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# Monohedral planar f-tilings

A. M. Azevedo Breda and A. J. Breda d'Azevedo[1]

**Abstract**

We classify the monohedral planar f-tilings and state some open problems related with folding maps.

## 1 Introduction

By an *$s-$monohedral planar $f$ tiling, $s$-mpf,* we mean a planar monohedral edge-to-edge tiling by convex $s$-polygons whose vertices are of even valency and satisfy the *angle folding relation,* that is, the sum of the alternating angles is $\pi$.

We begin by showing that $s$-mpf tilings are only possible for $s = 3$ or $s = 4$.

Let $\mathcal{T}$ be an $s$-monohedral edge-to-edge tiling and $P \in R^2$ be a fixed point. By $D(r, P)$ we mean the closed disc of radius $r$ centered at $P$. Considering the set of all tiles of $\mathcal{T}$ that meet $D(r, P)$ we obtain a set $A(r, P)$ called the patch generated by $D(r, P)$. Denote by $f(r, P)$, $e(r, P)$ and $v(r, P)$ the number of tiles, edges and vertices in $A(r, P)$ and by $E(r, P)$ the number of edges that lie round the boundary of the patch. In [3] Grünbaum and Shephard proved that

$$i)\ \ \forall_{\delta>0}\ \lim_{r\to\infty} \frac{f(r+\delta, P)}{f(r, P)} = 1;$$

$$ii)\ \ \lim_{r\to\infty} \frac{E(r, P)}{f(r, P)} = 0;$$

$$iii)\ \ \lim_{r\to\infty} \frac{v(r, P)}{f(r, P)}$$

and

$$\lim_{r\to\infty} \frac{e(r, P)}{f(r, P)}$$

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exist and are finite.

Let $\mathcal{T}$ be an $s$-mpf tiling and $P$ be a fixed point of $R^2$. Since every tile of $\mathcal{T}$ has $s$, $s \geq 3$, adjacents, for every $r > 0$ one has,

$$sf(r,P) = 2e(r,P) - E(r,P). \tag{1.1}$$

Also,

$$E(r,P) = v^*(r,P)$$

where $v^*(r,P)$ denotes the number of vertices which lie in $A(r,P) - \mathrm{int}D(r,P)$, where int stands for the topological interior. Besides, as $\mathcal{T}$ is an mpf tiling,

$$v(r,P) = \sum_{k\geq 2} v_{2k}(r,P) + \sum_{j\geq 2} v_j^*(r,P)$$

where $v_{2k}$ and $v_j^*$ denote respectively the number of vertices of valency $2k$ in $\mathrm{int}D(r,P)$ and of valency $j$ in $A(r,P) - \mathrm{int}D(r,P)$.

Thus,

$$\begin{aligned}
2e(r,P) &= \sum_{k\geq 2} 2kv_{2k}(r,P) + \sum_{j\geq 2} jv_j^*(r,P) \\
&\geq\ 4v(r,P) - 4\sum_{j\geq 2} v_j^*(r,P) + \sum_{j\geq 2} jv_j^*(r,P) \\
&\geq\ 4v(r,P) - 2\sum_{j\geq 2} v_j^*(r,P) \\
&\geq\ 4v(r,P) - 2E(r,P).
\end{aligned}$$

Having in account 1.1 one has,

$$sf(r,P) \geq 4v(r,P) - 3E(r,P)$$

or equivalently,

$$s \geq 4\frac{v(r,P)}{f(r,P)} - 3\frac{E(r,P)}{f(r,P)}. \tag{1.2}$$

Using 1.1 and the Euler's theorem, we obtain

$$\frac{v(r,P)}{f(r,P)} = \frac{E(r,P)}{2f(r,P)} + \frac{1}{f(r,P)} + \frac{s}{2} - 1.$$

Thus,

$$\lim_{r\to\infty} \frac{v(r,P)}{f(r,P)} = \frac{s-2}{2}$$

and so, in view of 1.2, $s = 3$ or $s = 4$.

## 2 Prototiles of mpf-tilings

We may construct, from any triangle $T$, an mpf-tiling, by the action of the group of isometries of $T$ generated by half turns around the mid points of its edges. Those tilings have vertex valency 6 and satisfy the angle folding relation. What can be said about the 4-polygons? Contrariwise not all quadrangles can be seen as prototiles of 4-mpf tilings. In fact, any quadrangle $Q$ of angles $\alpha = 120°$, $\beta = 110°$, $\gamma = \delta = 65°$ can not be a prototile of a mpf tiling since $2\alpha$, $\alpha+\beta$, $\alpha+\gamma$ and $\alpha+\delta$ are all greater than $\pi$. Now the question is to know which ones can be seen as prototiles of such tilings.

### 2.1 Quadrangular prototiles

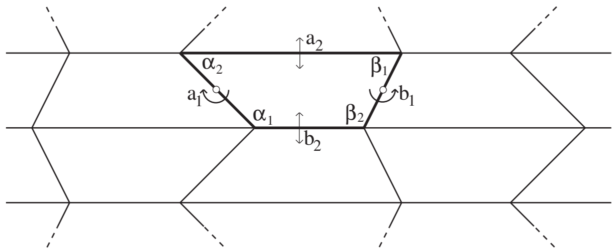
Throughout this section $Q$ is a quadrangle with edges $a_1$, $a_2$, $b_1$ and $b_2$ and angles $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ defined, respectively by $a_1$ and $b_2$, $a_1$ and $a_2$, $a_2$ and $b_1$ and by $b_1$ and $b_2$. Since $\alpha_1+\alpha_2+\beta_1+\beta_2 = 2\pi$ then up to a labelling either $\alpha_1+\alpha_2 \geq \pi$ or $\alpha_1+\beta_1 \geq \pi$. An mpf tiling of prototile $Q$ will be called a $Q$mpf tiling.

**Theorem 1** *If $\alpha_1+\alpha_2 = \pi$ it is always possible to construct a $Q$mpf tiling.*

**Proof:**

The action of the group generated by the reflections in the straight lines containing $a_2$ and $b_2$ and by the half-turn around the mid point of $a_1$ and $b_1$ induces a $Q$mpf-tiling (figure 1).

□




**Theorem 2** *If $\mathcal{T}$ is a $Q$mpf tiling and $\alpha_1+\alpha_2 > \pi$ then $\alpha_1+\beta_1 = \pi$ or $\alpha_1+\beta_2 = \pi$.*

**Proof:**

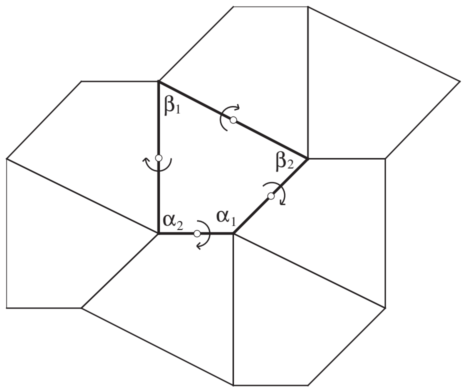
Without loss of generality we may assume that $\alpha_1 \geq \alpha_2$. Then $\alpha_1 > \frac{\pi}{2}$ and $\beta_1+\beta_2 < \pi$. A sequence of alternating angles containing $\alpha_1$ angles may only have $\beta_1$ and/or $\beta_2$ angles .

Assuming that there are positive integers $s$ and $t$ such that $\alpha_1+t\beta_1+s\beta_2=\pi$ we are drawn to a contradiction. In fact, since $\alpha_1+\alpha_2+\beta_1+\beta_2=2\pi$ one has,

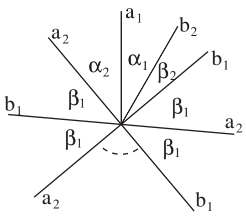
$$0 \leq \alpha_1-\alpha_2=(1-2t)\beta_1+(1-2s)\beta_2<0.$$

This contradiction means that vertices surrounded by $\alpha_1$ angles have or $\beta_1$ or $\beta_2$ as alternating angles.

If a vertex is surrounded by a combination of alternating angles $\alpha_1$ and $\beta_1$, that is there is a positive integer $q$ such that $\alpha_1+q\beta_1=\pi$. Now, if $q=1$, considering the action of the group generate by the half-turns around the mid points of $a_1$, $a_2$, $b_1$ and $b_2$ we are lead to a $Q$mpf-tiling (figure 2)




If $q>1$ then the vertex containing this combination of alternating angles is, up to a reverse ordering, surrounded as indicated in figure 3.

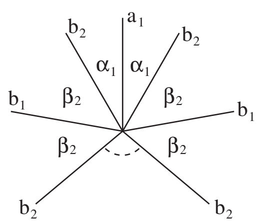



This means that $\alpha_2+\beta_2+(q-1)\beta_1=\pi$. But then,

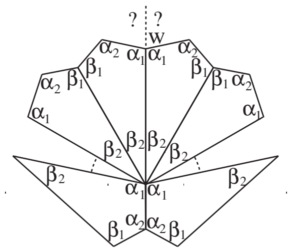
$$\alpha_1+t\beta_1+\alpha_2+\beta_2+(q-1)\beta_1=\alpha_1+\beta_1+\alpha_2+\beta_2,$$

and so, $q=1$, which is a contradiction.

Suppose now that there is a positive integer $p$ such that $\alpha_1+p\beta_2=\pi$. If $p=1$, it is enough to consider the action of the group generated by the reflections in the straight lines containing $a_1$ and $b_1$ and the half-turns around the mid points of $a_2$ and $b_2$, to construct a $Q$mpf-tiling. If $s>1$, a vertex containing $(\alpha_1, \beta_2, ..., \beta_2)$ as a sequence of alternating angles is, up to a reverse ordering, surrounded as indicated in figure 4.



Near a vertex $v$ surrounded as indicated in figure 4, the representation of a $Q$mpf-tiling would be the one illustrated in figure 5.



The vertex $w$ indicated above can not be encircled by a sequence of alternating angles obeying the angle-folding relation. Observe that from $a_1$ we may only adjust $a_2$ or $b_2$ obtaining $\alpha_1$ or $\alpha_2$ as alternating angles to $\alpha_1$.

□

**Theorem 3** *If $\alpha_1 + \beta_1 = \pi$ it is always possible to construct a Qmpf-tiling.*

**Proof:**

The action of the group generated by the half-turns around the mid points of the edges of $Q$ induces a $Q$mpf-tiling.

□

Using a similar reasoning to the one done in theorem 2, we may prove that

**Theorem 4** *If $\mathcal{T}$ is a Qmpf tiling and $\alpha_1 + \beta_1 > \pi$ then $\alpha_1 + \alpha_2 = \pi$ or $\alpha_1 + \beta_2 = \pi$.*

In the spherical case the prototiles of a monohedral $f$ tiling are triangles. A complete characterization of this class of tilings has been done in [1]. In the hyperbolic case we may have $s$-monohedral $f$ tilings for any $s$, being enough to consider an $s$ polygon $P$ of angles $\delta_j = \frac{\pi}{2s}$, $j = 1, ..., s$ and the action of the group generated by half-turns around the mid points of $P$.

## 3 Folding Maps

If one folds a Riemann surface $S$ and then unfold it back, the left over marks on the surface make a graph imbedded in $S$ which may not be connected. The edges of this graph are segments of geodesics and the angles around each vertex satisfies the *angle-folding relation* which states that the sum of the alternating angles is $\pi$. When this imbedded graph is connected we have a map which we call a *folding map*.

Any Qmpf-tiling is a folding map as any face-transitive folding map is a Qmpf-tiling. Yet, we don't know if every Qmpf-tiling is a face-transitive folding map.

To each folding map one may associate (not uniquely) a locally isometric function $\phi : S \longrightarrow S$, called *isometric folding*, where the set of singularities (the points where $\phi$ fails to be differentiable) is precisely the underlying graph of the folding map.

By an $s$-folding we mean an isometric folding whose set of singularities is an $s$-mpf tiling.

**Theorem 5** *Any* 3*-folding has bounded image.*

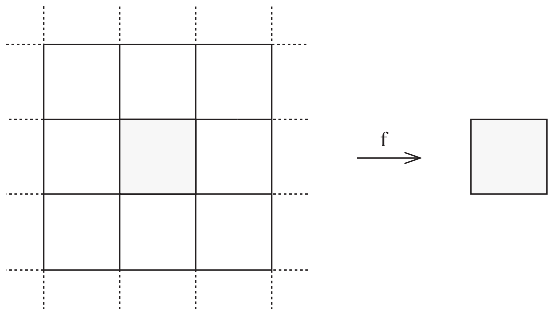
**Proof:**

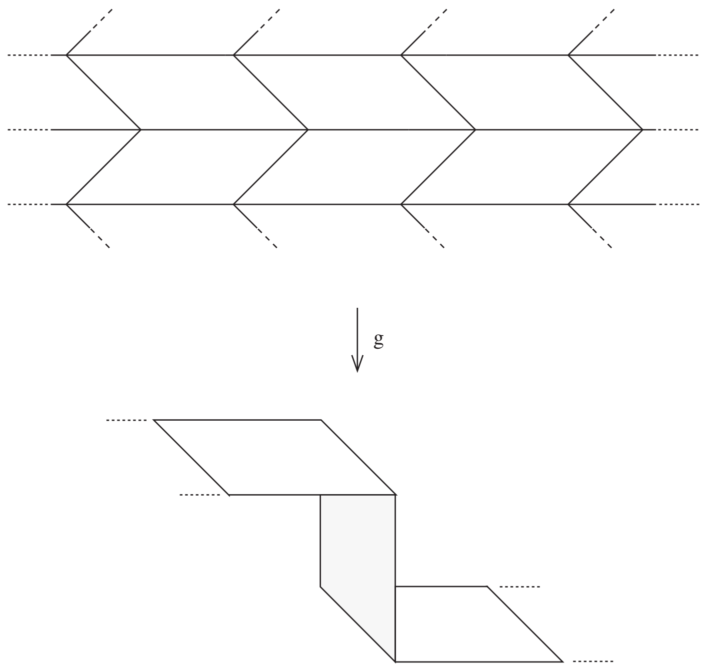
Let $f$ be a 3-folding and $T$ be a prototile of its singularity set, $\sum_f$. Without loss of generality we may assume that $f|_{convhull(T_0)}(x) = (x)$ for all $x$ in $T_0$, where $T_0$ is a copy of $T$, contained in $\sum_f$.

It is clear that for any 2 congruent triangles $T_1$ and $T_2$ in $\sum_f$ such that $T_1 \cap T_2$ is a side for both $T_1$ and $T_2$, the circumscribed circles to $T_1$ and $T_2^*$, where $T_2^*$ is the image of $T_2$ under the reflection in the straight line containing $T_1 \cap T_2$, are the same. Let $A$ be an arbitrary point of $\sum_f$, then $A$ lies in some triangular cell $S = convhull(T^\circ) \subset \sum_f$. Considering a chain $T_0, T_1, ..., T_n = T^\circ$ of triangles of $\sum_f$ such that for any $j = 0, 1, ....n-1$, $T_j \cap T_{j+1}$ is a side for both $T_j$ and $T_{j+1}$, we may conclude that $f(A)$ belongs to the circumscribed triangle to $T_0$.

$\square$

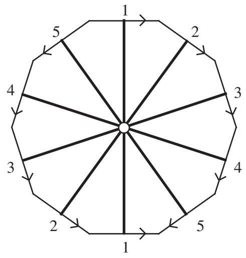
Contrariwise to 3-foldings we may have 4-foldings with bounded (figure 6) or unbounded (figure 7) image.






When we fold an universal Riemann surface (plane, sphere or hyperbolic plane) just once we get a circular rim in a sphere, which is a map with one vertex (chosen at will on the rim), one edge and two faces; or a straight line in the plane, or a geodesic in the hyperbolic plane, which are boths maps with one vertex, two edges and two faces.

In contrast with combinatorial (algebraic or topological) maps, the folding maps depend on the angles between their sucessive edges. By the angle-folding relation, the valency at each vertex must be even. So every folding map is a Riemann map but not every Riemann map is a folding map. However, rotary (and hence every reflexible) Riemann maps have all angles around a vertex equal, so evey rotary Riemann map with even vertex valency is a folding map. We can then bring to folding maps many results from map theory since each combinatorial (algebraic or topological) map represents a Riemann map. Thus we can find at least one reflexible folding map on an orientable surface for every genus $g$; one representation of such orientable folding map can be constructed by identifying orientably opposite edges of a $2k$-gon with a "star" graph imbeded as shown in figure 8; this gives a reflexible map $DM_k$, known as the dual of $M_k$ map, with $2k$ faces, $3k$ edges and 2 or 3 vertices according as $k$ is even or odd, and so of genus $g = \frac{k}{2}$ or $g = \frac{k-1}{2}$, as $k$ is even or odd.




In the non-orientable case, there are non-orientable surfaces that do not admit reflexible maps; for example the non-orientable surfaces of characteristic 0 and $-1$ [2], and $-16, -22, -25, -37$ and $-46$ [4]. Can we find a face-transitive folding map on a non-orientable surface for every genus $g$?

Each reflexible map represents only one reflexible folding map. Another open question is how many non-reflexible folding maps are represented by the same reflexible map. On the plane, for example, the folding maps of figure 1 and figure 2 are represented by the same reflexible map, so on the plane a reflexible map may represents infinitely many folding maps. Is the same true for any compact Riemann surface (orientable and non-orientable) of any genus?

## References

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António João Breda d'Azevedo and Ana Maria d'Azevedo Breda
Departamento de Matemática
Universidade de Aveiro
Campus de Santiago
3000 Aveiro
Portugal
Email: `breda@mat.ua.pt` and `ambreda@mat.ua.pt`