Proceedings Ist International Meeting
on Geometry and Topology
Braga (Portugal)

Public. Centro de Matemática
da Universidade do Minho
p. 1–13, 1998

On Lie Groups with Left Invariant semi-Riemannian Metric

R. P. Albuquerque

1 Introduction and General Results

J. Milnor in the well known [2] gave several results concerning curvatures of left invariant Riemannian metrics on Lie groups. Some of those results can be partial or totally generalized to indefinite metrics. We will first show three of those generalizations that we have obtained. These will serve our purposes later on.

Let G be a real Lie group of dimension n and $\mathfrak{g}$ its Lie algebra. Considering a left invariant semi-Riemannian structure on G, let $e_1, \ldots, e_n$ be an orthonormal basis of left invariant vector fields and α_{ijk} their structure constants, that is,

$$[e_i, e_j] = \sum_{k=1}^{n} \alpha_{ijk} e_k,$$

or equivalently, denoting $\epsilon_i = <e_i, e_i> \ \ (1 \leq i \leq n)$,

$$\alpha_{ijk} = \epsilon_k <[e_i, e_j], e_k> .$$

Lemma 1 *With structure constants α_{ijk} as above, the sectional curvature satisfies the formula, for $i \neq j$,*

$$K(e_i, e_j) = \epsilon_i \sum_{k=1}^{n} \left(\frac{1}{2}\alpha_{jik}(\alpha_{ikj} - \epsilon_j\epsilon_k\, \alpha_{jik} + \epsilon_j\epsilon_i\, \alpha_{kji}) - \epsilon_k\epsilon_i\, \alpha_{kii}\alpha_{kjj} \right.$$

$$\left. -\frac{1}{4}(\alpha_{jik} - \epsilon_k\epsilon_j\, \alpha_{ikj} + \epsilon_k\epsilon_i\, \alpha_{kji})(\alpha_{ikj} - \epsilon_j\epsilon_i\, \alpha_{kji} + \epsilon_j\epsilon_k\, \alpha_{jik}) \right) .$$

Proof. The Levi-Civita conexion given by Koszul formula verifies

$$\nabla_{e_i} e_j = \sum_k \frac{1}{2}(\alpha_{ijk} - \epsilon_k\epsilon_i\,\alpha_{jki} + \epsilon_k\epsilon_j\,\alpha_{kij})e_k.$$

Hence, denoting by R the semi-Riemannian curvature tensor

$$R(e_i, e_j)e_i = -\nabla_{e_i}\nabla_{e_j} e_i + \nabla_{e_j}\nabla_{e_i} e_i + \nabla_{[e_i,e_j]} e_i,$$

we get the desired result by inspection on the right side of the identity

$$\epsilon_i\epsilon_j K(e_i, e_j) = <R(e_i, e_j)e_i, e_j> .$$

□

As one can see, the curvatures depend continuously on the structure constants.

Lemma 2 *If the transformation* $\mathrm{ad}\,(\mathrm{e_i})$ *is skew-adjoint, then*

$$K(e_i, e_j) = \frac{\epsilon_i\epsilon_j}{4}\sum_{k=1}^{n} \epsilon_k\,\alpha_{kji}^2.$$

If e_i *is also orthogonal to* $[e_j, \mathfrak{g}]$*, then necessarily* $K(e_i, e_j) = 0$*. In the case of Riemannian metric this condition is also sufficient.*

Proof. If $\mathrm{ad}\,(\mathrm{e_i})$ is skew-adjoint, then

$$(\ only\ for\ indice\ i) \qquad <[e_i, e_j], e_k> = - <[e_i, e_k], e_j>,$$

that is, $\alpha_{ijk} = -\epsilon_j\epsilon_k\,\alpha_{ikj}$. It follows that

$$\alpha_{kii} = -\alpha_{iki} = \epsilon_i\epsilon_k\,\alpha_{iik} = 0.$$

These results make possible the simplification of the formula of lemma 1.

□

Recall that the Ricci curvature in a direction $x \in \mathfrak{g}$ and the scalar curvature are, respectively,

$$r(x) = \sum_{i=1}^{n} \epsilon_i <R(x, e_i)x, e_i> \qquad \text{and} \qquad S = \sum_{i=1}^{n} \epsilon_i r(e_i) = 2\sum_{i<j} K(e_i, e_j).$$

In a direction e_j, the Ricci curvature becomes $r(e_j) = \epsilon_j \sum_{i\neq j} K(e_j, e_i)$.

Let us denote by $(p, n-p)$ the signature $(-, \ldots, -, +, \ldots, +)$ of a metric with p minus signs.

Theorem 1 *If the Lie algebra $\mathfrak{g}$ contains linearly independent vector fields x, y, z so that*

$$[x, y] = z,$$

then there exist left invariant metrics on G, of signature $(p, n-p)$, such that:
(i) $0 \leq p \leq n$ and $r(x) < 0 < r(z)$ or $r(z) < 0 < r(x)$;
(ii) $0 \leq p < n$ and $S < 0$;
(iii) $0 < p \leq n$ and $S > 0$.

Proof. We will take the metric induced by a scalar product in $\mathfrak{g}$. Fix a basis $b_1, \ldots, b_n$ of $\mathfrak{g}$ such that $b_1 = x,\ b_2 = y,\ b_3 = z$. Let α_{ijk} be the structure constants of $\mathfrak{g}$ for $b_1, \ldots, b_n$. For any real number $\delta > 0$, consider an auxiliary basis $e_1, \ldots, e_n$ and the Lie algebras $\mathfrak{g}_\delta$ whose structure constants for $e_1, \ldots, e_n$ are those given by the bracket product of $\mathfrak{g}$ and the basis $e_1 = \delta b_1,\ e_2 = \delta b_2,\ e_i = \delta^2 b_i\ (i \geq 3)$. Computation shows

$$[e_1, e_2] = [\delta b_1, \delta b_2] = \delta^2 b_3 = e_3,$$

$$[e_i, e_j] = [\delta b_i, \delta^2 b_j] = \sum_k \delta^3 \alpha_{ijk} b_k = \delta^2 \alpha_{ij1} e_1 + \delta^2 \alpha_{ij2} e_2 + \delta \sum_{k \geq 3} \alpha_{ijk} e_k$$

for $i = 1, 2,\ j \geq 3$, and

$$[e_i, e_j] = \delta^3 \alpha_{ij1} e_1 + \delta^3 \alpha_{ij2} e_2 + \delta^2 \sum_{k \geq 3} \alpha_{ijk} e_k$$

for $i, j \geq 3$. Clearly $\mathfrak{g}_\delta \simeq \mathfrak{g}$ for $\delta > 0$ as Lie algebras, since we only made a change of basis. Now, for any $0 \leq p \leq n$, define the left invariant metric on $\mathfrak{g}$, with signature $(p, n-p)$, which makes $e_1, \ldots, e_n$ an orthonormal basis and so that $\epsilon_1 = \epsilon_3 = \epsilon_2$ or $\epsilon_1 = \epsilon_3 = -\epsilon_2$.

Once $\delta \to 0$, we get a limit Lie algebra $\mathfrak{g}_0$ defined by $[e_1, e_2] = -[e_2, e_1] = e_3$ and $[e_i, e_j] = 0$ otherwise. Using lemmas 1 and 2, one may check

$$K_0(e_3, e_1) = K_0(e_3, e_2) = \epsilon_2 \frac{1}{4},$$

$$K_0(e_1, e_2) = -\epsilon_2 \frac{3}{4}, \qquad K_0(e_i, e_j) = 0, \quad \text{for } \{i, j\} \not\subset \{1, 2, 3\}.$$

Hence

$$r_0(e_1) = \epsilon_1 \left(-\frac{3}{4} + \frac{1}{4}\right) \epsilon_2 = -\epsilon_1 \epsilon_2 \frac{1}{2},$$

$$r_0(e_3) = \epsilon_3 \left(\frac{1}{4} + \frac{1}{4}\right) \epsilon_2 = \epsilon_3 \epsilon_2 \frac{1}{2}.$$

With the prescribed metric, $r_0(e_1)$ and $r_0(e_3)$ clearly have different signs. With respect to scalar curvature it follows that

$$S_0 = 2(K(e_1, e_2) + K(e_1, e_3) + K(e_2, e_3)) = -\frac{1}{2} \epsilon_2.$$

From the continuous dependence of the Ricci and scalar curvatures on the structure constants, we get the desired results for the fixed values of p and for sufficiently small δ. □

2 The Special Class $\mathfrak{S}$

We will now study the Lie groups that do not satisfy the hypothesis of the last theorem. With K. Nomizu, we consider a special class $\mathfrak{S}$ of solvable Lie groups. A non-commutative Lie group G belongs to $\mathfrak{S}$ if its Lie algebra $\mathfrak{g}$ has the property that $[x, y]$ is a linear combination of x and y, for any $x, y \in \mathfrak{g}$.

In [2] it is shown that $G \in \mathfrak{S}$ if and only if there exists an abelian ideal $\mathfrak{n}$ of codimension 1 and an element $b \notin \mathfrak{n}$ such that $[b, u] = u$ for every $u \in \mathfrak{n}$. Furthermore, $G \in \mathfrak{S}$ if and only if every left invariant Riemannian metric on G has sectional curvatures of constant sign.

In order to prove our next theorem we deduced the following slight generalization of a lemma from [1]. The proof of this generalization is equal to the original.

Lemma 3 *Let G be a Lie group with a left invariant semi-Riemannian metric and such that its Lie algebra can be decomposed as*

$$\mathfrak{g} = < b > \oplus \mathfrak{n}$$

where b is orthogonal to $\mathfrak{n}$ and $< b, b >= \epsilon = \pm 1$. $\mathfrak{n}$ is an abelian ideal and $L = \mathrm{ad}\,(\mathrm{b})_{|\mathfrak{n}} = \lambda \mathrm{Id} + S$, where S is the skew-adjoint part of L and $\lambda \in \mathbb{R}$. Then G has constant sectional curvature $K = -\epsilon\lambda^2$.

The following is a generalization of [3,Theorem 1]. Notice the new demonstration of Case II.

Theorem 2 *Let G be a Lie group of dimension n belonging to the special class $\mathfrak{S}$. Then*

(i) Any left invariant semi-Riemannian structure on G, of signature $(p, n-p)$, has constant sectional curvature K.

In particular, K is negative constant, if $p = 0$, or positive constant, if $p = n$.

(ii) Given any $p \in \mathbb{N}$, $0 < p < n$, and any $K \in \mathbb{R}$, we can construct a left invariant metric of signature $(p, n-p)$ with K as constant sectional curvature. We may still conclude the same in the cases $p = 0$, $K < 0$ and $p = n$, $K > 0$.

Proof. Let $\mathfrak{g} = < b > \oplus \mathfrak{n}$ be the Lie algebra of the Lie group G.

$$[b, u] = u \qquad [u, v] = 0, \qquad \forall u, v \in \mathfrak{n}.$$

(i) Suppose $\mathfrak{g}$ has a scalar product.

Case I. $< \, , \, >_{|\mathfrak{n}\times\mathfrak{n}}$ is nondegenerate.
There exists an unitary vector $b' \notin \mathfrak{n}$ such that $<b', \mathfrak{n}>= 0$. Writing

$$b' = \lambda b + u_0, \ \ \lambda \in \mathbb{R} \setminus \{0\}, \ \ u_0 \in \mathfrak{n}$$

we have

$$[b', v] = \lambda v \qquad \forall v \in \mathfrak{n}.$$

Applying lemma 3 to this case in which the operator $S = 0$, we get the desired result, with the obvious particullarities for signatures $(0, n)$ and $(n, 0)$.

Case II. $< \, , \, >_{|\mathfrak{n}\times\mathfrak{n}}$ is degenerate.
There exists $e \in \mathfrak{n}$ such that $<e, \cdot>= 0$ all over $\mathfrak{n}$. So, since $< \, , \, >$ is nondegenerate on $\mathfrak{g}$, $<e, b>\neq 0$. Hence we may just suppose

$$<e, b>= 1.$$

Define two maps a and C by

$$a(x) =<e, x>, \qquad\qquad C(x, y) = a(x)y - a(y)x, \qquad\qquad \forall x, y \in \mathfrak{g}.$$

Immediately one recognizes the linearity and bilinearity, respectively, of a and C. From $\ker a = \mathfrak{n}$, the skew-adjointness of C and

$$C(b, u) = u = [b, u] \qquad C(u, v) = 0 = [u, v], \quad \forall u, v \in \mathfrak{n},$$

we find that $C = [\, , \,]$. Now we can compute, for any $x, y, z \in \mathfrak{g}$,

$$<\nabla_x y, z>= \frac{1}{2}(<[x, y], z> - <[y, z], x> + <[z, x], y>) =$$

$$= \frac{1}{2}\Big(a(x) <y, z> -a(y) <x, z> -a(y) <z, x> +a(z) <y, x>$$

$$-a(x) <z, y> +a(z) <x, y> \Big) = a(z) <x, y> -a(y) <x, z> .$$

Hence

$$\nabla_x y =<x, y> e - a(y)x$$

and then

$$R(x, y)z = -\nabla_x \nabla_y z + \nabla_y \nabla_x z + \nabla_{[x,y]} z =$$

$$= - <y, z> \nabla_x e + a(z)\nabla_x y - a(z)\nabla_y x + <x, z> \nabla_y e + a(x)\nabla_y z - a(y)\nabla_x z =$$

$$= - <y, z><x, e> e + a(z) <x, y> e - a(z)a(y)x - a(z) <y, x> e + a(z)a(x)y$$

$$+ <x, z><y, e> e + a(x) <y, z> e - a(x)a(z)y - a(y) <x, z> e + a(y)a(z)x = 0$$

(ii) If one wants sectional curvature $K > 0$ choose the following metric. Take $b' = \sqrt{K}b$ and define a scalar product $<\ ,\ >$ on $\mathfrak{g}$ satisfying

$$<b', b'> = -1, \qquad <b', \mathfrak{n}> = 0$$

$$<\ ,\ >_{|\mathfrak{n}\times\mathfrak{n}} \text{ of signature } (p-1, n-p) \quad (1 \leq p \leq n).$$

For what we have seen above, with the left invariant metric induced by this scalar product, G has constant sectional curvature K.

For $K < 0$, we do the same with $b' = \sqrt{-K}b$ and choosing a scalar product on $\mathfrak{g}$ satisfying

$$<b', b'> = 1, \qquad <b', \mathfrak{n}> = 0,$$

$$<\ ,\ >_{|\mathfrak{n}\times\mathfrak{n}} \text{ of signature } (p, n-1-p) \quad (0 \leq p \leq n-1).$$

Finaly, if one wants $K = 0$, it is sufficient to choose a left invariant metric on G that is degenerate on $\mathfrak{n}$. This must be an indefinite metric. □

By Theorems 1 and 2 we may conclude the following.

Proposition 1 *Every non-abelian Lie group admits left invariant metrics of signature $(p, n-p)$ such that $p < n$ and $S < 0$, or $0 < p$ and $S > 0$.*

Let us denote by $\mathfrak{F}(p)$ the class of Lie groups such that every left invariant metric of signature $(p, n-p)$ has sectional curvature of constant sign. As we said before $\mathfrak{F}(0) = \mathfrak{S}$. Looking at the proof of Theorem 1 we can establish

$$\mathfrak{S} = \mathfrak{F}(0) = \mathfrak{F}(1) = \mathfrak{F}(2) = \ldots = \mathfrak{F}(n).$$

In other words, it is useless to search for other Lie groups for which one has the same nice results of Theorem 2.

Notice that according to the well known Theorem of R.S. Kulkarni, which says that, for a connected manifold of dimension ≥ 3 and indefinite metric, if the sectional curvatures have an upper bound (or a lower) then they are constant, one becomes aware that looking for Lie groups in $\mathfrak{F}(p)$ $(0 < p < n,\ n \geq 3)$ is the same as looking for those that have constant K for all such metrics.

3 A Non-Complete semi-Riemannian Structure.

Let M be a simply connected semi-Riemannian manifold of dimension n and signature $(p, n-p)$, with constant sectional curvature K and geodesically complete — in the usual concept, M is a simply connected space form. Consulting, for example, [7], we note that a simply connected space form is diffeomorphic to the Euclidean space if and only if

(a) $p = n, n-1$ and $K > 0$;
(b) $K = 0$;
(c) $p = 0, 1$ and $K < 0$.
Otherwise, M is the product of the Euclidean space with a sphere.

Now suppose M is a simply connected Lie group $\widetilde{G}$, belonging to the special class $\mathfrak{S}$, provided with any metric given by part (ii) of Theorem 2 such that p and K are *out* of cases (a), (b) and (c) above. General Lie group theory says that a simply connected and solvable Lie group is diffeomorphic to the Euclidean space ([6]). Thus $\widetilde{G}$ is diffeomorphic to the Euclidean space. We derive from this that with the prescribed semi-Riemannian structure, completeness must fail in $\widetilde{G}$.

4 Example.

Fix $n \geq 2$. A simply connected Lie group in the special class $\mathfrak{S}$ is

$$G = \left\{ \begin{bmatrix} 1 & 0 \\ v & sI_{n-1} \end{bmatrix} \in GL(n, \mathbb{R}) : v \in \mathbb{R}^{n-1}, s > 0 \right\}$$

As a manifold this is just $M = \mathbb{R}^{n-1} \times \mathbb{R}^+$. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n = T_{(v,s)}M$ for all $(v, s) \in M$ (with the Lie product $(v, s) \cdot (u, t) = (v + su, st)$ it is easy to see that the $U_{i_{(v,s)}} = se_i$ are left invariant vector fields — in the above notation, $\mathfrak{n}$ is the ideal spanned by $U_1, \ldots, U_{n-1}$ and $b = U_n$).

We now define a Lorentzian metric[1] on M giving its components relative to the basis $e_1, \ldots, e_n$:

$$g_{ij} = \frac{\delta_{ij}}{s^2}, \text{ for } i < n, \qquad g_{nn} = -\frac{1}{s^2}$$

(this is just the left invariant metric which makes $U_1, \ldots, U_n$ an orthonormal basis so that $< U_n, U_n >= -1$). Now we can find the Christoffel symbols for the Levi-Civita connection. Calculations lead us to

$$\Gamma^n_{i\tilde{\jmath}} = \Gamma^h_{ij} = \Gamma^h_{in} = \Gamma^n_{in} = \Gamma^h_{nn} = 0, \qquad \Gamma^n_{ii} = \Gamma^i_{in} = \Gamma^n_{nn} = -\frac{1}{s},$$

for all $i, j, \tilde{\jmath}, h < n$, $\tilde{\jmath} \neq i$. Now let us find the geodesics of M. Suppose $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, \gamma)$ is such a curve so that (note $\gamma(t) > 0$)

$$\alpha_i(0) = v_{io}, \quad \gamma(0) = s_o > 0, \qquad \text{and} \qquad \alpha_i'(0) = \xi_{io}, \quad \gamma'(0) = \eta_o.$$

[1] notice the generalization, on both dimension and signature, of the "Poincaré half space" or "Lobatchevski Plane".

The system of ordinary differential equations of a geodesic gives:

$$\begin{cases} \gamma\alpha_i'' - 2\gamma'\alpha_i' = 0 \\ \gamma\gamma'' - \gamma'^2 - \sum_{i=1}^{n-1}\alpha_i'^2 = 0. \end{cases}$$

Denoting $\beta_i = \alpha_i'$, from the first equation we get $\frac{\beta_i'}{\beta_i} = \frac{2\gamma'}{\gamma}$. So $(\log|\beta_i|)' = (\log\gamma^2)'$, which gives, $\beta_i = \frac{\xi_{io}}{s_o^2}\gamma^2$. Henceforth, from the second equation we get

$$\gamma\gamma'' - \gamma'^2 - Q\gamma^4 = 0,$$

where $Q = s_o^{-4}\sum_{i=1}^{n-1}\xi_{io}^2$. We have $Q = 0$ if and only if all the $\xi_{io} = 0$. It is easy to see that in this case

$$\alpha(t) = (v_{1o}, \ldots, v_{n-1\,o}, s_o e^{(\frac{\eta_o}{s_o}t)})$$

is the desired geodesic, which happens to be the only complete one.

Now suppose $Q > 0$. Let us start by consider $\eta_o \neq 0$. Making the substitution $\gamma' = \gamma z$, and hence $\gamma'' = \gamma z^2 + \gamma^2 z\frac{dz}{d\gamma}$, the above differential equation symplifies to

$$z\frac{dz}{d\gamma} = Q\gamma \Longleftrightarrow z^2 = Q\gamma^2 + D \Longleftrightarrow \frac{\gamma'}{\gamma\sqrt{Q\gamma^2 + D}} = \pm 1 \tag{4.1}$$

($+$ or $-$ depending on the signal of η_o), where

$$D = \frac{\eta_o^2}{s_o^2} - Qs_o^2 = \frac{1}{s_o^2}(\eta_o^2 - \sum_{i=1}^{n-1}\xi_{io}^2).$$

Equation (4.1) is easely integrable, giving us different solutions for different values of D.

Case I. $D = 0$.

$$\frac{\gamma'}{\gamma^2} = \pm\sqrt{Q} \Longleftrightarrow \frac{1}{\gamma} = \mp\sqrt{Q}\,t + \frac{1}{s_o}.$$

Thus

$$\gamma(t) = \frac{s_o}{1 \mp s_o\sqrt{Q}\,t},$$

$$\alpha_i'(t) = \beta_i(t) = \frac{\xi_{io}}{(1 \mp s_o\sqrt{Q}\,t)^2}$$

and

$$\alpha_i(t) = \frac{\pm\xi_{io}}{s_o\sqrt{Q}(1 \mp s_o\sqrt{Q}\,t)} \mp \frac{\xi_{io}}{s_o\sqrt{Q}} + v_{io}.$$

Case II. $D > 0$. By integration of equation (4.1), we find

$$\frac{1}{\sqrt{D}} \log \frac{\sqrt{Q\gamma^2 + D} - \sqrt{D}}{\gamma} = \pm t + t_o,$$

where $t_o = \frac{1}{\sqrt{D}} \log \frac{\sqrt{Qs_o^2+D}-\sqrt{D}}{s_o} = \frac{1}{\sqrt{D}} \log \frac{|\eta_o|-s_o\sqrt{D}}{s_o^2}$. Solving for the implicit function, we get

$$\gamma(t) = \frac{2\sqrt{D}\, e^{\sqrt{D}(t_o \pm t)}}{Q - e^{2\sqrt{D}(t_o \pm t)}},$$

$$\beta_i(t) = \frac{4\xi_{io} D\, e^{2\sqrt{D}(t_o \pm t)}}{s_o^2(Q - e^{2\sqrt{D}(t_o \pm t)})^2},$$

$$\alpha_i(t) = \pm \frac{2\xi_{io}\sqrt{D}}{s_o^2(Q - e^{2\sqrt{D}(t_o \pm t)})} \mp \frac{2\xi_{io}\sqrt{D}}{s_o^2(Q - e^{2\sqrt{D}\, t_o})} + v_{io},$$

Case III. $D < 0$. Again, equation (4.1) gives

$$\frac{1}{\sqrt{-D}} \arccos\left(\frac{\sqrt{-D}}{\gamma\sqrt{Q}}\right) = \pm t + to,$$

where $t_o = \frac{1}{\sqrt{-D}} \arccos\left(\frac{\sqrt{-D}}{s_o\sqrt{Q}}\right)$. So

$$\gamma(t) = \frac{\sqrt{-D}}{\sqrt{Q}\cos(\sqrt{-D}(t_o \pm t))},$$

$$\beta_i(t) = \frac{-\xi_{io} D}{s_o^2 Q \cos^2(\sqrt{-D}(t_o \pm t))},$$

$$\alpha_i(t) = \pm \frac{\xi_{io}\sqrt{-D}}{s_o^2 Q} \mathrm{tg}(\sqrt{-D}(t_o \pm t)) \mp \frac{\xi_{io}\sqrt{-D}}{s_o^2 Q}\sqrt{s_o^2 Q + D} + v_{io}.$$

Finally, if one considers $\eta_o = 0$, which is equivalent to $s_o^2 Q + D = 0$, then the solutions of case III will adapt perfectly.

We can conclude that, on our example of a semi-Riemannian homogeneous space with constant sectional curvature 1, almost every geodesic is non complete.

There remains the question: Which is the condition on a Lie group with left invariant semi-Riemannian structure so that it is complete?

A theorem due to Marsden (cf.[4]) says that any compact homogeneous semi-Riemannian space is complete. When trying to see what happens on Lie groups we were led to the following interesting result which we have never heard about. Let G be a Lie group with left invariant metric.

Lemma 4 *Right invariant vector fields on G are Killing fields.*

Proof. In every manifold, the Lie derivative of a tensor A, with respect to a differentiable vector field X, verifies

$$\mathcal{L}_X A = \lim_{t\to 0} \frac{1}{t}(\psi_t^*(A) - A),$$

where $\{\psi_t\}$ is the (local) flow of X ([4]). Now let X be a right invariant vector field on G and let us determine its flow. A maximal integral curve α of X starting at e,

$$\alpha(0) = e, \qquad \alpha'(t) = X_{\alpha(t)},$$

is precisely the (unique) one-parameter subgroup of G associated to X (induced by the Lie homomorphism $t\frac{d}{dt} \mapsto tX$ between the Lie algebras $\mathbb{R}$ and the one consisting of right invariant vector fields on G). Let $\{\psi_t\}$ be the flow of X. We have $\psi_t(e) = \alpha(t)$. Given $g \in G$,

$$\left(\frac{dR_g \circ \alpha}{dt}\right)_0 = R_{g_*e}\left(\frac{d\alpha}{dt}(0)\right) = R_{g_*e}(X_e) = X_g,$$

hence $\psi_t(g) = \alpha(t)g$, i.e., $\psi_t = L_{\alpha(t)}$. Thus ψ_t is an isometry, or, in other words, $\psi_t^* <\, ,\, >=<\, ,\, >$. By the initial equality, $\mathcal{L}_X <\, ,\, >= 0$ as we wanted.

□

5 Bi-invariant Metrics.

Recall that a Lie algebra $\mathfrak{g}$ is *compact* if it is the Lie algebra of a compact Lie group. We say that $\mathfrak{g}$ is *simple* if it has no proper ideals other than 0. Recall also E. Cartan's criterion for semisimplicity: $\mathfrak{g}$ is semisimple if, and only if, its Killing form is nondegenerate.

Let us now recall some basic facts about the structure of a semisimple Lie algebra, that can be seen in [6]. Every Lie algebra $\mathfrak{g}$ admits a Cartan subalgebra, that is, a nilpotent subalgebra $\mathfrak{h}$ which coincides with its normalizer in $\mathfrak{g}$. All Cartan subalgebras of $\mathfrak{g}$ have the same dimension. This natural number is then called the *rank* of $\mathfrak{g}$. It is also well known that every complex semisimple Lie algebra $\mathfrak{g}$ admits a *root decomposition* relative to one of its Cartan subalgebras $\mathfrak{h}$, that is, $\mathfrak{g}$ can be decomposed as the direct sum

$$\mathfrak{g} = \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha = \{y \in \mathfrak{g} : [x, y] = \alpha(x)y,\ \forall x \in \mathfrak{h}\}$$

and Δ is a subset of $\mathfrak{h}^*$, complex dual of $\mathfrak{h}$, for which $\alpha \in \Delta$ iff $\mathfrak{g}_\alpha \neq 0$. $\Delta \setminus \{0\}$ is called a *root system.*

Lie group theory tells us that $\mathfrak{h} = \mathfrak{g}_0$, $\mathfrak{h}$ is maximal abelian (even in the real case) and that the root subspaces $\mathfrak{g}_\alpha$ $(\alpha \neq 0)$ have dimension 1.

When $\mathfrak{g}$ is real and semisimple then the complexification $\mathfrak{g}^c$ is semisimple[2] and, if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\mathfrak{h}^c$ is a Cartan subalgebra of $\mathfrak{g}^c$.

Lemma 5 *Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{h}$ one of its Cartan subalgebras.*

There exists a real linear form $\xi \neq 0$ on $\mathfrak{h}$ such that $\mathfrak{g}_\xi \neq 0$ ($\mathfrak{g}_\xi$ defined as above) if, and only if, there exists $y \in \mathfrak{g} \setminus \mathfrak{h}$ and a nonzero root α of $\mathfrak{g}^c$, relatve to $\mathfrak{h}^c$, such that $\mathfrak{g}^c_\alpha = \mathbb{C}y$. In this a case $\mathfrak{g}_\xi = \mathbb{R}y$.

The proof is immediate, for $\alpha = \xi^c$.

Now we are able to present the only theorem of this section. Its proof was mainly taken from [2,lemma 7.6], the particular case when $\mathfrak{g}$ is compact. First recall: if a left invariant metric on a Lie group is bi-invariant, then all the adjoint morphisms $\mathrm{ad}\,(x)$, $x \in \mathfrak{g}$, are skew-adjoint. In a Lie group G, the Killing form B of its Lie algebra is $\mathrm{Ad}\,(G)$-invariant, so, when G is semisimple, the left invariant metric induced by B is also right invariant.

Theorem 3 *Let G be a Lie group with simple Lie algebra $\mathfrak{g}$ of rank r. If $\mathfrak{g}$ satisfies one of the following conditions:*

(i) $\dim(\mathfrak{g})$ *or r are odd;*

(ii) for some Cartan subalgebra $\mathfrak{h}$, in the root decomposition of $(\mathfrak{g}^c, \mathfrak{h}^c)$ there is a root subspace of type $\mathbb{C}y$ with $y \in \mathfrak{g}$;

(iii) $\mathfrak{g}$ is compact;

then any bi-invariant metric on G is induced by a multiple of the Killing form.

Proof. Let $< \, , \, >$ denote any bi-invariant metric on G and B the Killing form. There is a linear bijection S of $\mathfrak{g}$ such that

$$<x, y>= B(S(x), y), \quad \forall x, y \in \mathfrak{g}.$$

From this we can deduce that S commutes with all the $\mathrm{ad}\,(x)$, $x \in \mathfrak{g}$.

Now, if $y \in \mathfrak{g}$ is an eigenvector of S associated to a real eigenvalue $\lambda \neq 0$, then $S[x, y] = [x, S(y)] = \lambda[x, y]$, so each eigenspace is an ideal. Since $\mathfrak{g}$ is simple, this eigenspace is all $\mathfrak{g}$ and so $< \, , \, >= \lambda B$ on $\mathfrak{g}$. Thus we must assure the existence of one real eigenvalue for S. If $\dim(\mathfrak{g})$ is odd this is trivial. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}$ is abelian and equals its normalizer, then $[S(\mathfrak{h}), \mathfrak{h}] \subset \mathfrak{h}$ and hence $S(\mathfrak{h}) = \mathfrak{h}$. Thus odd rank implies, as above, an eigenvalue for S.

[2]note: $\mathfrak{g}$ simple $\not\Rightarrow$ $\mathfrak{g}^c$ simple

If $\mathfrak{g}$ satisfies (ii), then by the lemma there is a real linear form $\xi \neq 0$ on $\mathfrak{h}$ with $\mathfrak{g}_\xi = \mathbb{R}y$, that is, $[x,y] = \xi(x)y$, for all $x \in \mathfrak{h}$. Then

$$[x, S(y)] = S[x,y] = \xi(x)S(y), \quad \forall x \in \mathfrak{h}$$

So $S(y) = \lambda y$ for some nonzero $\lambda \in \mathbb{R}$.

Finally, if $\mathfrak{g}$ is compact, then $-B$ is an inner product, S is symmetric and certainly diagonalizable.

□

Theorem 3 may be generalized to Lie groups with reductive Lie algebra ($[\mathfrak{g},\mathfrak{g}]$ semisimple), since, with bi-invariant metric, the simple components of $\mathfrak{g}$ and the center of $\mathfrak{g}$ are all orthogonal to each other.

There is a large class of Lie algebras satisfying condition (ii) of the theorem: the simple split Lie algebras. $\mathfrak{g}$ is said to be *split* if any of its maximal $\mathbb{R}$-diagonalizable subalgebras is a Cartan subalgebra. We have an $\mathbb{R}$-*diagonalizable* subalgebra $\mathfrak{a} \subset \mathfrak{g}$ when there exists a basis in $\mathfrak{g}$ with respect to which all operators $\mathrm{ad}\,(x)$ $(x \in \mathfrak{a})$ are expressed by diagonal matrices.

The following are examples of simple split Lie algebras ([5]): $\mathfrak{sl}_n$ $(n \geq 2)$, $\mathfrak{so}_{k,k+1}$ $(k \geq 1)$, $\mathfrak{so}_{k,k}$ $(k \geq 3)$, $\mathfrak{sp}_n$ $(n \geq 2)$.

In [3,remark 2] we find a Lorentz metric on SL_2 with constant sectional curvature -1 and that this "metric is essentially the same as the Killing-Cartan form". We can now stablish: all bi-invariant metrics on SL_2 have constant sectional curvature.

6 Remark on Complex Simple Lie Algebras.

Let $\mathfrak{g}$ be a complex Lie algebra and ρ a faithfull representation of $\mathfrak{g}$ in a complex vector space. Notice that such representations exist by the well known theorem of Ado. We call the bilinear and symmetric form on $\mathfrak{g}$

$$\beta(x,y) = \mathrm{tr}\,(\rho(x)\rho(y)).$$

a *trace form.*

With respect to β all endomorphisms $\mathrm{ad}\,(x)$ are skew-adjoint and it is proven like Cartan's semisimplicity criterion that, if $\mathfrak{g}$ is semisimple, then β is nondegenerate ([5]).

Proposition 2 *Every trace form on a complex simple Lie algebra is a multiple of the Killing form.*

The proof is obviously equal to the one of theorem 3. This time there is no problem with eigenvalues.

In other sense we have the following.

Corollary 1 *If* $\mathfrak{k}$ *is a simple Lie subalgebra of* $\mathfrak{g} \lll (n, \mathbb{C})$*, then its Killing form is a multiple of the trace form*

$$\mathrm{tr}\,(XY), \qquad X, Y \in \mathfrak{k}.$$

References

[1] F. Barnet: *On Lie groups that admit left-invariant Lorentz metrics of constant seccional curvature*, Illinois J. Math. 33 (1989), 631-642

[2] J. Milnor: *Curvatures of left invariant metrics on Lie Groups*, Advances in Mathematics 21 (1976), 293-329

[3] K. Nomizu: *Left invariant Lorentz metrics on Lie groups*, Osaka J. Math. 16 (1979), 143-150

[4] B. O'Neill: *Semi-Riemannian Geometry*, Academic, 1983

[5] A. L. Onishchik and E. B. Vinberg: *Lie groups and Lie algebras III*, EMS, vol. 41, Springer, 1991

[6] V. S. Varadarajan: *Lie groups, Lie algebras and their Representations*, Springer, 1965

[7] J. A. Wolf: *Spaces of Constant Curvature*, McGraw-Hill, 1967

Rui Pedro Albuquerque
Departamento de Matemática
Universidade de Évora
Portugal
Email: rpa@dmat.uevora.pt