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THE GEOMETRY OF NON-HOLONOMIC LAGRANGIAN SYSTEMS

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Abstract. A non-holonomic Lagrangian system may often be represented by an unconstrained system and a submanifold of a jet space (the constraint submanifold). We look here at the ways in which such a representation may arise, the conditions describing the regularity of such a system, and the different ways in which Hamiltonian representations may be obtained. We find that regularity of the unconstrained system is not a prerequisite for regularity of the constrained system. If the constraints have been derived from a connection then two different Hamiltonian representations may be obtained; for non-linear constraints, these give rise to two different regularity conditions.

1. Introduction

Non-holonomic mechanics is a subject which has attracted a certain amount of attention over the years, and has occasionally generated a significant degree of controversy. There has recently been a resurgence of interest in the geometrical background of the subject as attempts are made to place it within the geometrical framework now commonly used to understand classical holonomic mechanics (see, for example, [2, 3, 4, 5] and the references therein). This short note is a summary both of relevant known results, and of work currently in progress being carried out jointly with Willy Sarlet and Frans Cartrijn from the University of Gent.

Our work uses the formalism of jet bundles [8], and the basic geometrical structures involved may be seen in some of our earlier publications [6, 10, 7]. Essentially, we use a configuration space E (a smooth connected paracompact finite-dimensional differentiable manifold) fibred over R; the identity coordinate t on R represents time, and we adopt fibred local coordinates (t, q^A) $(1 \leq A \leq m+n)$ on E where q^A give the position of the system. Normally we take $\tau : E \to R$ to be a bundle (that is, we assume that it is locally trivial). The velocity space is then the first jet manifold $J^1\tau$ with local coordinates $(t, q^A, \dot{q}^A)$; the projection $J^1\tau \to E$ always defines an affine bundle. By a *non-holonomic constraint* we shall

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mean a submanifold $C \subset J^1\tau$ of codimension n, where the restricted projection $C \to E$ is a sub-bundle of $J^1\tau \to E$; locally the submanifold may be described by n functions $G^a(t, q^A, \dot{q}^A) = 0$. There are, of course, more general notions of non-holonomic constraint which have been considered in the literature: one might consider a foliation of $J^1\tau$, rather than an individual leaf C, or one might consider a distribution on $J^1\tau$ that need not even be integrable; we do not, however, discuss these more general situations.

We often wish to consider constrained Lagrangian systems and their possible Hamiltonian representations, so we need also to consider the dual jet representation. The jet manifold $J^1\tau$ is naturally a submanifold of TE of codimension 1, given by the equation $\dot{t} = 1$ (where local coordinates on TE are $(t, q^A, \dot{t}, \dot{q}^A)$; dually, the cotangent manifold T^*E has a quotient manifold $J^1\tau^*$ containing equivalence classes of cotangent vectors differing by multiples of dt. A Lagrangian function is a map $L : J^1\tau \to R$, whereas a Hamiltonian is not generally a function but is a section h of the projection $T^*E \to J^1\tau^*$. Local coordinates on T^*E are (t, q^A, p, p_A) and on $J^1\tau^*$ are (t, q^A, p_A), and the locally-defined function $H = p \circ h$ is the Hamiltonian function corresponding to the Hamiltonian h. (If the original bundle $E \to R$ is trivial then the coordinate p, and hence the Hamiltonian function H, will be global). Starting with a Lagrangian function L, we may construct its Legendre map $\mathrm{Leg}_L : J^1\tau \to T^*E$ in the usual way as a fibre derivative (or, equivalently, as a Poincaré-Cartan form) [9], and put leg_L as the composite map $J^1\tau \to J^1\tau^*$; if leg_L is a diffeomorphism then we say that L is *hyper-regular* and define a Hamiltonian h by $h = \mathrm{Leg}_L \circ \mathrm{leg}_L^{-1}$. For a local diffeomorphism leg_L we say that L is *regular* and define a local Hamiltonian in the same way.

The purpose of our work is to attempt to understand the rôle played by these geometrical structures in various different circumstances where constraints are involved. First, there is the case of *linear constraints* (or, more strictly, affine constraints) where $C \to E$ is an *affine* sub-bundle of $J^1\tau \to E$: this is, perhaps, the most simple situation. Related to this is the case where the constraint is *given by a connection*: here, there is an additional fibration $E \to M \to R$, and we are given a connection on $\pi : E \to M$. Such a connection defines a constraint on $J^1\tau$ which is necessarily linear (see [6]), and the existence of the manifold M suggests a possible formulation for a reduced dynamics. More generally we may look at constraints which are not necessarily linear, and related to this is the case where such a non-linear constraint is generated by what we call a *parametrised connection* (see [7]).

2. Linear constraints

A non-holonomic system is given by *linear constraints* if $C \to E$ is an affine sub-bundle of $J^1\tau \to E$. In these circumstances, we may solve the constraint equations $G^a(t, q^A) = 0$ locally to give n of the velocity coordinate functions $\dot{q}^a$ in terms of the remaining m velocity coordinates $\dot{q}^\alpha$ and the time and position coordinates t, q^A, to give

$$\dot{q}^a = B^a_\alpha(t, q^A)\dot{q}^\alpha + B^a(t, q^A) .$$

As $C \subset J^1\tau \subset TE$, we may consider the vector sub-bundle of $TE \to E$ spanned by C, and we shall denote this by $\overline{C}$; once again $\overline{C}$ has codimension n in TE. The annihilator $\overline{C}^{\circ}$ of $\overline{C}$ in T^*E then defines a bundle $\overline{C}^{\circ} \to E$ called the *Chetaev bundle*, and sections of this bundle may be considered to represent constraint forces. (We shall see later that there is another way to define the Chetaev bundle by using the "vertical endomorphism" S defined on $J^1\tau$, and this is normally more convenient for non-linear constraints: the two methods are equivalent.) The Chetaev bundle is spanned locally by *constraint forms*

$$\eta^a = dq^a - (B^a_\alpha dq^\alpha + B^a dt)\,.$$

Given a Lagrangian function L defined on $J^1\tau$ (or, at any rate, on a neighbourhood of C), we should like to find a definition of regularity such that there will be a unique constraint form η and a unique second-order vector field Γ_C, defined on C and tangent to it, satisfying

$$\Gamma_C \lrcorner\, d\theta_L = \eta$$

where θ_L is the Poincaré-Cartan form of L, and $\eta \in \overline{C}^{\circ}$ is a "constraint force". The definition we shall adopt is that L will be called *regular* when the symmetric bilinear form g whose Kähler lift is θ_L [1] is non-degenerate when restricted to vectors defined on C and tangent to C. In local coordinates, if we let $\theta^A = dq^A - \dot{q}^A dt$ be the contact forms on $J^1\tau$, then g is given by $g = g_{AB}\theta^A\theta^B$ where

$$g_{AB} = \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}$$

is the Hessian matrix of L. If we set $h_{\alpha\beta} = g_{\alpha\beta} + B^b_\beta g_{\alpha b}$, $h_{a\beta} = g_{a\beta} + B^b_\beta g_{ab}$ and $k_{\alpha\beta} = h_{\alpha\beta} + B^a_\alpha h_{a\beta}$ then the definition is equivalent to saying that L is regular when the matrix $k_{\alpha\beta}$ has maximal rank at points of C. In this linear case, the definition may also be expressed by saying that the Hessian of the reduced Lagrangian

$$\frac{\partial^2 \overline{L}}{\partial \dot{q}^\alpha \partial \dot{q}^\beta}$$

has maximal rank. A consequence of this definition is that the image $\mathrm{leg}_L(C)$ is immersed in $J^1\tau^*$ (note that the converse is not true, as $\mathrm{leg}_L(C)$ might be immersed in $J^1\tau^*$ even though L was not regular, because a non-zero element of TC might be mapped to an element of $T^{\circ}C$). The Hamiltonian representation of the system is then given by a (possibly locally-defined) map $\mathrm{leg}_L(C) \to T^*E$.

It is clear that a sufficient condition for L to be regular is that the Hessian g_{AB} is positive-definite. If this condition holds, or if (as a weaker condition) both the unconstrained and the constrained system are regular [2], then we may consider how solutions of the constrained and unconstrained systems are related. In this case, we may obtain the constrained vector field Γ_C from the unconstrained vector field Γ by projecting onto TC in a direction obtained by operating on the Chetaev bundle with the bilinear form g and then taking a vertical lift: details of this construction may be found in [10].

Suppose now that we have fibrations $E \xrightarrow{\pi} M \xrightarrow{\tau_0} R$ as described above. If we have a linear constraint manifold $C \subset J^1\tau$ then there is a possibility that the linear span $\overline{C}$ will be a direct complement to the vertical bundle $V\pi$ in TE; when this happens, the constraint defines a connection on the bundle $\pi : E \to M$ with horizontal bundle $\overline{C}$. Conversely, given such a connection with horizontal bundle H, we may set $C = H \cap J^1\tau$ to obtain a linear constraint manifold. We then have $\overline{C} \cong \pi^* TM$ and hence $C \cong \pi^* J^1\tau_0$. On the dual side, we have $T^*E \cong \overline{C}^\circ \oplus \pi^* T^* M$, and we therefore obtain a projection $T^*E \to \pi^* T^* M$ and consequently a projection $J^1\tau^* \to \pi^* J^1\tau_0^*$. This latter projection, composed with leg_L and the map $\pi^* J^1\tau_0 \to C$, allows us to consider a reduced problem on M whose solution may differ from Γ_C by means of a "curvature force" [2]. The Hamiltonian representation of this reduced problem is then a map $\pi^* J^1\tau_0^* \to \pi^* T^* M$.

3. Non-linear constraints

Many of the results for non-linear constraints run in parallel with those for linear constraints, with one important difference: whereas the theory for linear constraints frequently deals with objects defined on E, the corresponding theory for non-linear constraints requires these objects to be "parametrised by points of the constraint manifold C". We shall explain this distinction by looking at three aspects of the theory; we shall suppose that C has been given as a submanifold of $J^1\tau$ such that $C \to E$ is a (not necessarily affine) sub-bundle of $J^1\tau \to E$,

First, we consider the Chetaev bundle for non-linear constraints. A suitable definition (advertised earlier) is to consider the inclusion map $i : C \to J^1\tau$; TC is a sub-bundle of $i^*(TJ^1\tau)$, and so we may consider the annihilator $T^\circ C$ of TC in $i^*(T^*J^1\tau)$. The Chetaev bundle is then $S(T^\circ C)$, a sub-bundle of $i^*(T^*J^1\tau)$, where S is the vertical endomorphism on $J^1\tau$ [5]. If our choice of local coordinates on E is such that the constraint manifold C is given locally by $\dot{q}^a = g^a(\dot{q}^\beta, q^B, t)$ then the Chetaev bundle is spanned by the 1-forms

$$\eta^a = dq^a - \left(\frac{\partial g^a}{\partial \dot{q}^\alpha} dq^\alpha + \left(g^a - \dot{q}^\alpha \frac{\partial g^a}{\partial \dot{q}^\alpha} \right) dt \right) .$$

These forms, defined on C, are semi-basic over E. In the case of linear constraints the coefficients do not depend on the $\dot{q}^\alpha$ coordinates, so the forms are actually basic over E and hence define a sub-bundle $\overline{C}^\circ$ of T^*E as before. For non-linear constraints we have to think of $\overline{C}^\circ$ as being parametrised by points of C and hence as a sub-bundle of $C \times_E T^*E$. Once again, we say that a Lagrangian system is regular if there is a unique constraint form η and a unique second-order vector field Γ_C, defined on C and tangent to it, satisfying

$$\Gamma_C \lrcorner\, d\theta_L = \eta \,;$$

as before, the condition for regularity may be given by saying that the matrix $k_{\alpha\beta}$ has maximal rank at points of C.

Next, we consider the possibility that the constraint is given by a connection. It cannot be a true connection on $\pi : E \to M$, for this would give a linear constraint; instead it must be a connection "parametrised by C", in other words a map $C \to J^1\pi$ rather than a map $E \to J^1\pi$ [7]. The horizontal subspaces of this parametrised connection will again depend on C, and so the horizontal bundle $\overline{C}$ will be a sub-bundle of $C \times_E TE$ rather than of TE.

Finally, we look at the generalisation of the projection $J^1\tau^* \to \pi^* J^1\tau_0^*$ which we used earlier to describe reduced dynamics on M. Once again we have to incorporate C into the formalism, and we obtain a map $C \times_E J^1\tau^* \to \pi^* J^1\tau_0^*$. The regularity of the reduced dynamics will, as one might expect, depend on the non-degeneracy of the Hessian of the reduced Lagrangian $\partial^2\overline{L}/\partial\dot{q}^\alpha\partial\dot{q}^\beta$. For the non-linear case, this Hessian matrix is *not* the same as the matrix $k_{\alpha\beta}$, the difference being given by a term

$$\frac{\partial^2 g^a}{\partial\dot{q}^\alpha\partial\dot{q}^\beta}\, i^*\left(\frac{\partial L}{\partial\dot{q}^a}\right),$$

and so we have a second notion of regularity which is appropriate to the reduced problem. Work on the significance of these operations is still in progress.

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