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A REPRESENTATION OF THE 1ST-ORDER VARIATIONAL SEQUENCE IN FIELD THEORY

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ABSTRACT. Variational sequence of order r introduced by Krupka is defined on the r-jet prolongation of a fibered manifold $\pi : Y \to X$, as a quotient sequence of the de Rham sequence of sheaves. In this paper we describe explicitly the elements of the 1st-order variational sequence over an n-dimensional manifold X.

1. Introduction

The theory of variational bicomplexes and variational sequences is motivated by the idea to represent the Euler-Lagrange mapping as one of morphisms of a suitable "variational bicomplex" or the "variational sequence". This morphism is derived from the exterior derivative operator acting on differential forms.The idea that the Euler-Lagrange mapping should be obtained from the exterior derivative operator is appeared in papers of Lepage and Dedecker for the first time and it has been developed by Anderson [1], Krupka [6], [8], [9], Vinogradov[13], Vitolo [14], [15] and others. The results of this study are applied in the calculus of variations (see [9]), in the theory of differential equations (see [2]), in the differential geometry and in further branches of mathematics.

In this paper we investigate the variational sequence on the finite order jet space introduced by Krupka (see [12]), that is defined as a quotient sequence of de Rham sequence of sheaves over some special subsequence of de Rham sequence on the finite order jet space. This subsequence is formed by means of so called contact and strongly contact forms. The structure of contact and strongly contact forms was precisely described in [4], [12], where it is shown that they are generated by contact forms "ω" and "$d\omega$". The exactness of the variational sequence is discussed

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in [12]; it is shown the variational sequence is an acyclic resolution of the constant sheaf using the abstract de Rham theorem.

In Section 2 we state propositions which are used in Section 3. All the statements are given without proofs. The reader can find them in [4], [11], the trace decomposition problem and to problems of q-equations are defined in [11].

Section 3 is the main part of this paper. It contains the explicit description of the elements (Theorem (3.2)) of the variational sequence on J^1Y in the field theory. This calculation has been recently done in mechanics and we can see it in [8].

Throughout this paper we consider a fibered manifold $\pi : Y \to X$, where $\dim X = n$, $\dim Y = n + m$, the fibered chart (W, ψ), $\psi = (x^i, y^\sigma)$, $1 \leq i \leq n$, $1 \leq \sigma \leq m$, on Y associated with a fibered chart (U, φ), $\varphi = (x^i)$, $1 \leq i \leq n$, and the fibered chart (W^r, ψ^r) on J^rY associated with (W, ψ) such that for each $J^r_x\gamma \in J^rY$, $\psi^r(J^r_x\gamma) = (x^i, y^\sigma, y^\sigma_{i_1}, y^\sigma_{i_1 i_2}, ..., y^\sigma_{i_1 i_2 \dots i_p}, ..., y^\sigma_{i_1 i_2 \dots i_r})$, where $\forall p = 1, 2, ..., r$, $1 \leq i_1, i_2, ..., i_p \leq n$, $i_1 \leq i_2 \leq ... \leq i_p$, $y^\sigma_{i_1 i_2 \dots i_p}(J^r_x\gamma) = D_{i_1 i_2 \dots i_p}(y^\sigma \circ\gamma\circ\varphi^{-1})(\varphi(x))$, and $W^r = (\pi^{r,0})^{-1}(W)$. A ring of smooth functions on W^r is denoted by $\Omega^r_0 W$, $\Omega^r_q W$ is the $\Omega^r_0 W$-module of smooth differential q-forms on W^r.The mapping horizontalization, defined on $\Omega^r_q W$ is denoted by h. The total derivative of $f \in \Omega^r_0 W$ with respect to x^i is denoted by $d_i f$, using fibered system (W, ψ), $\psi = (x^i, y^\sigma)$. A k-contact component of the form $\rho \in \Omega^r_q W$ is denoted by $p_k\rho$ where k positive integer. We mean symetrization, resp. antisymetrization of the coeficient X_{j_1,j_2} in indices (j_1, j_2) if we write $\underset{(j_1,j_2)}{sym} X_{j_1,j_2}$, resp. $\underset{(j_1,j_2)}{alt} X_{j_1,j_2}$.

E^r_q is the vector space of systems of real numbers $A = A^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_q}$, where $1 \leq j_1, j_2, ..., j_r, i_1, i_2, ..., i_q \leq n$. The standard summation convention is used.

2. Prerequisites

It si well known that a q-form $\rho \in \Omega^r_q W$ is called *contact* if and only if $J^r\gamma^*\rho = 0$ for every smooth section γ of Y defined on an open subset of $\pi(W)$ and a q-form $\rho \in \Omega^r_q W$ is called *strongly contact* if $p_{q-n}\rho = 0$. The next statetment describes the structure of contact and strongly cantact forms on fibered manifolds.

2.1. Proposition. *Let $W \subset Y$ be an open set and let $\rho \in \Omega^r_q W$ be a form. Let (V, ψ), $\psi = (x^i, y^\sigma)$ be a fibered chart such that $V \subset W$.*

(a) *Let $q = 1$. Then ρ is contact if and only if*

$$\rho = \Phi^J_\sigma \omega^\sigma_J$$

for some functions $\Phi^J_\sigma : V^r \to R$, where $|J| \leq r - 1$.

(b) *Let $2 \leq q \leq n$. Then ρ is contact if and only if*

$$\rho = \omega^\sigma_J \wedge \Phi^J_\sigma + d\omega^\sigma_I \wedge \Psi^I_\sigma$$

where Φ^J_σ (resp. Ψ^I_σ) are some $(q-1)$-forms (resp. $(q-2)$-forms on V^r and $|J| \leq r - 1$, $|I| = r - 1$.

(c) *Let $n + 1 \leq q \leq \dim J^rY$. Then ρ is strongly contact if and only if*

$$\rho = \omega^{\sigma_1}_{J_1} \wedge \omega^{\sigma_2}_{J_2} \wedge ... \wedge \omega^{\sigma_p}_{J_p} \wedge d\omega^{\nu_1}_{I_1} \wedge d\omega^{\nu_2}_{I_2} \wedge ... \wedge d\omega^{\nu_s}_{I_s} \wedge \Phi^{J_1 J_2 ... J_p I_1 I_2 ... I_s}_{\sigma_1 \sigma_2 ... \sigma_p \nu_1 \nu_2 ... \nu_s}$$

for some $(q-p-2s)$*-forms* $\Phi^{J_1 J_2 ... J_p I_1 I_2 ... I_s}_{\sigma_1 \sigma_2 ... \sigma_p \nu_1 \nu_2 ... \nu_s}$ *on* V^r *such that* $p+s \geq q-n+1$, $|J_1|, |J_2|, ..., |J_p| \leq r-1$, $|I_1|, |I_2|, ..., |I_s| = r-1$.

For more complete discussion of the theory of contact and strongly contact forms and for the proofs, the reader is referred to [10] and [12].

The exposition of the $q-equation$ theory and proofs of the following propositions is given in [11], [12], as well as the definition of linear mappings tr_m, q_m, where m is some positive integer. You can also find in [12] the proof of the fact, that for every system of real number $A = A^{I_1 I_2 ... I_{m-1} I_m I_{m+1} ... I_r}{}_{i_1 i_2 ... i_q}$, $I_m = (j^m_1 j^m_1 ... j^m_{r_m})$, $m = 1, 2, ..., p$, $1 \leq j^m_1, j^m_2, ..., j^m_{r_m} \leq n, 1 \leq i_1, i_2, ..., i_q \leq n$ symmetric in superscripts entering the multiindex I_m of the length r_m, $m = 1, 2, ..., p$, and antisymmetric in subscripts, $r_1, r_2, ..., r_p, q, n > 0$ are integers, the following equations are satisfied:

(1) $$A = q\ {}_m tr_m A + tr_m\ q_m A$$

(2) $$q\ {}_m q_m A = 0$$

(3) $$tr_m tr_m A = 0$$

for any $m = 1, 2, ..., p$.

2.2. Proposition. (a) *Let* $r, q, n > 0$ *be integers and let* $A^{j_1 j_2 ... j_r}{}_{i_1 i_2 ... i_q} \in E^r_q$ $1 \leq j_1, j_2, ..., j_r, i_1, i_2, ..., i_q \leq n$ *be a system of real numbers symmetric in the superscripts, antisymmetric in the subscripts. Then the equation*

(4) $$qX = A$$

has a solution X *if and only if*

(5) $$qA = 0$$

(b) *Let* $p, q, n, r > 0$ *be integers,* $1 \leq p \leq q \leq n$. *Let*

$$Y = Y^{J_1 J_2 ... J_{m-1} J_m J_{m+1} ... J_p}{}_{i_{p+1} i_{p+2} ... i_{p+q}},$$
$$1 \leq i_{p+1}, i_{p+2}, ..., i_{p+q} \leq n,$$
$$|J_i| = r,\ \ 1 \leq i \leq p$$

be a traceless system of real numbers symmetric in the superscripts entering each of the multi-indices $J_1, J_2, ..., J_{m-1}, J_m, J_{m+1}, ..., J_p$. *If* X *satisfies the system of algebraic equations*

(6) $$\underset{(J_1 j_1)}{sym}\ \underset{(J_2 j_2)}{sym} \ldots \underset{(J_p j_p)}{sym}\ \underset{(i_1, i_2, ..., i_{p+q})}{alt} X^{J_1 J_2 ... J_{m-1} J_m J_{m+1} ... J_p}{}_{i_{p+1} i_{p+2} ... i_{p+q}} \delta^{j_1}_{i_1} \delta^{j_2}_{i_2} ... \delta^{j_p}_{i_p} = 0$$

then

$$X^{J_1 J_2 \dots J_{m-1} J_m J_{m+1} \dots J_p}{}_{i_{p+1} i_{p+2} \dots i_{p+q}} = 0 \tag{7}$$

The trace decomposition problem is formulated and proved in [11]. Now we give the statement, which will be used in the next chapter.

2.3. Proposition. (a) *Let r, q, n be a positive intgers, $r \leq q$, and let $A \in E_q^r$, $A = A^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_q}$, $1 \leq j_1, j_2, \dots, j_r, i_1, i_2, \dots, i_q \leq n$ be a system of real numbers. Then there exist a traceless system $B = B^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_q}$ and systems $B_{(s)}^{(p)} = B_{(s) i_1 i_2 \dots i_q}^{(p) j_1 j_2 \dots j_r}$, where $1 \leq p \leq r$, $1 \leq s \leq q$, such that*

$$\begin{aligned} &A^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_q} = \\ &B^{j_1 j_2 \dots j_r}{}_{i_1 i_2 \dots i_q} + \sum_{a=1}^{q} \sum_{b=1}^{r} \delta_{i_a}^{j_b} B_{(a) i_1 i_2 \dots i_{a-1} i_{a+1} \dots i_q}^{(b) j_1 j_2 \dots j_{b-1} j_{b+1} \dots j_r} \end{aligned} \tag{8}$$

The system B is unique.

(b) *Let r, q, n be a positive intgers, $r \leq q$ and let $A \in E_q^r$. There exist unique δ^k-generated, relatively traceless systems $A^(r) \in E_q^r$, where $0 \leq k \leq r-1$ and a δ^r-gererated system $A^{(r)} \in E_q^r$, such that*

$$A = A^{(0)} + A^{(1)} + A^{(2)} + \dots + A^{(r)} \tag{9}$$

3. Variational sequence of the 1st order

The variational sequence on the finite order jet space is defined as a quotient sequence of de Rham sequence of sheaves on $J^r Y$. It has been shown in [7] the exactness of the variational sequence and we can find there also the detailed definition of this concept.

3.1. Definition. Let

$$\begin{aligned} &0 \to R_Y \to \Omega_0^r \to \Omega_1^r \to \Omega_2^r \to \dots \to \Omega_n^r \to \Omega_{n+1}^r \to \Omega_{n+2}^r \to \dots \\ &\dots \to \Omega_M^r \to \Omega_{M+1}^r \to \dots \to \Omega_N^r \to \Omega_{N+1}^r \to 0 \end{aligned} \tag{10}$$

be de Rham sequence of sheaves defined on $J^r Y$, where

$$M = m \begin{pmatrix} n+r-1 \\ n \end{pmatrix} + 2(n-1) + 1, \tag{11}$$

$$N = \dim J^r Y = n + m \begin{pmatrix} n+r \\ n \end{pmatrix}. \tag{12}$$

Let

$$0 \to \Theta_1^r \to \Theta_2^r \to ... \to \Theta_n^r \to \Theta_{n+1}^r \to \Theta_{n+2}^r \to ... \to \Theta_M^r \to 0 \tag{13}$$

be the exact subsequence of (10), $\Theta_q^r \subset \Omega_q^r$,, $\Theta_q^r = \Omega_q^{r(c)} + d\Omega_{q-1}^{r(c)}$, where

$$\Omega_q^{r(c)} = \left[\begin{array}{ll} \ker h & 1 \leq q \leq n \\ \ker p_{q-n} & q \geq n+1 \end{array} \right.$$

h, p_{q-n}, are morphisms of sheaves induced by operators h, p_{q-n} acting on the differential forms on fibered manifold, d is the morphism of sheaves induced by a exterior derivative operator d acting on the differential forms on a manifold. So here we admit a denotation abuse, but we prefer this denomination since it leads to no misunderstanding. So $\Omega_q^{r(c)}$ are just the sheaves of contact and strongly contact forms described by Proposition (2.1).

The variational sequence of order r is defined as a the quotient sequence of (10), where we consider (13) as a subsequence:

$$0 \to R_Y \to \Omega_0^r \underset{E_0}{\to} \Omega_1^r/\Theta_1^r \underset{E_1}{\to} \Omega_2^r/\Theta_2^r \underset{E_2}{\to} ...$$

$$... \underset{E_{n-1}}{\to} \Omega_n^r/\Theta_n^r \underset{E_n}{\to} \Omega_{n+1}^r/\Theta_{n+1}^r \underset{E_{n+1}}{\to} \Omega_{n+2}^r/\Theta_{n+2}^r \underset{E_{n+2}}{\to} ...$$

$$... \to \Omega_M^r/\Theta_M^r \underset{E_M}{\to} \Omega_{M+1}^r \underset{E_{M+1}}{\to} ... \to \Omega_{N-1}^r \underset{E_{N-1}}{\to} \Omega_N^r \to 0 \tag{14}$$

Let us denote $\alpha_q : \Omega_q^r \to \Omega_q^r/\Theta_q^r$, $q = 0, 1, ..., N$ the quotient mappings between the sheaves, $E_q : \Omega_q^r/\Theta_q^r \to \Omega_{q+1}^r/\Theta_{q+1}^r$, $q = 0, 1, ..., N-1$ the sheaf morphisms between the quotient sheaves. We shall describe elements of quotient sheaves Ω_q^r/Θ_q^r, $q = 0, 1, ..., M$ (it is clear,that other cases are trivial, the appropriate morphisms α_q being the identity morphisms, for every $q \geq M+1$).

3.2. Theorem. *Let $\pi : Y \to X$ be a fibered manifold,* $\dim X = n$*,* $\dim Y = m$*, (V, ψ), $\psi = (x^i, y^\sigma)$ be a fibered chart on Y.*

(a) *Let $q \leq n$ and let*

$$\begin{aligned} \rho = & \sum_{k=0}^{q} \frac{1}{k!\,(q-k)!} A^{s_1 s_2 \ldots s_k}_{\sigma_1\sigma_2\ldots\sigma_k, i_{k+1} i_{k+2} \ldots i_q} dy^{\sigma_1}_{s_1} \wedge dy^{\sigma_2}_{s_2} \wedge ... \wedge dy^{\sigma_k}_{s_k} \\ & \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge ... \wedge dx^{i_q} \\ & \sum_{k=1}^{q} \sum_{l=0}^{q-k} \frac{1}{k!l!\,(q-k-l)!} B^{s_{k+1} s_{k+2} \ldots s_{k+l}}_{\sigma_1\sigma_2\ldots\sigma_k, \nu_{k+1}\nu_{k+2}\ldots\nu_{k+l}, i_{k+l+1} i_{k+l+2} \ldots i_q} \\ & \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge ... \wedge \omega^{\sigma_k} \wedge dy^{\nu_{k+1}}_{s_{k+1}} \wedge dy^{\nu_{k+2}}_{s_{k+2}} \wedge ... \wedge dy^{\nu_{k+l}}_{s_{k+l}} \wedge \\ & dx^{i_{k+l+1}} \wedge dx^{i_{k+l+2}} \wedge ... \wedge dx^{i_q} \end{aligned} \tag{15}$$

be a q-form defined on $V^1 \subset J^1Y$. Then $[\rho]$ can be represented by a horizontal q-form on $V^2 \subset J^2Y$

$$\frac{1}{q!}X_{i_1 i_2 \ldots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_q} \tag{16}$$

where

$$X_{i_1 i_2 \ldots i_q} = \operatorname*{alt}_{(i_1, i_2, \ldots, i_q)} \sum_{k=0}^{q} \binom{q}{k} A^{s_1 s_2 \ldots s_k}_{\sigma_1 \sigma_2 \ldots \sigma_k, i_{k+1} i_{k+2} \ldots i_q} y^{\sigma_1}_{s_1 i_1} y^{\sigma_2}_{s_2 i_2} \ldots y^{\sigma_k}_{s_k i_k} \tag{17}$$

(b) *Let*

$$\begin{aligned} \rho = {} & \sum_{m=0}^{n} \frac{1}{m!\,(n-m)!} A^{s_1 s_2 \ldots s_m}_{\sigma_1, \nu_1 \nu_2 \ldots \nu_m, i_{m+1} i_{m+2} \ldots i_n} \\ & \omega^{\sigma_1} \wedge dy^{\nu_1}_{s_1} \wedge dy^{\nu_2}_{s_2} \wedge \ldots \wedge dy^{\nu_m}_{s_m} \wedge dx^{i_{m+1}} \wedge dx^{i_{m+2}} \wedge \ldots \wedge dx^{i_n} + \\ & \sum_{m=0}^{n} \frac{1}{(m+1)!\,(n-m)!} A^{s_1 s_{l_1} s_{l_2} \ldots s_{l_m}}_{\sigma_1 \nu_{l_1} \nu_{l_2} \ldots \nu_{l_m}, i_{m+1} i_{m+2} \ldots i_n} \\ & dy^{\sigma_1}_{s_1} \wedge dy^{\nu_{l_1}}_{s_{l_1}} \wedge dy^{\nu_{l_2}}_{s_{l_2}} \wedge \ldots \wedge dy^{\nu_{l_m}}_{s_{l_m}} \wedge dx^{i_{m+1}} \wedge dx^{i_{m+2}} \wedge \ldots \wedge dx^{i_n} + \\ & \sum_{k=2}^{n+1} \sum_{m=0}^{n+1-k} \frac{1}{k!m!\,(n+1-k-m)!} B^{s_1 s_2 \ldots s_m}_{\nu_1 \nu_2 \ldots \nu_k, \sigma_1 \sigma_2 \ldots \sigma_m, i_{m+1} i_{m+2} \ldots i_{n+1-k}} \\ & \omega^{\nu_1} \wedge \omega^{\nu_2} \wedge \ldots \wedge \omega^{\nu_k} \wedge dy^{\sigma_1}_{s_1} \wedge dy^{\sigma_2}_{s_2} \wedge \ldots \wedge dy^{\sigma_m}_{s_m} \wedge dx^{i_{m+1}} \wedge \\ & dx^{i_{m+2}} \wedge \ldots \wedge dx^{i_{n+1-k}} \end{aligned} \tag{18}$$

be a $(n+1)$*-form defined on* $V^1 \subset J^1 Y$*. Then* $[\rho]$ *can be represented by a 1-contact,* $\pi^{3,0}$*-horizontal* $(n+1)$*-form on* $V^3 \subset J^3 Y$

$$\begin{aligned} [\rho] = {} & \frac{1}{n!} \left(X_{\sigma_1, i_1 i_2 \ldots i_n} - d_j X^j_{\sigma_1, i_1 i_2 \ldots i_n} \right) \\ & \omega^{\sigma_1} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_{n-1}} \wedge dx^{i_n} \end{aligned} \tag{19}$$

where

$$\begin{aligned} & X_{\sigma_1, i_1 i_2 \ldots i_n} = \\ & \operatorname*{alt}_{(i_1, i_2, \ldots, i_n)} \sum_{m=0}^{n} \binom{n}{m} A^{s_1 s_2 \ldots s_m}_{\sigma_1, \nu_1 \nu_2 \ldots \nu_m, i_{m+1} i_{m+2} \ldots i_n} y^{\nu_1}_{s_1 i_1} y^{\nu_2}_{s_2 i_2} \ldots y^{\nu_m}_{s_m i_m}, \\ & X^s_{\sigma_1, i_1 i_2 \ldots i_n} = \\ & \operatorname*{alt}_{(i_1, i_2, \ldots, i_n)} \sum_{m=0}^{n} \binom{n}{m} A^{s s_{l_1} s_{l_2} \ldots s_{l_m}}_{\sigma_1 \nu_{l_1} \nu_{l_2} \ldots \nu_{l_m}, i_{m+1} i_{m+2} \ldots i_n} y^{\nu_{l_1}}_{s_{l_1} i_1} y^{\nu_{l_2}}_{s_{l_2} i_2} \ldots y^{\nu_{l_m}}_{s_{l_m} i_m} \end{aligned} \tag{20}$$

(c) *Let* $q = n + l$ *where* $2 \le l \le M - n$*. Let*

$$(21)\qquad \rho = \sum_{m=0}^{q-l}\sum_{k=0}^{l}\frac{1}{k!\,(m+l-k)!\,(n-m)!}$$
$$A^{s_{k+1}s_{k+2}\dots s_l s_{l_1} s_{l_2}\dots s_{l_m}}_{\sigma_1\sigma_2\dots\sigma_k,\nu_{k+1}\nu_{k+2}\dots\nu_l\nu_{l_1}\nu_{l_2}\dots\nu_{l_m},i_{m+1}i_{m+2}\dots i_n}$$
$$\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\dots\wedge\omega^{\sigma_k}\wedge dy^{\nu_{k+1}}_{s_{k+1}}\wedge dy^{\nu_{k+2}}_{s_{k+2}}\wedge\dots\wedge dy^{\nu_l}_{s_l}$$
$$\wedge dy^{\nu_{l_1}}_{s_{l_1}}\wedge dy^{\nu_{l_2}}_{s_{l_2}}\wedge\dots\wedge dy^{\nu_{l_m}}_{s_{l_m}}\wedge dx^{i_{m+1}}\wedge dx^{i_{m+2}}\wedge\dots\wedge dx^{i_n}+$$
$$\sum_{k=l+1}^{q}\sum_{m=0}^{q-k}\frac{1}{k!m!\,(q-k-m)!}B^{s_1s_2\dots s_m}_{\nu_1\nu_2\dots\nu_k,\sigma_1\sigma_2\dots\sigma_m,i_{m+1}i_{m+2}\dots i_{q-k}}$$
$$\omega^{\nu_1}\wedge\omega^{\nu_2}\wedge\dots\wedge\omega^{\nu_k}\wedge dy^{\sigma_1}_{s_1}\wedge dy^{\sigma_2}_{s_2}\wedge\dots\wedge dy^{\sigma_m}_{s_m}\wedge$$
$$dx^{i_{m+1}}\wedge dx^{i_{m+2}}\wedge\dots\wedge dx^{i_{q-k}}$$

be a q-form defined on $V^1\subset J^1Y$. Then $[\rho]$ can be represented by an l-contact, $\pi^{3,0}$-horizontal q-form on $V^3\subset J^3Y$

$$(22)\qquad [\rho]=\frac{1}{l!n!}\left(X_{\sigma_1\sigma_2\dots\sigma_l,i_1i_2\dots i_n}-d_j\underset{(\sigma_1\sigma_2,\dots,\sigma_l)}{alt}X^j_{\sigma_1\sigma_2\dots\sigma_{l-1},\sigma_l,i_1i_2\dots i_n}\right)$$
$$\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\dots\wedge\omega^{\sigma_l}\wedge dx^{i_1}\wedge dx^{i_2}\wedge\dots dx^{i_n}+$$
$$\frac{1}{(l-1)!n!}\frac{2}{l}\underset{(\sigma_{l-1},\sigma_l)}{sym}X^{s_l}_{\sigma_1\sigma_2\dots\sigma_{l-1},\sigma_l,i_1i_2\dots i_n}$$
$$\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\dots\wedge\omega^{\sigma_{l-1}}\wedge\omega^{\sigma_l}_{s_l}\wedge dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge dx^{i_n}$$
$$\sum_{k=0}^{l-2}\frac{1}{k!\,(l-k)!n!}X^{s_{k+1}s_{k+2}\dots s_l}_{\sigma_1\sigma_2\dots\sigma_k,\nu_{k+1}\nu_{k+2}\dots\nu_l,i_1i_2\dots i_n}$$
$$\omega^{\sigma_1}\wedge\omega^{\sigma_2}\wedge\dots\wedge\omega^{\sigma_k}\wedge\omega^{\nu_{k+1}}_{s_{k+1}}\wedge\omega^{\nu_{k+2}}_{s_{k+2}}\wedge\dots\wedge\omega^{\nu_l}_{s_l}\wedge dx^{i_1}\wedge dx^{i_2}\wedge\dots\wedge dx^{i_n}$$

where

$$(23)\qquad X^{s_{k+1}s_{k+2}\dots s_l}_{\sigma_1\sigma_2\dots\sigma_k,\nu_{k+1}\nu_{k+2}\dots\nu_l,i_1i_2\dots i_n}=$$
$$\underset{(i_1,i_2,\dots,i_n)}{alt}\sum_{m=0}^{n}\binom{n}{m}A^{s_{k+1}s_{k+2}\dots s_l s_{l_1}s_{l_2}\dots s_{l_m}}_{\sigma_1\sigma_2\dots\sigma_k,\nu_{k+1}\nu_{k+2}\dots\nu_l\nu_{l_1}\nu_{l_2}\dots\nu_{l_m},i_{m+1}i_{m+2}\dots i_n}$$
$$y^{\nu_{l_1}}_{s_{l_1}i_1}y^{\nu_{l_2}}_{s_{l_2}i_2}\dots y^{\nu_{l_m}}_{s_{l_m}i_m}$$
$$1\le k\le l$$

Proof. We prove the equivalence $[\rho]=0\Leftrightarrow\rho\in\Theta^r_q$ in parts (a), (b), and (c). This yields the fact, that the forms (16), (19), (22), represent just the elements of the quotient sheaves in question. This representation is considered up to an isomorphism.

(a) Putting $[\rho]=0$, some coefficients become immediately equal to zero and at the same time we obtain a system of q-equation. We have used the fact that the equation $[\rho]=0$ is a polynomial in " y^{σ}_{si}" here. We solve the q-equation by means of Proposition (2.3) throwing off the δ-generated part of some coefficients "A". We can do it, because δ^1-generated part of the coefficient "A" plays no role in q-equation because of antisymmetry properties of coefficients. So only the traceless coefficients enter q-equations and Proposition (2.2)(b) can be applied since all its assumptions are satisfied. Finally, it is easy to see that $\rho \in \Theta^r_q$, using Proposition (2.1). Further, let us have $\rho \in \Theta^r_q$. Using Proposition (2.1) and the fact that the set of contact forms is endowed with a structure of the ideal (in the exterior algebra of forms on J^1Y) we obtain $[\rho]=0$.

(b) The first part of the proof is obtained by the similar procedure as in (a). Proposition (2.2)(a) and equations (1), (2), (3) must be applied. We must solve the cases $l+1 \leq n-m$, $l+1 > n-m$, apart.

Further, let us have an element $\rho \in \Theta^1_{n+1}$. We use the structure of the sheaf Θ^1_{n+1} and the linearity of a sheaf morphism α_{n+1} here (see (19), (20)). First, let us have a strongly contact $(n+1)$-form; applying α_{n+1} to it we get immediately zero as it is claimed. Second, let $\kappa \in \theta^1_n$ be a contact n-form. Proposition (2.1)(b) provides

$$d\kappa = d\omega^{\sigma} \wedge \Phi_{\sigma} - \omega^{\sigma} \wedge d\Phi_{\sigma} + d\omega^{\sigma} \wedge d\Psi_{\sigma},$$

where Φ_{σ} is some $(n-1)$-form and Ψ_{σ} is some $(n-2)$-form on V^1. We want to show that $[d\kappa]=0$. Let us take the form $d\omega^{\sigma} \wedge \Phi_{\sigma} - \omega^{\sigma} \wedge d\Phi_{\sigma}$. The coefficients of the form Φ_{σ} can be decomposed by Proposition (2.3) and thus the form Φ_{σ} is splitted into two parts. The first one, denoted Φ'_{σ}, is generated by forms ω and $d\omega$ and the second one Φ''_{σ} is of the form

$$\Phi''_{\sigma} = \sum_{k=0}^{n-1} \Phi^{s_1 s_2 \dots s_k}_{\sigma,\sigma_1\sigma_2\dots\sigma_k, i_{k+1} i_{k+2} \dots i_{n-1}} dy^{\sigma_1}_{s_1} \wedge dy^{\sigma_2}_{s_2} \wedge \dots \wedge dy^{\sigma_k}_{s_k}$$
$$\wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \dots \wedge dx^{i_{n-1}}$$

where all coefficients $\Phi^{s_1 s_2 \dots s_k}_{\sigma,\sigma_1\sigma_2\dots\sigma_k, i_{k+1} i_{k+2} \dots i_{n-1}}$ are traceless. It is clear that

$$d\omega^{\sigma} \wedge \Phi'_{\sigma} - \omega^{\sigma} \wedge d\Phi'_{\sigma} \in \ker p_1.$$

using Proposition (2.3) again. Further, we have

$$\alpha_{n+1}(d\omega^{\sigma} \wedge \Phi''_{\sigma} - \omega^{\sigma} \wedge d\Phi''_{\sigma}) = 0,$$

applying antisymmetry properties of some coefficients. Now let us concern with the form $d\omega^{\sigma} \wedge d\Psi_{\sigma}$. By the same arguments as above we can decompose the form Ψ_{σ} to the part Ψ'_{σ}, which is generated by ω and $d\omega$ and the second part Ψ''_{σ} which has a form

$$\Psi''_\sigma = \sum_{k=0}^{n-1} \frac{1}{k!\,(n-1-k)!} \Psi''^{s_1 s_2 \ldots s_k}_{\sigma,\sigma_1\sigma_2\ldots\sigma_k, i_{k+1} i_{k+2}\ldots i_{n-1}}$$
$$dy^{\sigma_1}_{s_1} \wedge dy^{\sigma_2}_{s_2} \wedge \ldots \wedge dy^{\sigma_k}_{s_k} \wedge dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \ldots \wedge dx^{i_{n-1}}$$

where $\Psi''^{s_1 s_2 \ldots s_k}_{\sigma,\sigma_1\sigma_2\ldots\sigma_k, i_{k+1} i_{k+2}\ldots i_{n-1}}$ are traceless. It is obvious that $\alpha_{n+1}(d\omega^\sigma \wedge d\Psi'_\sigma) = 0$ by Proposition (2.3), the conclusion $\alpha_{n+1}(d\omega^\sigma \wedge \Psi''_\sigma) = 0$ is obtained by the direct calculation, applying antisymmetry properties of coefficients.

(c) The first part of the equivalence $[\rho] = 0 \Leftrightarrow \rho \in \Theta^r_q$ is obtained by the similar way as in (b). So let us have $\rho \in \Theta^r_q$. We use the linearity of the sheaf morphism α_q, $q = n+2, n+3, \ldots, M$. First, let us have a strongly contact q-form, where $q = n+2, n+3, \ldots, M$. Its class is equal to zero as can be shown by a straightforward calculation. Second, let $\kappa \in \theta^1_{q-1}$ be a strongly contact $(q-1)$-form. Using Proposition (2.1) we see that

$$\kappa = \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \ldots \wedge \omega^{\sigma_p} \wedge d\omega^{\nu_1} \wedge d\omega^{\nu_2} \wedge \ldots \wedge d\omega^{\nu_k}$$
$$\wedge \Phi_{\sigma_1\sigma_2\ldots\sigma_p,\nu_1\nu_2\ldots\nu_k}$$

where $p + k = l$ and $\Phi_{\sigma_1\sigma_2\ldots\sigma_p,\nu_1\nu_2\ldots\nu_k}$ are some $(q-1-p-2k)$-forms on V^1. We show that $[d\kappa] = 0$ by the similar method as in (b), but we must solve the cases $l = 2$, $p = 0,1,2$; $l = 3$, $p = 0,1,2,3$; $l = 4,5,\ldots,M-n$, $p = 0$, $p = 1,2,\ldots,l-3$, $p = l-2$, $p = l-1$, $p = l$ apart.

3.3. Remark. Elements of Ω^1_n/Θ^1_n represent some Lagrangians of the order two, affine in "y^σ_{si}". We can see that the morphism of sheaves

$$E_n : \Omega^1_n/\Theta^1_n \to \Omega^1_{n+1}/\Theta^1_{n+1}$$

can be locally represented by the following equation for any form ρ described by (15), $q = n$ and $X_{i_1 i_2 \ldots i_n}$ described by (17):

$$(24)\quad E_n\alpha_n(\rho) = \alpha_{n+1}(d\rho) =$$
$$\frac{1}{n!}\left(\frac{\partial X_{i_1 i_2\ldots i_n}}{\partial y^\sigma} - d_j \frac{\partial X_{i_1 i_2\ldots i_n}}{\partial y^\sigma_j} + d_j d_i \frac{\partial X_{i_1 i_2\ldots i_n}}{\partial y^\sigma_{ji}}\right)\omega^\sigma \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots dx^{i_n}$$

and thus the morphism of sheaves E_n can be represented as the *Euler-Lagrange mapping* which assigns to each Lagrangian $\frac{1}{n!} X_{i_1 i_2\ldots i_n} dx^{i_1} \wedge dx^{i_2} \wedge \ldots dx^{i_n}$ its Euler-Lagrange form.

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