z) = 1$$

and a family of Bonnet surfaces is corresponding to

$$g(z) = -i\,\alpha \tanh\left(\frac{z}{2}\right) \qquad \text{and} \qquad f(z) = \frac{2\,i}{\alpha}\cosh^2\left(\frac{z}{2}\right),$$

with

$$\alpha = \sqrt{\frac{1+a}{1-a}},\ \ z \in \mathbb{C}.$$

Inspired on Bryant's paper, Umehara and Yamada ([UY-1]) gave a representation for mean curvature one surfaces in hyperbolic space in terms of the functions f and g of the Weierstrass functions of the associated minimal surface.

Locally, the corresponding hyperbolic mean curvature one immersion $X : M \longrightarrow \mathbb{H}^3$ is given by means of the null curve $F : M \longrightarrow Sl(2, \mathbb{C})$,

$$F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C, D are holomorphic functions defined in a domain of $\mathbb{C}$, verifying the relations

$$AD - BC = 1, \qquad A'D' = B'C',$$

A, C are solutions of the differential equation

$$Y'' - \frac{f'}{f}Y' - fg'Y = 0 \tag{1}$$

and B, D are solutions of

$$Z'' - \frac{(g^2 f)'}{(g^2 f)} Z' - f g' Z = 0. \tag{2}$$

The induced metric of the hyperbolic immersion will be

$$ds^2 = |f|^2 \, (1 + |g|^2)^2 \, |dz|^2,$$

that is, the hyperbolic immersion X is isometric to the homothety $2x$ of the minimal immersion x given by the Weierstrass data f and g.

The called hyperbolic Gauss map is described in this context by the function

$$h(z) = \frac{A'(z)}{C'(z)} = \frac{B'(z)}{D'(z)}.$$

Revisiting Bryant's paper, in [GG] we study the geometry of the mean curvature one surfaces in the Minkowsky model for the hyperbolic space . This hyperbolic Gauss map behaves as the classical Gauss map for minimal surfaces in an euclidean space, that is, we have the following theorem first proved by Bryant:

Theorem 1. *Let $n : M \longrightarrow \partial D$ be the hyperbolic Gauss map of a surface $X : M \longrightarrow \mathbf{H}^3$, n non constant. The map $n : M \longrightarrow \partial D$ is conformal iff the immersion X either has mean curvature H constant and equal to one (in which case n preserves the orientation) or X is totally umbilic (in which case n reverses the orientation).*

We can now state our result:

Theorem 2. *Let $X : M \longrightarrow \mathbf{H}^3$ be a non-umbilic immersion in $\mathbf{H}^3$ with mean curvature one, $X(z) = (x_0(z), x_1(z), x_2(z), x_3(z))$ and*

$$n(z) = \left(1, \frac{2\Re\mathrm{e} h}{1+|h|^2}, \frac{2\Im\mathrm{m} h}{1+|h|^2}, \frac{|h|^2-1}{|h|^2+1}\right)$$

its hyperbolic Gauss map. The real functions $\phi_1(z) = x_0(z) + x_3(z)$ and $\phi_2(z) = x_0(z) - x_3(z)$ and the complex function $\phi_3(z) = x_1(z) + i\, x_2(z)$ satisfy

$$\begin{cases} \phi_1 \, \phi_2 = 1 + |\phi_3|^2 \\[2ex] \dfrac{\partial \phi_1}{\partial z} = h \dfrac{\partial \overline{\phi}_3}{\partial z} \\[2ex] \dfrac{\partial \phi_2}{\partial z} = \dfrac{1}{h} \dfrac{\partial \phi_3}{\partial z} \end{cases} \tag{$*$}$$

Conversely, given a holomorphic non-constant function $h : U \subset \mathbf{C} \longrightarrow \mathbf{C}$, *two real functions* ϕ_1 *and* ϕ_2 ($\phi_2 > 0$) *and a complex function* ϕ_3 *satisfying* $(*)$ *in the simply connected domain* U, *then*

$$X(z) = \left(\frac{\phi_1(z) + \phi_2(z)}{2}, \Re\mathrm{e}\, \phi_3(z), \Im\mathrm{m}\, \phi_3(z), \frac{\phi_1(z) - \phi_2(z)}{2} \right)$$

defines a conformal immersion in $\mathbf{H}^3$ *with constant mean curvature one and hyperbolic Gauss map* n *given by* h *as above.*

Remarks.

1. The compatibility condition for the two partial differential equations in $(*)$ is the same and writes

$$\Im\mathrm{m}\{\overline{h}\ \Delta\phi_3\} = 0. \tag{3}$$

2. An integral formula can be written:

$$X = \left(\Re\mathrm{e} \int_{z_0}^{z} \left(h \frac{\partial \overline{\phi}_3}{\partial z} + \frac{1}{h} \frac{\partial \phi_3}{\partial z} \right) dz, \Re\mathrm{e}\phi_3, \Im\mathrm{m}\phi_3, \Re\mathrm{e} \int_{z_0}^{z} \left(h \frac{\partial \overline{\phi}_3}{\partial z} - \frac{1}{h} \frac{\partial \phi_3}{\partial z} \right) dz \right).$$

Examples.

To obtain examples we have two possibilities:

- to solve the differential equations (1) and (2) given by Umehara and Yamada for the three choices of f and g corresponding to the catenoid, Enneper and Bonnet surfaces

or

- to integrate the system $(*)$ given in the representation theorem.

For the first two examples , the hyperbolic Gauss map was given by Bryant. For the third it was obtained by solving one of the differential equations ([GN]).

Given the hyperbolic Gauss map, to get the examples we can find solutions of (3):

$$\Im\mathrm{m}\,\{\overline{h}\,\Delta\phi_3\,\} = 0$$

and a linear combination of this solutions in order to have

$$\phi_1\phi_2 = 1 + |\phi_3|^2,$$

that is, in order to have the corresponding immersion in $\mathbf{L}^4$ contained in $\mathbf{H}^3$.

1. Given the function h we can search solutions as

$$\phi_3 = h(z)F(|z|^2)$$

with F a one real variable differentiable function. We have an immersion with constant mean curvature one solving $(*)$ with $h(z) = z$, $F_\alpha(t) = t^\alpha$ and

$$\phi_3(z) = h(z)\ [\ A\, F_\alpha(\,|z|^2) + B\, F_\beta(\,|z|^2)\ ].$$

The condition $\phi_1\,\phi_2 = 1 + |\phi_3|^2$ is verified under the restrictions:

$$\alpha + \beta = -1, \qquad A\,B\left(\frac{2\alpha+1}{\alpha\,(\alpha+1)}\right)^2 = 1, \qquad 2\alpha + 1 \neq 0.$$

These surfaces are called " cathenoid cousins".

2. The system $(*)$ also admits solutions as

$$\phi_3(z) = F(z)\,\overline{G(z)}$$

with F, G and h holomorphic functions. In this case if

$$F'(z) = h(z)G'(z)$$

the integrability condition (3) is verified. The surfaces called " Enneper Cousins" are corresponding to

$$\phi_3 = A\,F_1\,\overline{G}_1 + B\,F_2\,\overline{G}_2,$$

$$h(z) = \tanh\lambda z, \quad G_1'(z) = \cosh\lambda z, \quad G_2'(z) = z\cosh\lambda z, \quad AB = |\lambda|^6,\ \lambda \in \mathbf{C}.$$

More details can be find in [GG].

3. By taking

$$h(z) = \tanh\left(\frac{\sqrt{5}}{2}z\right) = \frac{\sinh(\alpha_1 z) + \sinh(\alpha_2 z)}{\cosh(\alpha_1 z) + \cosh(\alpha_2 z)},$$

with $\alpha_1 = \dfrac{\sqrt{5}-1}{2}$ and $\alpha_2 = \dfrac{\sqrt{5}+1}{2}$ and

$$\phi_3 = A\,F_1\,\overline{G}_1 + \ B\,F_2\,\overline{G}_2,$$

we can obtain the " Bonnet Cousins " ([GN]) corresponding to the solutions:

$$F_1(z) = \frac{1}{\alpha_1}\cosh(\alpha_1 z) + \frac{1}{\alpha_2}\cosh(\alpha_2 z)$$

$$G_1(z) = \frac{1}{\alpha_1}\sinh(\alpha_1 z) + \frac{1}{\alpha_2}\sinh(\alpha_2 z)$$
$$F_2(z) = \frac{1}{\alpha_1}\sinh(\alpha_1 z) - \frac{1}{\alpha_2}\sinh(\alpha_2 z)$$
$$G_2(z) = \frac{1}{\alpha_1}\cosh(\alpha_1 z) - \frac{1}{\alpha_2}\cosh(\alpha_2 z),$$
$$AB = \frac{1}{(\alpha_2^2 - \alpha_1^2)^2}.$$

Finally we can remark that minimal surfaces have some nice symmetries corresponding to geodesics that are also curvature lines or assymptotic lines ([DHKW], [N]).

In the first case we have a planar curve and the surface is invariant under an orthogonal symmetry with respect to this plane. In the second case, we have a straight line and the surface is symmetric with respect a reflexion about this line.

Some geodesics in a surface are fixed points of an isommetry of the surface metric. These curves are also geodesics on the associated isometric minimal and mean curvature one surfaces. From the fact that the second fundamental forms of these corresponding surfaces differ by a diagonal matrix, the curvature lines will be also corresponding. Earp and Toubiana ([ET]) showed that the planar euclidean reflexions correspond to hyperbolic planar reflexions. It follows that our surfaces wil have some hyperbolic symmetries.

We can also verify directly in the three families of examples that euclidean planar curvature lines correspond to hyperbolic curvature lines contained in umbilical surfaces.

It remains the question about a hyperbolic version of the classification theorem.

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