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The parallel group of a plane curve

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Abstract:
For any smooth immersion f of the circle in the plane, the parallel group $P(f)$ consists of all self-diffeomorphisms of the circle such that the normal lines at points of each orbit are parallel. The action of $P(f)$ on S^1 cannot be transitive. Thus, for example, $P(f) \neq SO(2)$. We construct examples where $P(f)$ contains a subgroup isomorphic to the group of self-diffeomorphisms of a closed interval (fixing the end-points), is isomorphic to the cyclic group $\mathbf{Z}_n$ for any $n \epsilon \mathbf{N}$, and to the dihedral group D_n, for any $n \epsilon \mathbf{N}$. If the curvature of f is nowhere zero, however, then $P(f)$ is cyclic of even order.

1 The parallel group

By a **plane curve** we mean any C^2 immersion of the circle S^1 into Euclidean plane E^2. It is convenient to regard S^1 as the set of real numbers, modulo 1. For any such plane curve f, and for any $t \epsilon S^1$, there is a unique affine line $\nu(t)$ normal to f at t, and a unique affine line $\tau(t)$ tangent to f at t.

We consider the set $P(f)$ of all diffeomorphisms $\delta : S^1 \to S^1$ such that, for all $t \epsilon S^1, \nu(t)$ is parallel to $\nu(\delta(t))$. Then $P(f)$ is a subgroup of the self-diffeomorphism group of S^1, which we call the **parallel group** of f. The following example shows that $P(f)$ can be very large.

(i) Let f be a plane curve such that for any $t \epsilon [0, 1/4], f(t) = (t, 1)$, and for any $t \epsilon [1/2, 3/4], f(t) = (3/4 - t, 3/4)$. Let k be any self- diffeomorphism of $[0, 1/4]$ such that $k(0) = 0, k(1/4) = 1/4$, and $k'(0) = k'(1/4) = 1$ and $k''(0) = k''(1/4) = 0$. Let δ be the self-diffeomorphism of S^1 that is k on $[0, 1/4]$, is the identity on $[1/4, 1/2]$ and $[3/4, 1]$, and $\delta(t) = k(t - 1/2) + 1/2$ for $t \epsilon [1/2, 3/4]$. Then $\delta \epsilon P(f)$.

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Thus $P(f)$ contains the group of all such diffeomorphisms δ. This example is illustrated in Figure 1.

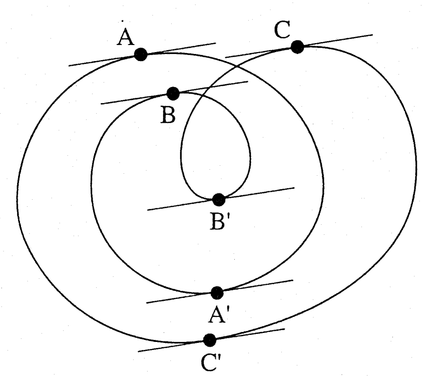

Figure 1. A curve with infinite parallel group

(ii) Next we observe that if $f(t) = (cos2\pi nt, sin2\pi nt)$, then $P(f)$ is cyclic of order $2n$.

(iii) To construct an example for which $P(f)$ is cyclic of odd order, consider the immersion illustrated in Figure 2, where the curvature of f does not change sign, and is zero only at A, B and C. Then $P(f)$ is cyclic of order 3, where we assume that the image is traversed only once. This is because for all $\delta \epsilon P(f), \kappa(t) = 0$ iff $\kappa(\delta(t)) = 0$, where κ denotes the curvature function. Thus no element of $P(f)$ takes A, B, C to A', B', C'. We thank I.J.Leary for this example.

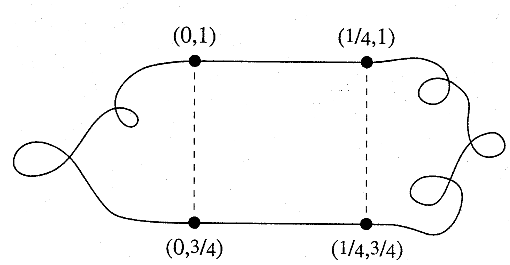

Figure 2. A curve with parallel group of order 3

Similar examples are easily described with $(P(f)$ cyclic of any odd order.

(iv) For any $n\epsilon\mathbf{N}$, the curve f given by $f(t) = (sin4\pi nt, sin2\pi nt)$ has $P(f)$ isomorphic to the dihedral group D_n of degree n. The image of f is the 'figure eight' shown in Figure 3.

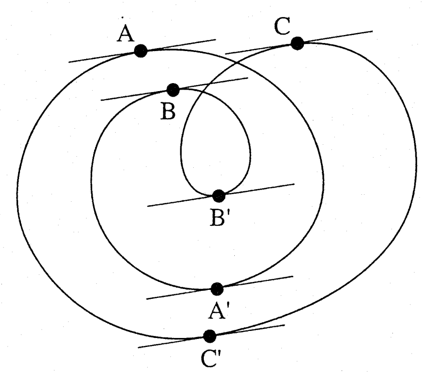

Figure 3. A curve with dihedral parallel group

It is impossible for all the normal lines $\nu(t)$ to be parallel, so $P(f)$ cannot act transitively on S^1. In particular, as remarked above, $P(f)$ cannot be $SO(2)$.

The group $P(f)$ has subgroups $N(f)$ and $T(f)$ defined as follows. We say that a self-diffeomorphism δ of S^1 is in the **normal group** $N(f)$ of f iff for all $t\epsilon S^1, \nu(t) = \nu(\delta(t)$, and is in the **tangent group** $T(f)$ of f iff for all $t\epsilon S^1, \tau(t) = \tau(\delta(t)$. The first example above shows that $T(f)$ may be just as complicated as $P(f)$. In [**2**] it is shown that, in case f is an embedding, $N(f)$ is either trivial or cyclic of order 2. (See also [**4,5**].)

2 Curves with nonvanishing curvature

Let us now restrict attention to immersions $f : S^1 \to E^2$ for which the curvature function is nowhere zero. Such a curve is sometimes called a **rosette** (See, for example, [**1**]).

Proposition 1 *Let f be a rosette. Then $P(f)$ acts properly.*

Proof: Let $\delta\epsilon P(f)$, and suppose that for some $t\epsilon S^1, \delta(t) = t$. Denote by τ^* the map which assigns to each $t\epsilon S^1$ the line through O parallel to $\tau(t)$.Then τ^* is an immersion and therefore is injective on some open arc A_t containing t. Then for all $s\epsilon A_t \bigcap \delta^{-1}(A_t), \delta(s) = s$. Thus the set F_δ of fixed points of δ is open in S^1. Since F_δ is also closed in S^1, either f has no fixed points, or f is the identity. We conclude that $P(f)$ acts freely on S^1. Moreover, if $A_t \bigcap \delta(A_t) \neq \emptyset$ then δ is the identity. This proves that $P(f)$ acts properly discontinuously. □

Proposition 2 *Let f be a rosette of rotation number $n\epsilon\mathbf{Z}\backslash 0$. Then $P(f)$ is isomorphic to $\mathbf{Z}_{2m}$, where $m = |n|$.*

Proof: Under these hypotheses, τ^* is a covering map of order $2m$, and $P(f)$ is the associated group of covering transformations. □

We conclude this section by proving the following fact about the group $T(f)$ for a rosette f, where we say that $y\epsilon S^1$ is **simple** iff $f^{-1}f(y) = \{y\}$.

Proposition 3 *Let f be a rosette which has at least one simple point. Then $T(f)$ is trivial.*

Proof: Let $\delta\epsilon T(f)$. Then for some C^1-function $\phi : S^1 \to \mathbf{R}$, and $s\epsilon S^1$,

$$f(s) - f(\delta(s) = \phi(s)\mathbf{t}(s),$$

where $\mathbf{t}$ denotes the unit tangent vector field for f. It follows that

$$v(s)\mathbf{t}(s) - \delta'(s)v(\delta(s)\mathbf{t}(\delta(s)) = \phi'(s)\mathbf{t}(s) + \phi(s)v(s)\kappa(s)\mathbf{n}(s)$$

where v denotes the velocity function, and $\mathbf{n}$ the unit normal vector field for f. The left-hand side of the equation is a multiple of $\mathbf{t}(s)$, and so $\phi(s) = 0$, for all $s\epsilon S^1$. We conclude that $f(s) = f(\delta(s))$ for all $s\epsilon S^1$. Now there is a simple point y,say, of f. Thus $\delta(y) = y$. It follows by Proposition 1 that δ is the identity. □

3 Concluding remarks

In the definition of the parallel group, we could relax the requirement that f be C^2 and the elements of $P(f)$ should be diffeomorphisms of S^1, and allow f to be C^1, while the elements of $P(f)$ are merely homeomorphisms. The investigation of the possibilities for $P(f)$ becomes more difficult under such a change. Some idea of the difference between the two concepts may be obtained by noting that in the example illustrated in Figure 2, the group $P(f)$ is isomorphic to $\mathbf{Z}_3$ in the 'diffeomorphism' case, and to $\mathbf{Z}_6$ in the 'homeomorphism' case.

We could also replace the domain S^1 of our plane curves by the real line $\mathbf{R}$. In this case, Propositions 1 and 3 remain true, and there is an analogue of Proposition 2, illustrated by the example of the logarithmic spiral, an embedding of $\mathbf{R}$ whose parallel group is not finite (in fact, isomorphic to $\mathbf{Z}$).

The concept of parallel group, whether defined in terms of homeomorphisms or diffeomorphisms, may be extended to apply to immersions of n-dimensional manifolds in Euclidean $(n+k)$-space E^{n+k}, for any $n, k \epsilon \mathbf{N}$. This area of ideas remains to be explored in full generality, although the theory of transnormality [**3**] and of parallel immersions [**2,4,5**] indicates its attraction.

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