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INTEGRATION OF THE RIGID BODY EQUATIONS WITH QUADRATIC CONTROLS

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ABSTRACT. We prove the integrability of the rigid body equations with one and two quadratic controls via Jacobi functions.

1. Introduction

In the last decade there was a great deal of interest in the study of the rigid body equations with one and two quadratic controls. We can mention here the papers of Bloch and Marsden [1], Holm and Marsden [2] and Puta [4], [5].

The goal of our paper is to prove their integrability via Jacobi elliptic functions.

2. Jacobi elliptic functions

Let us remind for beginning some general facts on Jacobi elliptic functions which will be frequently used in all that follows.

Let τ be a complex number such that its imaginary part is positive and

$$q \stackrel{def}{=} e^{i\pi\tau} .$$

Then the theta functions $\theta_1, \theta_2, \theta_3, \theta_4$ are respectively given by:

$$\begin{aligned} \theta_1(z) &= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z\,, \\ \theta_2(z) &= 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos(n+1)z \end{aligned}$$

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$$\begin{aligned}
\theta_3(z) &= 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz\,,\\
\theta_4(z) &= 1+2\sum_{n=1}^{\infty} (-1)^n q^{n^2}\cos 2nz\,.
\end{aligned}$$

q is called the nome of the theta functions and τ is its parameter.

The Jacobi elliptic functions are defined as ratio of theta functions as follows:

$$\begin{aligned}
sn\ u &= \frac{\theta_3(0)}{\theta_2(0)}\cdot\frac{\theta_1(z)}{\theta_4(z)}\,,\\
cn\ u &= \frac{\theta_4(0)}{\theta_2(0)}\cdot\frac{\theta_2(z)}{\theta_4(z)}\,,\\
dn\ u &= \frac{\theta_4(0)}{\theta_3(0)}\cdot\frac{\theta_3(z)}{\theta_4(z)}\,,
\end{aligned}$$

where

$$z=\frac{u}{\theta_3^2(0)}\,.$$

If the parameter τ is purely imaginary, i.e. q is real, the Jacobi elliptic functions are all real for real values of u. Moreover, the following equalities hold:

(i) $sn^2u+cn^2u=1$,

(ii) $dn^2u+\frac{\theta_2^4(0)}{\theta_3^4(0)}sn^2u=1$,

(iii) $dn^2u-\frac{\theta_2^4(0)}{\theta_3^4(0)}cn^2u=\frac{\theta_4^4(0)}{\theta_3^4(0)}$.

The first relation is the counterpart for the elliptic sine and cosine of a familiar trigonometric identity and explains the presence of the numerical factor $\frac{\theta_3(0)}{\theta_2(0)}$.

Let us define the parameters k and k' by the equations:

$$k=\frac{\theta_2^2(0)}{\theta_3^2(0)};\ k'=\frac{\theta_4^2(0)}{\theta_3^2(0)}\,.$$

Then we have:

$$k^2+k^{'2}=1 \tag{2.1}$$

$$sn^2u+cn^2u=1 \tag{2.2}$$

$$dn^2u+k^2sn^2u=1 \tag{2.3}$$

$$dn^2u-k^2cn^2u=k^{'2}\,. \tag{2.4}$$

k is termed the modulus of the Jacobi elliptic functions and k' is the complementary modulus.

Using the definitions of the Jacobi elliptic functions we are lead immediately to the following formulas:

$$\frac{d}{du} sn\ u = cn\ u \cdot dn\ u\,, \tag{2.5}$$

$$\frac{d}{du} cn\ u = -sn\ u \cdot dn\ u\,, \tag{2.6}$$

$$\frac{d}{du} dn\ u = -k^2 sn\ u \cdot cn\ u\,. \tag{2.7}$$

Higher-order derivatives are now calculable by repeated application of these relations and then using MacLaurin's theorem, power series expansions can be found for these functions which are regular in the neighborhood of $u = 0$. More exactly we have:

$$sn\ u = u - \frac{1}{3!}(1+k^2)u^3 + \frac{1}{5!}(1+14k^2+k^4)u^5 - \dots\,, \tag{2.8}$$

$$cn\ u = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}(1+4k^2)u^4 - \dots\,, \tag{2.9}$$

$$dn\ u = 1 - \frac{1}{2!}k^2u^2 + \frac{1}{4!}(4k^2+k^4)u^4 - \dots\,. \tag{2.10}$$

3. Integrability of the rigid body with one control

The rigid body equations with a single control about the minor axis are given by

$$\begin{cases} I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 \\ I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3 \\ I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + u\,, \end{cases} \tag{3.1}$$

where the I_i are the principal moments of inertia and we take $I_1 > I_2 > I_3$. Note that the control is taken about the minor axis, but all our results carry over without essential change to the case where the control is about the major axis.

Now we employ the feedback:

$$u = -lI_1I_2\omega_1\omega_2\,, \tag{3.2}$$

where l is the feedback gain parameter. We refer to the system with this feedback as the controlled system. It is now easy to show that:

Proposition 3.1 ([1]) *For the controlled system (3.1), (3.2) the quantities*

$$T(\omega_1,\omega_2,\omega_3) = I_1\omega_1^2 + I_2\omega_2^2 + \frac{I_3(I_1 - I_2)}{I_1 - I_2 - lI_1I_2}\omega_3^2 \tag{3.3}$$

and

$$M^2(\omega_1,\omega_2,\omega_3) = I_1^2\omega_1^2 + I_2^2\omega_2^2 + \frac{I_3^2(I_1-I_2)}{I_1-I_2-lI_1I_2}\omega_3^2 \tag{3.4}$$

are constants of the motion.

Proof. Note that

$$\frac{d}{dt}\omega_3^2 = 2\frac{I_1-I_2-lI_1I_2}{I_3}\omega_1\omega_2\omega_3\,.$$

Then the calculations reduces to the standard rigid body calculations. □

Theorem 3.1 *Under the conditions*

$$(i)\ I_1 > I_2 > \frac{M^2}{T} > I_3$$

and

$$(ii)\ l < \frac{1}{I_2} - \frac{1}{I_1},$$

the controlled system (3.1), (3.2) is integrable via Jacobi's elliptic functions.

Proof. Let us define

$$\left\{\begin{array}{l} \omega_1 = P\ cn\ p(t-t_0) \\ \omega_2 = -Q\ sn\ p(t-t_0) \\ \omega_3 = R\ dn\ p(t-t_0)\,, \end{array}\right. \tag{3.5}$$

where P, Q, R, p, k are five unknown quantities which will be determined from the relations (3.1)-(3.4). For beginning let us observe that:

$$\left\{\begin{array}{l} \dot\omega_1 = -Pp\ sn\ p(t-t_0)\ dn\ p(t-t_0) \\ \dot\omega_2 = -Qp\ cn\ p(t-t_0)\ dn\ p(t-t_0) \\ \dot\omega_3 = -k^2Rp\ sn\ p(t-t_0)\ cn\ p(t-t_0) \end{array}\right.$$

and then if we plug these quantities in (3.1), (3.2) we obtain:

$$\left\{\begin{array}{l} CRk^2p = (I_1-I_2-lI_1I_2)PQ \\ APp = (I_2-I_3)QR \\ BQp = (I_1-I_3)PR\,. \end{array}\right. \tag{3.6}$$

On the other hand if we put (3.5) in (3.3) and (3.4) and make $t = t_0$, we obtain:

$$\left\{\begin{array}{l} AP^2 + \frac{I_3(I_1-I_2)}{I_1-I_2-lI_1I_2}R^2 = T \\ A^2P^2 + \frac{I_3^2(I_1-I_2)}{I_1-I_2-lI_1I_2}R^2 = M^2\,. \end{array}\right. \tag{3.7}$$

Using now the relations (3.6), (3.7) we are lead immediately to:

$$\begin{aligned}
p^2 &= \frac{(I_2-I_3)(I_1T-M^2)(I_1-I_2-lI_1I_2)}{I_1I_2I_3(I_1-I_2)}, \\
k^2 &= \frac{(I_1-I_2)(M^2-I_3T)}{(I_2-I_3)(I_1T-M^2)}, \\
P^2 &= \frac{M^2-I_3T}{I_1(I_1-I_3)}, \\
Q^2 &= \frac{M^2-I_3T}{I_2(I_2-I_3)}, \\
R^2 &= \frac{M^2-I_3T}{I_1-I_3}.
\end{aligned}$$

The conditions for P, Q, R, p, k to be real are:

$$\begin{cases} I_1-I_2-lI_1I_2>0 \\ I_1T-M^2>0 \\ M^2-I_3T>0 \\ k^2<1 \end{cases}$$

or equivalent:

$$\begin{cases} I_1>I_2>\frac{M^2}{T}>I_3 \\ l<\frac{1}{I_2}-\frac{1}{I_1} \end{cases}$$

as required. □

If $\frac{M^2}{T}=I_2$, then $k=1$. In this case, as t increases from $-\infty$ to ∞, the body with one control moves from a state in which it is rotating about O_2 with angular velocity Q to a state in which it is rotating about this axis with angular velocity $-Q$, for each

$$l<\frac{1}{I_2}-\frac{1}{I_1}.$$

If $\frac{M^2}{T}=C$, then $P=Q=k=0$ and the body with one control rotates with constant angular velocity about its axis of minimum moment for each

$$l<\frac{1}{I_2}-\frac{1}{I_1}.$$

The stability of motion depends obviously by the parameter l.

In the special case $I_1=I_2$ the modulus k vanishes and we have only trigonometric functions in the solutions. In this case, equations (3.5) shows that ω_1 and ω_2 oscillate sinusoidally with the same amplitude, but their phases are in quadrature. ω_3 is constant if and only if $l=0$ as is evident from the third relation of the equations (3.1).

Remark 3.1 In the particular case $l=0$, we refined the well known case of the free rigid body [3].

4. Integrability of the rigid body with two controls

The rigid body equations with two controls about the minor and the major axes can be written in the following form:

$$\left\{\begin{array}{l} I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + u_1 \\ I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3 \\ I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + u_2\,. \end{array}\right. \tag{4.1}$$

Now we employ the feedbacks:

$$u_1 = -l\omega_2\omega_3 \tag{4.2}$$

and

$$u_3 = l\omega_1\omega_2\,, \tag{4.3}$$

where l is the feedback gain parameter. We refer to the system (4.1)-(4.3) as the controlled system. It is easy to show that:

Proposition 4.1 *For the controlled system (4.1)-(4.3) the quantities:*

$$T(\omega_1, \omega_2, \omega_3) = I_1\omega_1^2 + (I_2 + lI_2^2)\omega_2^2 + I_3\omega_3^2 \tag{4.4}$$

and

$$M^2(\omega_1, \omega_2, \omega_3) = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \tag{4.5}$$

are constants of the motion.

Proof. The proof is a straightforward computation and we shall omit any other details. □

Theorem 4.2 *Under the conditions:*

$$I_3 < \frac{M^2}{T} < \frac{I_2}{1 + lI_2} < A\,,$$

the controlled system (4.1)-(4.3) is integrable via Jacobi's elliptic functions.

Proof. Let us define:

$$\left\{\begin{array}{l} \omega_1 = P\ cn\ p(t - t_0) \\ \omega_2 = -Q\ sn\ p(t - t_0) \\ \omega_3 = R\ dn\ p(t - t_0)\,, \end{array}\right. \tag{4.6}$$

where P, Q, R, p, k are five unknown quantities which will be determined from the relations (4.1)-(4.5).

Using the same technique as in the previous section we are lead to:

$$\begin{aligned}
P^2 &= \frac{M^2 - I_3 T}{I_1(I_1 - I_3)} \\
Q^2 &= \frac{M^2 - I_3 T}{I_2(I_2 - I_3 - l I_2 I_3)} \\
R^2 &= \frac{I_1 T - M^2}{I_3(I_1 - I_3)} \\
p^2 &= \frac{(I_1 T - M^2)(I_2 - I_3 - l I_2 I_3)}{I_1 I_2 I_3} \\
k^2 &= \frac{(I_1 - I_2 + l I_1 I_2)(M^2 - I_3 T)}{(I_2 - I_3 - l I_2 I_3)(I_1 T - M^2)} .
\end{aligned}$$

The conditions for P, Q, R, p, k to be real are

$$\begin{cases}
I_1 T - M^2 > 0 \\
M^2 - I_3 T > 0 \\
I_2 - I_3 - l I_2 I_3 > 0 \\
I_1 - I_2 + l I_1 I_2 > 0 \\
k^2 < 1
\end{cases}$$

or equivalent

$$I_3 < \frac{M^2}{T} < \frac{I_2}{1 + l I_2} < I_1$$

as required. □

Remark 4.1 In the particular case $l = 0$ we refined the well known result of the free rigid body [3].

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