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Minimal Lagrangian submanifolds in complex Euclidean space

I. Castro[1]

1 Introduction.

In general, not much is known about minimal submanifolds of Euclidean space of high codimension.

An approach about this topic is due to Anderson, who in [1] studies complete minimal submanifolds of Euclidean space with finite total scalar curvature, trying to generalize classical results of minimal surfaces of Chern and Osserman (see [6]) on minimal surfaces in Euclidean space with finite total curvature. There were a number of difficulties in carrying out such a program. For example, one no longer has the tools of complex function theory and the topological structure of the submanifold, particularly regarding the asymptotic behaviour is more complicated. He had to replace the complex variable techniques by real variable techniques, namely geometric measure theory and use of elliptic PDE to derive some estimates on the curvature decay of minimal submanifolds.

Harvey and Lawson also study in their classical paper about calibrations (see [10]) a particular family of minimal submanifolds of complex Euclidean space, *the special Lagrangian submanifolds*, i.e. oriented minimal Lagrangian submanifolds. They have the property of being absolutely volume minimizing. Among other things, they construct important examples of the previously mentioned minimal Lagrangian submanifolds.

Following their ideas, new examples of this kind of submanifolds are also obtained by Borisenko in [2]. This family is well known in the case of surfaces, thanks to a theorem of Chen and Morvan (see [5]) that says that an orientable minimal surface of $\mathbf{C}^2$ is Lagrangian if and only if it is holomorphic with respect to some orthogonal complex structure on $\mathbf{R}^4$.

[1]Part of this work is joint with Francisco Urbano, from the Departament of Geometry and Topology, University of Granada.

2 The Lagrangian catenoid.

Among the examples constructed by Harvey and Lawson in [10], we emphasize the one given in Theorem 3.5 and, concretely, one of its connected components, which is defined by

$$M_0 = \{(x,y) \in \mathbf{C}^n \equiv \mathbf{R}^n \times \mathbf{R}^n;\ |x|y = |y|x; \mathrm{Im}(|x| + i|y|)^n = 1; |y| < |x| \tan\frac{\pi}{n}\}.$$

Besides being a minimal Lagrangian submanifold of complex Euclidean space $\mathbf{C}^n$, M_0 is invariant under the diagonal action of $SO(n)$ on $\mathbf{C}^n \equiv \mathbf{R}^n \times \mathbf{R}^n$.

The first part of this talk is devoted to this example and we are going to show that it is a very regular example with many similar properties to the classical catenoid. So, from now on we will refer to M_0 as the *Lagrangian catenoid.*

Let $\mathbf{C}^n$ be complex Euclidean space with complex coordinates z_i, $i = 1, \ldots, n$, and

$$\Omega = \frac{i}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$$

its symplectic structure. If $\langle , \rangle$ denotes the Euclidean metric and J the standard complex structure on $\mathbf{C}^n$, then $\Omega(u,v) = \langle Ju, v\rangle$ for vectors u, v.

Let $\phi : M \longrightarrow \mathbf{C}^n$ be an immersion of an n-dimensional manifold M. ϕ is called *Lagrangian* if $\phi^*\Omega \equiv 0$. This means that the complex structure J of $\mathbf{C}^n$ carries each tangent space of M into its corresponding normal space.

Suppose now that ϕ is a minimal immersion, i.e. the mean curvature vector of ϕ vanishes identically, $H = 0$. If τ is the scalar curvature of the submanifold M, then $\tau = -|\sigma|^2$, where σ denotes the second fundamental form of ϕ. The total scalar curvature of M is defined by (see [1])

$$\int_M |\sigma|^n dV,$$

where dV is the measure associated to the metric $\langle , \rangle$.

On the other hand, let $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$, and

$$\phi_0 : \mathbf{R} \times \mathbf{S}^{n-1} \longrightarrow \mathbf{C}^n \equiv \mathbf{R}^n \times \mathbf{R}^n$$

the map given by

$$\phi_0(t,p) = \cosh^{\frac{1}{n}}(nt)\, e^{i\beta(t)}\, (p, 0)$$

where $\beta(t) = \frac{\pi}{2n} - \frac{2}{n}\arctan(\tanh\frac{nt}{2})$. We point out that $\beta(t) \in (0, \pi/n)$.

The main properties of the Lagrangian catenoid are collected in the following proposition.

Proposition 1 (a) *ϕ_0 is a minimal Lagrangian embedding such that*

$$\phi_0(\mathbf{R} \times \mathbf{S}^{n-1}) = M_0.$$

(b) *When n=2, M_0 can be identified with the holomorphic surface*

$$\{(z, \frac{1}{z});\ z \in \mathbf{C}^*\}.$$

(c) *The induced metric by ϕ_0 on $\mathbf{R} \times \mathbf{S}^{n-1}$ is*

$$\langle , \rangle = \cosh^{\frac{2}{n}}(nt)(dt^2 + g_0),$$

where g_0 is the standard metric on $\mathbf{S}^{n-1}$ of constant curvature 1.
In particular, $(M_0, \langle , \rangle)$ is complete.

(d) *M_0 is foliated by round $(n-1)$-spheres of $\mathbf{C}^n$.*

(e) *M_0 has finite total scalar curvature.*
In fact, if σ is the second fundamental form of M_0, then

$$\int_{M_0} |\sigma|^n dV = ((n+2)(n-1))^{\frac{n}{2}} c_{n-1} I_n,$$

where c_{n-1} is the volume of the $(n-1)$-dimensional unit sphere, $I_2 = 1$, $I_3 = \pi/6$ and $I_n = \frac{n-2}{n-1} I_{n-2}$, $n \geq 4$.

In this way, topologically M_0 is $\mathbf{R} \times \mathbf{S}^{n-1}$ and geometrically it is foliated by $(n-1)$-dimensional round spheres of $\mathbf{C}^n$ and it has finite total scalar curvature.

When $n = 2$ it has total curvature exactly -4π, being one of the examples described by Hoffman and Osserman in [11]. In this case, the Lagrangian catenoid can be written as the following holomorphic surface of $\mathbf{C}^2$:

$$M_0 \equiv \{(z, \frac{1}{z}) \in \mathbf{C}^2;\ z \in \mathbf{C}^*\}.$$

It is known that Enneper [7] (for surfaces) and Jagy [9] (for higher dimension) proved that the catenoid, the Riemann surfaces and the higher dimensional catenoid are the only (non flat) minimal hypersurfaces of $\mathbf{R}^{n+1}$ foliated by pieces of $(n-1)$-dimensional round spheres of $\mathbf{R}^{n+1}$.

In Theorem 1, we obtain the Lagrangian version of the above result, showing that here there are no examples similar to the Riemann minimal surfaces.

Concretely we prove the following result.

Theorem 1 *Let $\phi : M \longrightarrow \mathbf{C}^n$ be a minimal (non flat) Lagrangian immersion of an n-dimensional manifold M. Then M is foliated by pieces of round (n-1)-spheres of $\mathbf{C}^n$ if and only if ϕ is congruent (up to dilations) to an open subset of the Lagrangian catenoid.*

On the other hand, using results of Anderson [1] and Enomoto [8], we find that any complete minimal submanifold M (of dimension $n \geq 3$) of finite total scalar curvature and properly immersed in Euclidean space $\mathbf{R}^m$ has a compactification by an inversion. This means that there exists a compact n-dimensional submanifold $\overline{M}$ of $\mathbf{R}^m$ passing through the origin such that M is the image of $\overline{M} - \{0\}$ by the inversion $F : \mathbf{R}^m \cup \{\infty\} \longrightarrow \mathbf{R}^m \cup \{\infty\}$ defined by $F(p) = p/|p|^2$.

Following these ideas, we also obtain a global characterization of the Lagrangian catenoid in the family of the submanifolds with finite total scalar curvature, which is given in the following result.

Theorem 2 *Let M be an n-dimensional, $n \geq 3$, complete minimal (non flat) submanifold with finite total scalar curvature properly immersed in Euclidean space $\mathbf{R}^{2n}$. Then the compactification by the inversion of M is Lagrangian for a certain orthogonal complex structure on $\mathbf{R}^{2n}$ if and only if M is (up to dilations) the Lagrangian catenoid.*

When n=2, and assuming that the minimal surface admits a compactification by the inversion, the proof also works and the result is true.

The method used to prove these two theorems is different to the one used in the case of Euclidean space ([7],[9]) and, in some sense, it follows some ideas developed in [12]. Basically it consists in proving that the foliation of the submanifold $\phi : M \longrightarrow \mathbf{C}^n$ is invariant under the action of a uniparametric group of conformal transformations of M, and that the associated conformal vector field X satisfies the vectorial equation $\phi = f\,\phi_* X$ where f is a smooth $\mathbf{C}$-valuated function of constant length. Integrating this equation we obtain our uniqueness results.

3 Lagrangian spheres in C^n.

Precisely the compactification by the inversion of the Lagrangian catenoid is one of the most interesting Lagrangian compact submanifold of complex Euclidean space: the Whitney sphere, which is defined by the Whitney immersion

$$\Phi : \mathbf{S}^n \longrightarrow \mathbf{C}^n,$$

given by

$$\Phi(x_1, \ldots, x_n, x_{n+1}) = \frac{1 + i x_{n+1}}{1 + x_{n+1}^2} (x_1, \ldots, x_n) .$$

After the Lagrangian subspaces of $\mathbf{C}^n$, the Whitney sphere is probably the most regular example of Lagrangian submanifold in $\mathbf{C}^n$ because of its many interesting properties. In fact, Urbano and the author —when $n = 2$— and later Ros and Urbano —in arbitrary dimension— state in [4] and [12] respectively that, in a certain sense, the Whitney spheres play the role of umbilical hypersurfaces of the Euclidean space $\mathbf{R}^{n+1}$ inside the family of Lagrangian submanifolds.

For a Lagrangian submanifold of $\mathbf{C}^n$, we emphasize the inequality

$$\tau \leq \frac{n^2(n-1)}{n+2}|H|^2,$$

involving the scalar curvature τ and the norm of the mean curvature vector H. Besides the totally geodesic Lagrangian submanifolds, the Whitney spheres are the only ones that reach the equality in the above inequality (Corollary 3 of [12]; see also [3] for a different method). So it is natural to raise the following question:

> *Given any positive $\eta \leq n^2(n-1)/(n+2)$, does a Lagrangian immersion $\phi = \phi(\eta) : M \longrightarrow \mathbf{C}^n$ exist in such a way that $\tau = \eta|H|^2$?.*

We can even ask about Lagrangian *spheres* satisfying the above equality.

The second part of this talk is devoted to give an affirmative answer to the latter problem and for this purpose we will explicitly construct a one-parameter family of Lagrangian spheres

$$\Phi_q : \mathbf{S}^n \longrightarrow \mathbf{C}^n,\ q \geq 2,$$

such that the scalar curvature $\tau = \tau_q$ and the mean curvature vector $H = H_q$ of Φ_q safisfies the geometric equality

$$\tau = \frac{(2q+n-2)(n-1)n^2}{(n+q)^2}|H|^2.$$

They can be considered as a generalization of the Whitney sphere since this corresponds to the immersion Φ_2. In fact, all the spheres Φ_q have the same topological behaviour that the Whitney sphere and the same happens with their Ricci and scalar curvatures.

About the construction of $\Phi_q : \mathbf{S}^n \longrightarrow \mathbf{C}^n$, $q \geq 2$, we make use of a method introduced in [12] that provides examples of Lagrangian submanifolds of $\mathbf{C}^n$ in terms of Lagrangian submanifolds of the complex projective space $\mathbf{CP}^{n-1}$ and a regular plane curve γ. In this context, the Whitney sphere corresponds to the example constructed with the totally geodesic Lagrangian submanifold of $\mathbf{CP}^{n-1}$ and the “lemniscata” (i.e. the curve that in polar coordinates (r, θ) is given by $r^2 = \cos 2\theta$, whose picture is the figure of a “eight lying down”). For the rest of spheres Φ_q we take again the totally geodesic Lagrangian submanifold of $\mathbf{CP}^{n-1}$ and now consider the curve γ_q, $q \geq 2 : r^q = \cos(q\theta)$, $\theta \in (-\pi/2q, \pi/2q)$, whose picture is also like a “eight lying down” but with a different opening. Exactly π/q is the angle between the tangents at the selfintersection point of γ_q.

This work is clearly inspired in the very interesting paper [12] mentioned above and we also follow their ideas in order to characterize our examples Φ_q. In this way, we obtain a local characterization of Φ_q in terms of the second fundamental form and an immediate consequence of this is the following result, where we characterize locally our examples Φ_q, $q \geq 2$, in terms of the above geometric equality.

Theorem 3 *Let $\phi : M \longrightarrow \mathbf{C}^n$ be a Lagrangian immersion of an n-dimensional manifold such that $\tau = \eta |H|^2$, for some $\eta \in]0, n^2(n-1)/(n+2)]$.*

Then:

$$\langle \sigma(JH, JH), H \rangle \leq \frac{n(q_\eta + 1)}{n + q_\eta} |H|^4,$$

and the equality holds if and only if ϕ is totally geodesic or locally congruent to Φ_{q_η}, where

$$q_\eta = \frac{n}{\eta} \left(n(n-1) - \eta + \sqrt{n^2(n-1)^2 - \eta(n-1)(n+2)} \right).$$

We finally proceed to characterize globally our Lagrangian spheres Φ_q inside a particular family of compact Lagrangian submanifolds in terms of a geometric differential equation satisfied for the associated Maslov form α_H (it is the one-form on M dual to the tangent vector field JH, i.e. $\alpha_H = \Omega(-, H)$) .

Theorem 4 *Let $\phi : M \longrightarrow \mathbf{C}^n$ be a Lagrangian immersion of an n-dimensional compact manifold M with null first Betti number. If the Maslov form α_H of ϕ satisfies*

$$\nabla \alpha_H = \frac{\delta \alpha_H}{n + q - 2} \left(\langle \, , \, \rangle + \frac{q-2}{|\alpha_H|^2} (\alpha_H \otimes \alpha_H) \right)$$

for some $q \geq 2$, then ϕ is congruent to Φ_q.

Remark 1 This result is a generalization of Corollary 4 in [12], where the Whitney sphere Φ_2 was characterized as the only Lagrangian compact submanifold with null first Betti number and conformal Maslov form.

Note that, when $q = 2$, the hypothesis of Theorem 4 about the Maslov form just means that α_H is conformal.

Remark 2 The material of this talk can be found in detail in the two following papers:

1. I. Castro, *Lagrangian spheres in complex Euclidean space satisfying a geometric equality.* Geometriae Dedicata (in press).
2. I. Castro, F. Urbano, *On a minimal Lagrangian submanifold of* $\mathbf{C}^n$ *foliated by spheres.* Preprint 1997.

References

[1] M. T. Anderson, *The compactification of a minimal submanifold in Euclidean space by the Gauss map.* Preprint.

[2] A. Borisenko, *Ruled special Lagrangian surfaces.* Advances in Soviet Math. **15** (1993), 269–285.

[3] V. Borrelli, B.-Y. Chen, J.M.Morvan, *Une caracterisation geometrique de la sphere de Whitney.* C.R.Acad.Sci., Paris, Ser. I **321** (1995), 1485–1490.

[4] I. Castro, F. Urbano, *Lagrangian surfaces in the complex Euclidean plane with conformal Maslov form.* Tôhoku Math. J. **45** (1993), 565–582.

[5] B. Y. Chen, J. M. Morvan, *Geometrie des surfaces lagrangiennes de* $\mathbf{C}^2$. J.Math.Pures Appl. **66** (1987), 321–325.

[6] S. S. Chern, R. Osserman, *Complete minimal surfaces in* $\mathbf{E}^n$. J. d'Analyse Math. **19** (1967), 15–34.

[7] A. Enneper, *Die cyklischen Flachen.* Z.Math.Phys. **14** (1869), 393–421.

[8] K. Enomoto, *Compactification of submanifolds in Euclidean space by the inversion.* Advanced Stud.Pure Math. **22** (1993), Progress In Diff.Geom., 1–11.

[9] W. C. Jagy, *Minimal hypersurfaces foliated by spheres.* Michigan Math.J. **38** (1991), 255–270.

[10] R. Harvey, H. B. Lawson, *Calibrated geometries.* Acta Math. **148** (1982), 47–157.

[11] D. A. Hoffman, R. Osserman, *The Geometry of the generalized Gauss map.* Memoirs A.M.S. **236** (1980).

[12] A. Ros, F. Urbano, *Lagrangian submanifolds of* $\mathbf{C}^n$ *with conformal Maslov form and the Whitney sphere.* J.Math.Soc. Japan, to appear.

Ildefonso Castro
Departamento de Matemáticas, EPS
Universidad de Jaén
23071 Jaén
Spain
Email: `icastro@ujaen.es`