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NONEXISTENCE OF FINITE-DIMENSIONAL QUANTIZATIONS OF A NONCOMPACT SYMPLECTIC MANIFOLD

MARK J. GOTAY AND HENDRIK B. GRUNDLING

ABSTRACT. We prove that there is no faithful representation by skew-hermitian matrices of a "basic algebra of observables" $\mathfrak{b}$ on a noncompact symplectic manifold M. Consequently there exists no finite-dimensional quantization of *any* Lie subalgebra of the Poisson algebra $C^\infty(M)$ containing $\mathfrak{b}$.

1. Introduction

Let M be a connected noncompact symplectic manifold. On physical grounds one expects a quantization of M, if it exists, to be infinite-dimensional. This is what we rigorously prove here, in the framework of the paper [Go1]. Our precise hypotheses are spelled out below.

A key ingredient in the quantization process is the choice of a *basic algebra of observables* in the Poisson algebra $C^\infty(M)$. This is a Lie subalgebra $\mathfrak{b}$ of $C^\infty(M)$ such that

(B1) $\mathfrak{b}$ is finitely generated,

(B2) the Hamiltonian vector fields $X_f, f \in \mathfrak{b}$, are complete,

(B3) $\mathfrak{b}$ is transitive and separating, and

(B4) $\mathfrak{b}$ is minimal with respect to these conditions.

A Lie subalgebra $\mathfrak{b} \subset C^\infty(M)$ is "transitive" if $\{X_f(m) \mid f \in \mathfrak{b}\}$ spans T_mM at every point. $\mathfrak{b}$ is "separating" provided its elements globally separate points of M. Throughout this paper, we assume that $\mathfrak{b}$ is finite-dimensional.

Now fix a basic algebra $\mathfrak{b}$, and let $\mathcal{O}$ be any Lie subalgebra of $C^\infty(M)$ containing 1 and $\mathfrak{b}$. Then by a *finite-dimensional quantization* of the pair $(\mathcal{O}, \mathfrak{b})$ we mean a Lie representation $\mathcal{Q}$ of $\mathcal{O}$ by skew-hermitian matrices on some $\mathbb{C}^k$ such that

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(Q1) $\mathcal{Q}(1) = I$,

(Q2) $\mathcal{Q} \restriction \mathfrak{b}$ is irreducible, and

(Q3) $\mathcal{Q} \restriction \mathfrak{b}$ is faithful.

We refer the reader to [Go1] for a detailed discussion of these matters. We remark that in the infinite-dimensional case there are additional conditions which must be imposed upon a quantization. We also elaborate briefly on (Q3). Although faithfulness is not usually assumed in the definition of a quantization, it seems to us a reasonable requirement in that a classical observable can hardly be regarded as "basic" in a physical sense if it is in the kernel of a quantization map. In this case, it cannot be obtained in any classical limit from the quantum theory.

2. The Obstruction

Given the definitions above, we state our main result:

Theorem 1. *Let $\mathfrak{b}$ be a finite-dimensional basic algebra on a noncompact symplectic manifold M. Then $\mathfrak{b}$ has no faithful representations by skew-hermitian matrices.*

Proof. We argue by contradiction. Suppose there exists such a representation $\mathcal{Q}$ of $\mathfrak{b}$ on some $\mathbb{C}^k$. As $\mathcal{Q}(\mathfrak{b})$ consists of skew-hermitian matrices, $\mathcal{Q}$ is completely reducible. Since $\mathcal{Q}$ is faithful, one deduces from [Va, Thm. 3.16.3] that $\mathfrak{b}$ is reductive, i.e. $\mathfrak{b} = \mathfrak{z} \oplus \mathfrak{s}$ where $\mathfrak{z}$ is the center of $\mathfrak{b}$ and $\mathfrak{s}$ is semisimple. We show that $\mathfrak{z} = \{0\}$. Indeed, by the transitivity condition (B3), the elements of $\mathfrak{z}$ must be constant but, if these are nonzero, then $\mathfrak{s}$ alone would serve as a basic algebra, contradicting the minimality condition (B4). Thus $\mathfrak{z} = \{0\}$ and $\mathfrak{b} = \mathfrak{s}$ is semisimple.

Let B be the connected, simply connected Lie group with Lie algebra $\mathfrak{b}$. We claim that B is noncompact. Now the map $f \mapsto X_f$ can be thought of as an action of $\mathfrak{b}$ on M. By (B2) the vector fields X_f are complete, so by a theorem of Palais [Va, Thm. 2.16.13] this action of $\mathfrak{b}$ can be integrated to an action of the group B on M. Condition (B3) implies that this action is locally transitive and thus globally transitive as M is connected. Thus the noncompact manifold M is a homogeneous space for B, and so B must be noncompact as well.

Now consider a unitary representation U of B on $\mathbb{C}^k$. Decompose B into a product $B_1 \times \cdots \times B_N$ of connected simple groups. Then (at least) one of these, say B_1, must be noncompact. But it is well-known that a connected, simple, noncompact Lie group has no nontrivial finite-dimensional unitary representations [BR, Thm. 8.1.2]. Thus $U(b) = I$ for all $b \in B_1$. Since every representation $\mathcal{Q}$ of $\mathfrak{b}$ by skew-hermitian matrices is a derived representation of some finite-dimensional unitary representation U of B, it follows that $\mathcal{Q} \restriction \mathfrak{b}_1 = 0$, and so $\mathcal{Q}$ cannot be faithful. □

From this we immediately have

Corollary 2. *Let M be a noncompact symplectic manifold, $\mathfrak{b}$ a finite-dimensional basic algebra on M, and $\mathcal{O}$ any Lie subalgebra of $C^\infty(M)$ containing 1 and $\mathfrak{b}$. Then there is no finite-dimensional quantization of $(\mathcal{O}, \mathfrak{b})$.*

As the proof above shows, we do not need conditions (Q1) or (Q2) to obtain this result. Moreover, the subalgebra $\mathcal{O}$ is irrelevant since the proof depends only on the Lie theoretical properties of the basic algebra $\mathfrak{b}$ and its action on M.

3. Discussion

Corollary 2 is complementary to the recent result of [GGG] which states that there are no nontrivial quantizations (finite- *or* infinite-dimensional) of $(P(\mathfrak{b}), \mathfrak{b})$ on a *compact* symplectic manifold M, where $P(\mathfrak{b})$ is the Poisson algebra of polynomials generated by the basic algebra $\mathfrak{b}$. The proof of that result leaned heavily on the algebraic structure of $P(\mathfrak{b})$; indeed, when M is compact, it turns out that $\mathfrak{b}$ must be compact semisimple, and such algebras *do* have faithful representations by skew-hermitian matrices. Thus in the compact case, the obstruction to the existence of a quantization is Poisson, rather than Lie theoretical. Combining [GGG] with Corollary 2, we can now assert, roughly speaking, that *no symplectic manifold with a (finite-dimensional) basic algebra has a finite-dimensional quantization.*

Neither Theorem 1 nor Corollary 2 are valid when the representation is infinite-dimensional. For instance, in [GGra] an explicit quantization of the polynomial algebra generated by the "'affine" basic algebra span$\{pq, q^2\}$ on $T^*\mathbb{R}_+ = \big\{(q,p) \in \mathbb{R}^2 \,|\, q > 0\big\}$ is constructed on the Hilbert space $L^2(\mathbb{R}_+, dq/q)$. On the other hand, there do exist obstructions to obtaining infinite-dimensional quantizations of other noncompact symplectic manifolds, such as $\mathbb{R}^{2n}$ [Go2] and T^*S^1 [GGru]. We hope to determine the circumstances under which such obstructions occur in future work. This appears to be a difficult problem; still, some results are already known in this direction [Go1].

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Mark J. Gotay
Department of Mathematics, University of Hawai'i
2565 The Mall, Honolulu, HI 96822 USA
E-mail: gotay@math.hawaii.edu
Home Page: www.math.hawaii.edu/~gotay

Hendrik B. Grundling
Department of Pure Mathematics, University of New South Wales
P.O. Box 1, Kensington, NSW 2033 AUSTRALIA
E-mail: hendrik@maths.unsw.edu.au