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INTEGRABILITY ASPECTS OF THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS

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ABSTRACT. For a long time, a paper by J. Douglas of 1941 has been the only contribution to the question of classifying second-order ordinary differential equations for which a non-singular multiplier matrix exists which turns the given system into an equivalent system of Euler-Lagrange equations. It was based on the Riquier-Janet theory of formal integrability of partial differential equations and limited to systems with two degrees of freedom. Quite recently, a geometrical calculus of derivations of tensor fields along projections has been developed, which in the study of second-order differential equations is primarily related to the existence of a canonically defined linear connection on a suitable bundle. It turns out that this calculus provides the right tools for closely monitoring the process of Douglas's analysis in a coordinate free way. After a survey of the integrability analysis which can be carried out this way, we briefly sketch how subcases belonging to each of the three classes in the main classification scheme of Douglas can be generalised to an arbitrary number of degrees of freedom.

1. Introduction: the inverse problem for second-order ordinary differential equations

Considering a given system of differential equations of the form

$$\ddot{x}^i = f^i(t, x, \dot{x}), \qquad i = 1, \dots, n$$

whose vector field representation reads

$$\Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} + f^i \frac{\partial}{\partial v^i},$$

the inverse problem of the calculus of variations concerns the investigation of the existence of a non-singular, symmetric matrix $(g_{ij}(t, x, \dot{x}))$, such that

$$g_{ij}(\ddot{x}^j - f^j) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}$$

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for some function $L(t, x, \dot{x})$.

In analytical terms, the conditions to be satisfied by such g_{ij}, known as the Helmholtz conditions, are the following (see e.g. [6, 13]):

$$\begin{aligned} \Gamma(g_{ij}) &= g_{ik}\Gamma^k_j + g_{jk}\Gamma^k_i \,, \\ \frac{\partial g_{ij}}{\partial v^k} &= \frac{\partial g_{ik}}{\partial v^j} \,, \\ g_{ij}\Phi^j_k &= g_{kj}\Phi^j_i \,, \end{aligned}$$

where

$$\Gamma^i_k = -\tfrac{1}{2}\frac{\partial f^i}{\partial v^k} \,, \qquad \Phi^i_j = -\frac{\partial f^i}{\partial x^j} - \Gamma^i_k\Gamma^k_j - \Gamma(\Gamma^i_j) \,.$$

The ultimate aim is to characterize the existence of such a multiplier matrix by conditions involving the given f^i only. One may expect this to lead to a sort of classification problem, based on an integrability analysis of the Helmholtz conditions.

Remarks. Until fairly recently, only Douglas [6] approached this aspect of the inverse problem, his analysis being limited to the case $n = 2$. The memoir by Anderson and Thompson [1] is a first recent contribution to the problem. We will report here on a new systematic approach, which has been developed in joint work, mainly with M. Crampin, E. Martínez, G. Prince and G. Thompson [5, 14, 3]. A completely different method of investigation has been developed simultaneously by Grifone and Muzsnay [7, 10]. These authors do not take the Helmholtz conditions as their starting point. Instead, for autonomous systems, they investigate the existence of function a $L(x, \dot{x})$, such that

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} f^j + \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j}\dot{x}^j - \frac{\partial L}{\partial x^i} \equiv 0 \,.$$

A coordinate free version of this second-order partial differential equation for L and the Spencer-Goldschmidt approach to formally integrable partial differential equations is the basis for their work.

2. Geometrical version of the Helmholtz conditions

A geometrical version of the inverse problem which can be found in [4] reads as follows: find a suitable Cartan 2-form for Γ, i.e. a 2-form ω, such that

$$\mathcal{L}_\Gamma \omega = 0, \quad d\omega = 0, \quad \omega(X^V, Y^V) = 0 \,.$$

ω lives on $J^1\pi$ say (coordinates (t, x, v)), with $\pi : E \to I\!R$, and X^V, Y^V are arbitrary vector fields which are vertical with respect to the fibration $\pi^0_1 : J^1\pi \to E$ and can be regarded therefore as vertical lifts of vector fields along π^0_1.

One can readily see that this setting is somehow too large: indeed, with respect to the coframe adapted to the connection coming with the second-order equation field Γ, ω is of the form

$$\omega = g_{ij}\eta^i \wedge \theta^j \,, \qquad g_{ij} = g_{ji} \,,$$

with
$$\theta^j = dx^j - v^j\,dt, \qquad \eta^i = dv^i - f^i\,dt + \Gamma^i_k \theta^k\,.$$
Hence, although this geometrical description takes place on a space with dimension $2n+1$, we truly investigate, as expected, the existence of a symmetric $n\times n$ matrix.

A good interpretation of what is happening here is the following (cf. [15]) : ω is the 'Kähler lift' of a symmetric, covariant 2-tensor along $\pi^0_1 : J^1\pi \to E$, namely
$$g = g_{ij}\theta^i \otimes \theta^j\,.$$
Schematically, ω is related to g in the following way:
$$\omega = \left(\begin{array}{c|c|c} 0 & g & 0 \\ \hline -g & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right).$$
Hence, if a calculus can be developed for g directly, it will be more economical: each formula for g will correspond somehow to two formulas for ω. Such a calculus has been developed, first for autonomous equations in [8, 9], and later for time-dependent equations in [15, 2]. It leads to the following concise transcription of the Helmholtz conditions in a coordinate free form:
$$\begin{aligned} \nabla g &= 0 \\ D^V g\,(Z, X, Y) &= D^V g\,(Y, X, Z) \\ g\,(\Phi(X), Y) &= g\,(\Phi(Y), X)\,. \end{aligned}$$
Here, Φ is to be thought of as a (1,1) tensor along π^0_1, called the Jacobi endomorphism, with the matrix Φ^i_j mentioned before as components:
$$\Phi = \Phi^i_j\,\theta^j \otimes \frac{\partial}{\partial x^i}\,.$$
To give an idea of the meaning of the symbols which enter this description, we first make a short digression. Let $\mathcal{X}(\pi^0_1)$ denote the set of vector fields along $\pi^0_1 : J^1\pi \to E$. There is a natural splitting $\mathcal{X}(\pi^0_1) = \langle\mathbf{T}\rangle \oplus \overline{\mathcal{X}}(\pi^0_1)$, where $\mathbf{T} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}$ is the canonical vector field along π^0_1. So, each $X \in \mathcal{X}(\pi^0_1)$ is of the form
$$X = X^0\mathbf{T} + \overline{X}, \qquad \overline{X} = X^i\frac{\partial}{\partial x^i}\,, \qquad X^0, X^i \in C^\infty(J^1\pi)\,.$$
All vector fields along π^0_1 in what follows belong to $\overline{\mathcal{X}}(\pi^0_1)$ and we henceforth omit the bar.

There are three important so-called self-dual, degree zero derivations, denoted by ∇, D^V_X, D^H_X, X here being an arbitrary element of $\overline{\mathcal{X}}(\pi^0_1)$. To see how they act on arbitrary tensor fields along π^0_1, it suffices to specify their action on functions, vector fields and 1-forms along π^0_1. This can be done intrinsically, of course, but for our present needs, it suffices to know that on $C^\infty(J^1\pi)$ we have,
$$\nabla F = \Gamma(F), \quad D^V_X F = X^V(F)\,, \quad D^H_X F = X^H(F)\,,$$

on the local basis for $\overline{\mathcal{X}}(\pi_1^0)$:

$$\nabla\left(\frac{\partial}{\partial x^j}\right) = \Gamma^i_j \frac{\partial}{\partial x^i}\,, \quad D^V_X\left(\frac{\partial}{\partial x^j}\right) = 0\,, \quad D^H_X\left(\frac{\partial}{\partial x^j}\right) = \left(X^k \frac{\partial \Gamma^i_j}{\partial v^k}\right)\frac{\partial}{\partial x^i}$$

and then by duality,

$$\nabla\theta^i = -\Gamma^i_j\theta^j\,, \quad D^V_X\theta^i = -X^i\,dt\,, \quad D^H_X\theta^i = -\left(X^k\frac{\partial\Gamma^i_j}{\partial v^k}\right)\theta^j\,.$$

To give a clue about the structure behind these derivations, observe that vector fields along π_1^0 are sections of the bundle $\pi_1^{0^*}(\tau_E)$ over $J^1\pi$ and that Γ determines a linear connection on $\pi_1^{0^*}(\tau_E) \to J^1\pi$, which roughly has D^V_X, D^H_X and ∇ (the 'dynamical covariant derivative') as its independent components (see [2] for details). Many of the formulas and properties to follow in fact relate to properties of the curvature of this linear connection, plus its first and second Bianchi identities.

Finally, to give a meaning to the coordinate free Helmholtz conditions presented above, it suffices to introduce covariant differentials D^V and D^H defined, for any tensor field T along π_1^0, by

$$D^V T(Z,\dots) = D^V_Z T(\dots)\,, \quad D^H T(Z,\dots) = D^H_Z T(\dots)\,.$$

3. Some basic ideas about Riquier-Janet theory

In rudimentary terms, checking 'integrability' of systems of partial differential equations means: looking for potential incompatibilities in the prolongations of the equations, i.e. differential consequences of the given equations which reveal new restrictions on the unknown functions. In the Riquier-Janet approach, one can distinguish two ways in which such supplementary restrictions can arise. Basically, what one will attempt first is to see whether combinations of prolonged equations exist in which all highest-order derivatives cancel out. This may give rise to new equations, of the same order as the original ones, which are not just linear combinations of these and therefore have to be added on to the given system. Integrability conditions of this kind will be encountered in any theory of formal integrability of pde's.

An example in the present context is readily encountered as follows. We have the following prolongations of our coordinate free Helmholtz conditions:

$$\begin{aligned} D^V\nabla g &= 0 \\ \nabla D^V g(Z,X,Y) &= \nabla D^V g(Y,X,Z)\,, \end{aligned}$$

but we know about the identity

$$[\nabla, D^V] = -D^H\,.$$

Hence, a combination of these prolongations imposes that

$$D^H g(Z,X,Y) = D^H g(Y,X,Z)\,.$$

This is an equation which cannot be obtained from combining the original ones. Obviously, every solution of the original equations will automatically satisfy the newly derived one. However, in the process of verifying that a solution exists (at least a formal solution, represented by its formal Taylor expansion), we are looking at new information here. Accordingly, the equation has to be added on to the system and the process can start allover again.

The algorithmic procedure by which such integrability conditions have to be looked for, is not the same in different approaches. In the more modern Spencer-Goldschmidt theory, for example, one would check first at each level whether the symbol of the system is 'involutive'. If then no new integrability condition emerges from projecting the next prolongation onto the system under investigation at that stage, the process has come to an end and one can conclude for formal integrability. In the spirit of the Riquier theory, however, the other aspect which plays a role in the algorithm is 'orthonomicity'. Briefly, this requires that one solves all equations for so-called 'principal' derivatives; all derivatives of these are principal also and all others are said to be 'parametric'. Orthonomicity means that one must be able to write all equations with one principal derivative in the left-hand side and only parametric derivatives on the right. To do this without internal inconsistencies, one has to set up ordering rules so that all elected principal derivatives are 'higher in rank' than everything in the right-hand side. It would lead us too far to enter into the details of this mechanism here, but roughly what it means is that the parametric derivatives will determine in the end how much freedom there is in selecting arbitarily certain coefficients in the formal Taylor expansion of the solution. The idea clearly originates from the Cauchy-Kowalevski theorem.

What is important now is the following. In the Riquier approach, if an orthonomic system is prolonged, something else can happen re compatibility: combinations of prolonged equations may be found which eliminate all principal derivatives of highest order, leaving us with a relation (of the same order of the prolongation still) between parametric derivatives. Obviously, such relations contain new information as well: parametric derivatives, previously thought to be unrestricted, now are no longer independent so that there may be less freedom in the formal solution. To distinguish them from the other integrability conditions, we use the terminology initiated by Riquier and call them 'passivity conditions'. They also have to be added on to the system, meaning that one of the parametric derivatives will be promoted to the rank of principal derivative and the process must be re-initiated again. A fairly comprehensive survey of the Riquier method can be found in [6]. A computer algebra implementation was described by Reid [11].

Concerning the inverse problem for an arbitrary number of degrees of freedom, we have computed in [14], in a coordinate free way, all integrability and passivity conditions which one can obtain without entering the discussion of the ordering process which accompanies the procedure for writing everything in orthonomic form. These results are briefly summarised in the next section.

4. The complete set of passivity conditions

From $\nabla g = 0$ and the algebraic equation, it follows that we must satisfy the full hierarchy of algebraic conditions

$$g\,(\nabla^k \Phi(X), Y) = g\,(\nabla^k \Phi(Y), X)\,, \qquad \forall k \geq 0\,.$$

From the D^V equation and the D^V-prolongation of the original algebraic equation follows a new algebraic integrability condition, namely ($\sum$ stands for 'cyclic sum'):

$$\sum g\,(R(X,Y),Z) = 0\,,$$

where R is the curvature of the non-linear connection. This is a result of the property

$$D^V \Phi(X,Y) - D^V \Phi(Y,X) = 3\,R(X,Y)\,.$$

Subsequently, we of course obtain a second hierarchy of algebraic conditions:

$$\sum g\,(\nabla^k R(X,Y),Z) = 0\,, \qquad \forall k\,.$$

As already indicated, the ∇ and D^V equation produce the D^H equation

$$D^H g\,(Z,X,Y) = D^H g\,(Y,X,Z)\,.$$

Going through the same process with D^H instead of D^V, nothing new comes from the combination of D^H and Φ, because

$$D^H \Phi(X,Y) - D^H \Phi(Y,X) = \nabla R(X,Y)\,,$$

and we obtain this way a condition which was already accounted for. Also, D^V and R, and D^H and R give nothing new as a result of the properties

$$\sum D^V R(X,Y,Z) = 0, \qquad \sum D^H R(X,Y,Z) = 0\,.$$

Observe that all extra properties which have so far been exploited are first Bianchi identities of the curvature of the linear connection. Finally, it can be shown (see Appendix A of [14]) that combining D^V and $\nabla^k\Phi$ or $\nabla^k R$ for $k \geq 1$, and also D^H and $\nabla^k\Phi$ or $\nabla^k R$ for $k \geq 1$ produce no further algebraic requirements. This exhausts all possible algebraic conditions (generically).

Consider now further prolongations to second-order equations. One easily finds that the ∇ and D^H equation reproduce the curvature condition again, as a result of the property of the commutator $[\nabla, D^H]$. Finally, the process of comparing corresponding prolongations of the D^V and D^H equations is extremely tedious. Among other things, one has to make use of the following identities, which essentially define components of the curvature of the linear connection on ${\pi_1^0}^*(\tau_E) \to J^1\pi$: for any covariant tensor field T along π_1^0, we have

$$\begin{aligned} D^V D^V T(X,Y,Z,\ldots) - D^V D^V T(Y,X,Z,\ldots) &= 0 \\ D^V D^H T(X,Y,Z,\ldots) - D^H D^V T(Y,X,Z,\ldots) &= -i_{\theta(X,Y)} T(Z,\ldots)\,, \end{aligned}$$

and finally

$$D^H D^H T(X,Y,Z,\dots) - D^H D^H T(Y,X,Z,\dots) = \\ D^V T(R(X,Y),Z,\dots) + i_{D^V R(X,Y)} T(Z,\dots) .$$

The first of these identities expresses that the linear connection has no 'vertical curvature'; the second one can be regarded as the defining relation of the (1,3) tensor field θ, which in view of the first Bianchi identities is completely symmetric:

$$\theta(X,Y)Z = \theta(Y,X)Z = \theta(X,Z)Y .$$

No passivity conditions can be created from prolonging D^V equations among themselves, precisely because of the zero 'vertical curvature'. More difficult to prove is that nothing comes from prolonging D^H equations among themselves either. From the D^V and D^H equations, however, we do obtain potential passivity or integrability conditions. They can be expressed as follows:

$$\begin{aligned} 0 &= \mathcal{A}(X,Y)g(U,V) \equiv \\ &D^H D^V g\,(Y,X,U,V) - D^H D^V g\,(X,Y,U,V) + g\,(\theta(U,V)X,Y) - g\,(\theta(U,V)Y,X) . \end{aligned}$$

The common arguments U, V in the leading terms indicate that the highest-order terms will concern the same components of g. But now, whatever ordering process one could think of for writing the first-order equations in orthonomic form, one easily understands that not all g_{ij} can have all their first-order D^V and D^H derivatives as principal derivatives. We may be facing here, therefore, second-order passivity conditions.

On the other hand, in many cases, depending on the structure of g as determined by the algebraic restrictions, all parametric second-order terms may vanish, giving rise this way to a situation where the $\mathcal{A}$-conditions will degenerate into first-order or even algebraic conditions. The further analysis of such situations can only be dealt with in a case by case investigation.

For the non-degenerate situation, we know that certain second-order conditions will have to be added to the system and we face the problem of re-initiating the prolongation and compatibility procedure. One of the main achievements in [14] is that we have succeeded in pushing the further passivity analysis to the very end. We have shown in fact that all further potential passivity conditions are identically satisfied in view of the second Bianchi identities of the linear connection.

In summary, the situation concerning formal integrability of the Helmholtz conditions, to the extent that it is possible to state general conclusions, is the following. The original equations, let us refer to them as the ∇, D^V and Φ equation, have to be extended with:

- the double hierarchy of algebraic conditions involving $\nabla^k \Phi$ and $\nabla^k R$,
- the first-order D^H equation,
- the second-order $\mathcal{A}$ conditions.

If the latter are truly passivity conditions of second-order, no further integrability or passivity conditions can be obtained, and it remains to verify that the complete set of equations can be written in orthonomic form. If, on the contrary, the $\mathcal{A}$ conditions degenerate into integrability conditions of first or zeroth order, then a further detailed study is required into the nature of these conditions and the possible generation of more integrability conditions.

A natural way to make use of these general results is to start by imposing the restrictions on the multiplier matrix g_{ij} coming from the algebraic conditions involving the Jacobi endomorphism Φ. Obviously, since the module of type (1,1) tensor fields along π_1^0 is finite dimensional, only a certain number of these will play a role. It should come as no surprise, therefore, that the three main cases which are distinguished in Douglas's classification for $n = 2$, precisely correspond to the assumptions: Φ and the identity tensor I are linearly dependent (case I); $\nabla\Phi$, Φ and I are linearly dependent (case II); $\nabla^2\Phi$, $\nabla\Phi$, Φ and I are linearly dependent (case III). Perhaps this pattern cannot be exactly followed for $n > 2$. For example, Douglas could not be worried about the algebraic conditions involving the curvature, as these are void for $n = 2$. But it seems plausible that any form of classification will be related to different assumptions on Φ, for example with respect to its Jordan canonical form. For each case, if there is any algebraic freedom left over in g after imposing the algebraic restrictions, one could, for example, first look at the $\mathcal{A}$ conditions to see whether these need any further attention with regard to computing more integrability requirements (many of the nested subcases of case II in Douglas fall into this category), and then verify whether the full set of differential equations can be written in orthonomic form.

Let us repeat here that the main merit of the results which were reviewed in this section is that they take us quite a long way already into the integrability analysis, in a manner which is completely independent of technicalities such as deciding in each case which derivatives are going to be selected as principal derivatives for checking orthonomicity. But there is a substantial part of the problem which can only be dealt with further in a case by case investigation.

5. Application: two variational cases for general n

Assume first that

$$\Phi = \mu\, I$$

for some function $\mu \in C^\infty(J^1\pi)$. This is case I in [6] and we will refer to it as case I also for arbitrary n. It is easy to verify that all algebraic requirements are trivially satisfied, so g can still be an arbitrary (non-singular) symmetric matrix up to this stage. As a result, there is no degeneracy in the second-order passivity conditions and we know that we are left with the question of orthonomicity. This has been discussed in great detail in [14]. What was perhaps not sufficiently exploited there is the fact that also this technical discussion greatly simplifies if one works with a local frame of vector fields adapted to the geometry coming with the given vector field Γ, rather than with the coordinate derivatives, which a strict application of the original Riquier theory would require. So, we will try to explain this aspect now.

Replacing coordinate vector fields by the more geometrical frame coming from the connection defined by Γ, schematically means the following:

$$\begin{aligned}
\frac{\partial}{\partial t} \quad &\rightsquigarrow \quad \Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} + f^i \frac{\partial}{\partial v^i} \\
\frac{\partial}{\partial x^i} \quad &\rightsquigarrow \quad H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial v^j} \\
\frac{\partial}{\partial v^i} \quad &\rightsquigarrow \quad V_i = \frac{\partial}{\partial v^i}\,.
\end{aligned}$$

Verifying orthonomicity will always require that the first order D^V equations, which in coordinates involve the operators V_i only, are orthonomic by themselves. It is known (see e.g. [12]) that all derivatives of principal derivatives in an orthonomic system can be substituted in terms of parametric derivatives. Suppose then that in a standard application of Riquier theory, a term like $\partial g_{ij}/\partial x^k$, or in a second-order equation a term like $\partial^2 g_{ij}/\partial x^k \partial v^l$ would be selected as principal derivative (in a way which is consistent with all the rules which the assignment of 'cotes' must satisfy). Then, strictly speaking, an equation of the form $H_k(g_{ij}) = \dots$, or likewise $H_k V_l(g_{ij}) = \dots$, should be thought of as having only the coordinate derivatives just mentioned on the left. But we know from the above argument that the other terms on the left, even if they originally involve principal V_l-derivatives or derivatives thereof, can be replaced after proper substitutions by parametric derivatives. So, we need not bother about these terms: they will not interfere with the reasoning which will lead to the conclusion that the full system can be written in orthonomic form or not. In more technical terms: the 'cote' of the variables t and x^k can be thought of as being associated to the operators Γ and H_k, rather than to $\partial/\partial t$ and $\partial/\partial x^k$. It is perhaps interesting to observe here that this argumentation in fact takes an idea of Douglas one step further. Indeed, Douglas already used the same argument for replacing $\partial/\partial t$ by Γ.

We now briefly sketch how one can verify the orthonomicity requirement for case I. To fix the idea, we take $n = 3$ and choose for example the diagonalisewise ordering of the $g_{ij},\ i \leq j$. We note first that $\nabla g = 0$ will produce a Γ (or $\partial/\partial t$) equation for each g_{ij}, meaning that all terms in the formal Taylor series solution involving a t-derivative are determined. We can, therefore, safely disregard the equation $\nabla g = 0$ from the analysis which follows. We further choose an ordering for the remaining 'independent variables' as follows: $H_1, H_2, H_3, V_1, V_2, V_3$. In symbolic notations which are fairly self-evident now, an orthonomic representation of the full system of equations is the following one (for conciseness, an equation like $11, 2 = 12, 1$ actually represents both the equation $V_2(g_{11}) = V_1(g_{12})$ and its analogue coming from the D^H equation).

11,2 = 12,1	12, H_1V_2 = 12, H_2V_1
11,3 = 13,1	23, H_2V_3 = 23, H_3V_2
22,1 = 12,2	13, H_1V_2 = 13, H_2V_1
22,3 = 23,2	13, H_1V_3 = 13, H_3V_1
33,1 = 13,3	13, H_2V_3 = 13, H_3V_2
33,2 = 23,3	
12,3 = 13,2	
23,1 = 13,2	

The second-order equations in the second column, which are formally represented by their leading terms only, come from those $\mathcal{A}$-expressions in which none of the second-order terms is a derivative of a principal derivative in the first-order equations.

Such a procedure works for all n, hence case I is variational. The same conclusion was obtained before by Anderson and Thompson [1] and recently also by Grifone and Muzsnay [7].

Remark: counting the freedom in the general solution is a delicate matter which does not lead to unique answers, because there may be many different ways of representing the solution (see [16]). Only the number of arbitrary functions of the maximal number of arguments has an invariant meaning. The freedom we find from our Riquier approach here is given by the formula:

$$\sum_{k=1}^{n} k\,(n-k+1)\,f(k+1)\,,$$

$f(i)$ being shorthand for an arbitrary function of i variables. It contains $n\,f(n+1)$ for the prediction of the functions with the highest number of arguments, which is in agreement with the result in [1].

For distinguishing a second case, we start by assuming that $[\nabla\Phi, \Phi] = 0$ and that Φ is diagonalisable with distinct (real) eigenvalues λ_i. Note hereby that for $n = 2$, $[\nabla\Phi, \Phi] = 0$ is equivalent to $\nabla\Phi = \alpha\Phi + \beta\, I$, which was the assumption of Douglas in his case II. The former condition is weaker, however, for larger n and will be sufficient for our purposes.

With respect to a basis X_i of eigenvectors of Φ, which can be rescaled to be ∇ invariant, g will be diagonal, say

$$g = \sum_i \rho_i\, \vartheta^i \otimes \vartheta^i$$

(the ϑ^i are dual to the X_i). The original D^{v} conditions then become:

$$D^{\mathrm{v}}_{X_i}(\rho_j) = (2\tau^j_{ij} - \tau^j_{ji})\rho_j - \tau^i_{jj}\rho_i, \quad i \neq j$$

and, with i, j, k different

$$\tau^k_{ij}\rho_k + \tau^j_{ik}\rho_j = \tau^k_{ji}\rho_k + \tau^i_{jk}\rho_i\,.$$

The functions τ^i_{jk} are known, in principle; they are defined by:

$$D^V_{X_j} X_k = \sum_s \tau^s_{jk}\, X_s\,.$$

A further, tensorial assumption $H_\Phi = 0$ now has the following effect:

$$\tau^k_{ij} = 0 \quad \text{for} \quad k \neq i \text{ and } k \neq j\,.$$

Here, H_Φ is the type (1,2) tensor field, determined by:

$$\begin{aligned} H_\Phi(X,Y) \;\; &= \;\; D^V\Phi(\Phi(X),\Phi(Y)) - \Phi(D^V\Phi(X,\Phi(Y))) \\ &\quad - \Phi(D^V\Phi(\Phi(X),Y)) + \Phi^2(D^V\Phi(X,Y))\,. \end{aligned}$$

The simplification which results is that all above algebraic equations disappear and that the remaining D^V equations decouple:

$$D^V_{X_i}(\rho_j) = (2\tau^j_{ij} - \tau^j_{ji})\rho_j \qquad i \neq j\,.$$

For this reason, Douglas called this the 'separated case' for $n = 2$ (it is his case IIa1).

Since g is diagonal, all potential second-order passivity conditions degenerate here. One can prove, however, that fortunately all terms cancel out, so that no further integrability analysis is required. Orthonomicity then follows easily, and the freedom in the solution is easily seen to be n functions of 2 variables.

The details of this case can be found in [3], where it is also shown that the extra assumption $H_\Phi = 0$ has the following interesting interpretation: there exist coordinates, with respect to which the original n second-order equations decouple into n separate systems of two first-order equations (not in general n separate second-order equations).

We conclude with a few words about case III in [6] and what our approach can tell about this case. As said before, in case III (for $n = 2$) we have for a start that $\nabla^2\Phi = a\,\nabla\Phi + b\,\Phi + c\,I$ and if the system is variational, Φ will again be diagonalisable. The algebraic freedom in g is reduced to a scalar factor; with respect to a suitable basis of eigenforms of Φ, we have

$$g = \mu\left(\vartheta^1 \otimes \vartheta^1 - \alpha\,\vartheta^2 \otimes \vartheta^2\right)$$

where α is a (known) first integral. Douglas showed that the existence of a solution for μ is reduced to the closedness of a certain 1-form on $J^1\pi$. Our approach can make this much more precise. Indeed, the $\mathcal{A}$-conditions here degenerate into a single algebraic requirement of the form $(\ldots)\mu = 0$. The coefficient of μ involves α, the structure functions τ^i_{jk} and their derivatives. We conclude that vanishing of this coefficient is the only obstruction to variationality (and hence to the closedness of the 1-form considered by Douglas). Explicit calculations would show, by the way, that $H_\Phi = 0$ is sufficient again for this to happen. Work is in progress to describe an n-dimensional analogue of this situation as well.

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