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FOCK SPACE METHODS AND THE LAGRANGIAN FORMALISM ON FINITE JET BUNDLE EXTENSIONS

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Abstract. The geometric Lagrangian theory is formulated in the language of jet bundle extensions. The use of finite jet bundle extensions corresponds more closely to spirit of the original Lagrangian theory as studied in physical literature corresponding to a Lagrangian theory of arbitrary, but finite order.

The underlying structure is based on the analysis of some basic mathematical objects such as: the contact ideal and the (exact) variational sequence. In this paper we give new and much simpler proofs for the whole theory using Fock space methods. For instance, in a given chart, one finds out quite naturally that some global differential forms involved in the description of the variational sequence (the so-called contact forms) can be locally written in terms of expressions having some symmetry and/or antisymmetry properties. These expressions can be regarded as tensors in some Fock space and some very complicated identities become very simple if one uses the creation and annihilation operators.

Using these results we give the most general expression for a variationally trivial Lagrangian.

1. Introduction

The modern Lagrangian theory is most suitable analyzed in the language of finite order jet bundle extensions [11] - [13]. Some very hard combinatorial problems must be solved and methods inspired by quantum mechanics i.e. Fock space methods proved to be extremely helpful. The central observation is that various function coefficients of the various differential forms appearing in the proofs have some symmetry or antisymmetry properties which make them as elements in some Fock space so, most of the hard combinatorial relations can be written with the help of the creation and annihilation operators and solved using elementary properties of these Fock space operators. In [8] we have used this idea to lay the foundations of the whole theory i.e. we have given new proofs for the structure formula of the contact and respectively the strong contact forms, for the existence

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of the Euler-Lagrange and of the Helmholtz-Sonin forms and for the exactness of the variational sequence. In [9] the analysis of the most general form of a variationally trivial Lagrangian of arbitrary order and the generic form of a locally variational differential equation was provided. (We mention that an independent proof recently appeared in [15].) In both cases the dependence on the highest order derivatives is through some polynomial expressions, called hyper-Jacobians [4], [10], but in the first case we are also able to explicitate the form of the coefficients of these polynomial expressions. These results generalize results from the previous references (see for instance [5], [5], [7]).

2. Jet Bundles and the Lagrangian Formalism

2.1. Basic Facts about Jet Bundle Extensions. The kinematical structure of a classical field theory is described by a fibred bundle structures $\pi : Y \mapsto X$ where X , Y are differentiable manifolds of dimensions $\dim(X) = n, \quad \dim(Y) = m + n$ and π is the canonical projection of the fibration. We use standard notations for the *adapted charts* (V, ψ) where $\psi = (x^i, y^\sigma) \quad (i = 1, ..., n;\ \sigma = 1, ..., m)$ and on the *r-jet bundle extension* $J^r_n Y \mapsto X \quad (r \in \mathbb{N})$ we use the *adapted charts* (V^r, ψ^r), where $\psi = (x^i, y^\sigma, y^\sigma_j, ..., y^\sigma_{j_1,...,j_k}, ..., y^\sigma_{j_1,...,j_r}), \quad j_1 \leq j_2 \leq \cdots \leq j_k, \quad k = 1, ..., r$. Some other standard notations [2] are useful: $\partial^{j_1,...,j_k}_\sigma \equiv \frac{r_1!\ldots r_n!}{k!} \frac{\partial}{\partial y^\sigma_{j_1,...,j_k}}, \quad k = 1, ..., r$ and the *formal derivatives* $d^r_i \equiv \frac{\partial}{\partial x^i} + \sum_{k=0}^{r-1} y^\sigma_{i,j_1,...,j_k} \partial^{j_1,...,j_k}_\sigma = \frac{\partial}{\partial x^i} + \sum_{|J| \leq r-1} y_{iJ} \partial^J_\sigma$; (multi-index notation will be frequently used.)

We denote the forms of degree q on $J^r Y$ by Ω^r_q. A form $\rho \in \Omega^r_q$ is called π^r-*horizontal* (or *basic*) if it has the following local expression: $\rho = B_{i_1,...,i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}$ with $B_{i_1,...,i_q}$ smooth skew-symmetric functions on V^r. We denote the set of basic forms of degree q by $\Omega^r_{q,X}$. The elements of $\lambda \in \Omega^r_{n,X}$ are called *Lagrange forms*. They have the local expression $\lambda = L\theta_0$ where L is a smooth function on V^r and $\theta_0 \equiv dx^1 \wedge \cdots \wedge dx^n$.

By a *contact form* we mean any form $\rho \in \Omega^r_q$ verifying $(j^r\gamma)^* \rho = 0$ for any section γ. We denote by $\Omega^r_{q(c)}$ the set of contact forms of degree $q \leq n$. By elementary computations one finds out that for any chart (V, ψ) on Y, every element of the set $\Omega^r_{1(c)}(V)$ is a linear combination of the following expressions:

$$\omega^\sigma_{j_1,...,j_k} \equiv dy^\sigma_{j_1,...,j_k} - y^\sigma_{i,j_1,...,j_k} dx^i, \quad k = 0, ..., r - 1\,. \tag{2.1}$$

If $q = n + 1, ..., N \equiv \dim(J^r Y) = m\binom{n+r}{n}$ then $\rho \in \Omega^r_q$ is a *strongly contact form iff* its contact component of order $q - n$ vanishes i.e. $p_{q-n}\rho = 0$. For more details, we refer to [8], [9] and [13].

2.2. Fock Space methods. The rôle of the Fock space methods emerges from a canonical expression for any form $\rho \in \Omega^r_q$. We start from the fact that the forms $dx^i, \quad \omega^\sigma_J \quad (|J| \leq r - 1)$ and $dy^\sigma_I \quad (|I| = r)$ are a basis in the linear space of 1-forms. The form ρ is a polynomial of degree q in these forms (with respect to the wedge product $\wedge$). We separate all the terms containing at least a factor ω^σ_J and get a decomposition $\rho = \rho_0 + \rho'$ where ρ_0 has the structure $\rho_0 = \sum_{|J| \leq r-1} \omega^\sigma_J \wedge \Phi^J_\sigma$

and ρ' is a polynomial of degree q only in dx^i and dy^σ_I $\quad (|I| = r)$. It is clear the one can write it as follows:

$$\rho' = \sum_{s=0}^{q} \frac{1}{s!(q-s)!} \sum_{|I_1|,\ldots,|I_s|=r} A^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_q}$$
$$dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_q} \,, \tag{2.2}$$

where $A^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_q}$ are smooth functions on V^r and can be assumed to verify the following symmetry property:

$$A^{I_{P(1)},\ldots,I_{P(s)}}_{\sigma_{P(1)},\ldots,\sigma_{P(s)},i_{Q(s+1)},\ldots,i_{Q(q)}} = (-1)^{|P|+|Q|} A^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_q} \tag{2.3}$$

where P and Q are permutations and $|P|, |Q|$ are their signature.

Let us define the following Hilbert spaces of Fock type:

$$\mathcal{H}_s \equiv \mathcal{F}^{(-)}(\mathbb{R}^n) \otimes \underbrace{\mathcal{F}^{(+)}(\mathbb{R}^n) \otimes \cdots \otimes \mathcal{F}^{(+)}(\mathbb{R}^n)}_{s-\text{times}}$$

where $\mathcal{F}^{(\pm)}(\mathbb{R}^n)$ are the symmetric (corresp.. $+$) and the antisymmetric (corresp.. $-$) Fock spaces. Then we have the well known decomposition in subspaces with fixed number of "Bosons" and "Fermions": $\mathcal{H}_s = \oplus \mathcal{H}_{k,l_1,\ldots,l_s}$ with the convention: $\mathcal{H}_{k,l_1,\ldots,l_s} \equiv 0$ if any one of the indices $k, l_1, \ldots, l_q$ is negative or, if $k > n$. Then we can consider $A^{I_1,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_{s+1},\ldots,i_q}$ as the components of a tensor $A_{\sigma_1,\ldots,\sigma_s} \in \mathcal{H}_{q-s,\underbrace{r,\ldots,r}_{s-\text{times}}}$.

Let us denote the creation and the annihilation Fermionic operators a^{*i}, a_i, $(i = 1, \ldots, n)$ and the corresponding creation and annihilation Bosonic operators $b^*_{(\alpha)i}$, $b^i_{(\alpha)}$ $(\alpha = 1, \ldots, s;\ i = 1, \ldots, n)$.

One introduces the operators

$$B_\alpha : \mathcal{H}_{k,l_1,\ldots,l_s} \to \mathcal{H}_{k+1,l_1,\ldots,l_{\alpha-1},l_\alpha+1,l_{\alpha+1},\ldots,l_s} \quad (\alpha = 1, \ldots, s)$$

according to the formula

$$B_\alpha \equiv b^*_{(\alpha)i} a^{*i} \,, \quad (\alpha = 1, \ldots, s) \,. \tag{2.4}$$

It is convenient to rescale these operators (compare with [8]) such that we have for instance:

$$(B_1 A)^{\{j_0,\ldots,j_k\},I_2,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_0,\ldots,i_l} = (-1)^{s-1} \mathcal{S}^+_{j_0,\ldots,j_k} \mathcal{S}^-_{i_0,\ldots,i_l} A^{\{j_1,\ldots,j_k\},I_2,\ldots,I_s}_{\sigma_1,\ldots,\sigma_s,i_1,\ldots,i_l} \delta^{j_0}_{i_0} \tag{2.5}$$

and similarly for $B_2, \ldots, B_s$; here $\mathcal{S}^\pm_J$ is the symmetrization (resp. antisymmetrization) operator in the indices J. The central property of these operators is (see [8], lemma 3.6):

Lemma 2.1. *Let $X \in \mathcal{H}_{k,r_1,\ldots,r_s}$ Then X verifies the equation*

$$B_1 \cdots B_s X = 0 \tag{2.6}$$

iff it is of the form

$$X = \sum_{\alpha=1}^{s} B_\alpha X_\alpha \tag{2.7}$$

for some $X_\alpha \in \mathcal{H}_{k-1,r_1,\ldots r_{\alpha-1},r_\alpha-1,r_{\alpha+1},\ldots,r_s}$.

This key result can be used (see [8]) to obtain a structure formula for an arbitrary contact form generalizing the expression (2.1) for contact forms of degree 1. Explicitly, let (V,ψ) an adapted chart on the fibre bundle Y and let $\rho \in \Omega^r_q(Y), \quad q = 2,\ldots,n$. Then ρ is contact *iff* in the associated chart is sum a sum of terms each of them having at least a factor which is a contact form of degree 1 i.e. $\omega^\sigma_J, \quad |J| \leq r-1$ or an exterior differential of the form $d\omega^\sigma_I, \quad |I| = r$. Applying the same methods one obtains the structure theorem for strongly contact forms. Let $n+1 \leq q \leq N$ and $\rho \in \Omega^r_q$. Let (V,ψ) be a chart on Y Then ρ is a strongly contact form *iff* in the associated chart it is sum of terms each of one having at least $q-n+1$ factors of the type $\omega^\sigma_J, \quad |J| \leq r-1$ or $d\omega^\sigma_I, \quad |I| = r$. We mention that the original proof of this structure formula, based on a different treatment of the combinatorial aspects, appears in [14].

We also have a dual result of the lemma (2.1), namely:

Lemma 2.2. *The tensor* $X \in \mathcal{H}_{s,\underbrace{r,\ldots,r}_{k-\text{times}}}$ *verifies the system of equations*

$$B_\alpha X = 0 \quad (\alpha = 1,\ldots,k). \tag{2.8}$$

iff is of the following form:

$$X = \left(\prod_{\alpha=1}^{k} B_\alpha\right) X_0 \tag{2.9}$$

where X_0 *is arbitrary.*

2.3. Euler-Lagrange and Helmholtz-Sonin forms. An element $T \in \Omega^s_{n+1,X}$ is called a *differential equation* (or a *source form* in the terminology of [1]) if it has the following local expression in every associated chart:

$$T = T_\sigma \omega^\sigma \wedge \theta_0\,. \tag{2.10}$$

Let us take $\lambda \in \Omega^r_{n,X}$ with the local expression (2.1). Then one defines on any chart $(V^s,\psi^s), \quad (s \geq 2r)$ the so-called *Lie-Euler expressions* [1] according to:

$$E^I_\sigma(L) \equiv \sum_{|J|\leq r-|I|} (-1)^{|J|} \binom{|I|+|J|}{|I|} d_J \partial^{IJ}_\sigma L\,. \tag{2.11}$$

In particular, for $I = \emptyset$ we have the *Euler-Lagrange expressions*:

$$E_\sigma(L) \equiv \sum_{|J|\leq r} (-1)^{|J|} d_J \partial^J_\sigma L\,. \tag{2.12}$$

Then there exists a globally defined $(n+1)$-form, denoted by $E(\lambda)$ such that in V^s:

$$E(\lambda) = E_\sigma(L)\omega^\sigma \wedge \theta_0 \,. \tag{2.13}$$

One calls this form the *Euler-Lagrange form* associated to λ and notes that it is a differential equation (see formula (2.10)). In general, a differential equation $T \in \Omega^s_{n+1,X}$ is called *(locally) variational* or *of Euler-Lagrange type iff* there exists a (local) Lagrange form $\lambda \in \Omega^r_{n,X}$ such that

$$T = E(\lambda), \quad \text{up to a pull} - \text{back} \,. \tag{2.14}$$

One notices that in this case the general form of a differential equation coincides with the well-known form of the Euler-Lagrange equations.

A similar construction gives the so-called Helmholtz-Sonin form. Let $T \in \Omega^s_{n+1,X}$ be a differential equation with the local form given by (2.10). We define the following expressions in any chart $V^t, \quad t \geq 2s$:

$$H^J_{\sigma\nu} \equiv \partial^J_\nu T_\sigma - (-1)^{|J|} E^J_\sigma(T_\nu), \quad |J| \leq s \,. \tag{2.15}$$

Then there exists a globally defined $(n+2)$-form, denoted by $H(T)$ such that on V^t:

$$H(T) = \sum_{|J|\leq s} H^J_{\sigma\nu}\omega^\nu_J \wedge \omega^\sigma \wedge \theta_0 \,. \tag{2.16}$$

2.4. The Exact Variational Sequence. The basic constructions goes along the following lines. If $\pi : Y \to X$ is a fibre bundle and $U, V \subset Y$ are charts such that $U \subset V$, we denote by $i_{U,V} : U^r \to V^r$ the canonical inclusion. Then, the collection $\{\Omega^r_q(V)\} \quad (q \geq 0), \quad \{i^*_{U,V}\}$ is a presheaf denoted by Ω^r_q. One also needs the subspaces $\theta^r_q \in \Omega^r_{q(c)}$ by:

$$\theta^r_1 \equiv \Omega^r_{1(c)}, \quad \theta^r_q \equiv d\Omega^r_{q-1(c)} + \Omega^r_{q(c)} \quad (q = 2, ..., N = \dim(J^r Y)) \,. \tag{2.17}$$

Next, one introduces the so-called contact homotopy operator. The construction is the following. Let $U \subset \mathbb{R}^n$ (resp. $V \subset \mathbb{R}^m$) an open set (resp. a ball centred in $0 \in \mathbb{R}^m$) and $W = U \times V$. One considers the operator χ_r as a map $\chi^r_t : [0,1] \times J^r W \to J^r W$ given by:

$$\chi_r(t, (x^i, y^\sigma, y^\sigma_j, ..., y^\sigma_{j_1,\ldots,j_s})) = (x^i, ty^\sigma, ty^\sigma_j, ..., ty^\sigma_{j_1,\ldots,j_s}) \,. \tag{2.18}$$

Then for any $\rho \in \Omega^r_q(W)$ we have the *unique* decomposition

$$(\chi^r_t)^*\rho = dt \wedge \rho_0(t) + \rho_1(t) \tag{2.19}$$

where $\rho_0(t)$ (resp. $\rho_1(t)$) are $q-1$ (resp. q) forms which do not contain the differential dt. Then the *contact homotopy operator* is by definition the map $A : \Omega^r_q(V) \to \Omega^r_{q-1}(V)$ given by:

$$A\rho \equiv \int_0^1 \rho_0(t) \,. \tag{2.20}$$

This definition is justified by the following result resembling the key idea of the Poincaré lemma. Let $\rho \in \Omega^r W$ be arbitrary. The following formula is true:

$$\rho = Ad\rho + dA\rho + \rho_1(0)\,. \tag{2.21}$$

The central result of [12] (the contact homotopy is essential for the proof) is the following one.

Theorem 2.3. *Let $E_q : \Omega^r_q/\theta^r_q \to \Omega^r_{q+1}/\theta^r_{q+1}$ be given by*

$$E_q([\rho]) \equiv [d\rho] \tag{2.22}$$

where $[\rho]$ is the class of ρ modulo θ^r_q. Then the quotient sequence

$$\begin{aligned} 0 \longrightarrow \mathbb{R} \longrightarrow \Omega^r_0 \xrightarrow{E_0} \Omega^r_1/\theta^r_1 \xrightarrow{E_1} \cdots \xrightarrow{E_{P-1}} \Omega^r_P/\theta^r_P \xrightarrow{E_P} \\ \Omega^r_{P+1} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^r_N \longrightarrow 0 \end{aligned} \tag{2.23}$$

is a acyclic resolution of the constant sheaf $\mathbb{R}$. In particular it is exact.

One calls this the *variational sequence of order r over Y* and denotes for simplicity: $\mathcal{V}^r_q \equiv \Omega^r_q/\theta^r_q$. Some special classes have distinct names. So, if $\lambda \in \Omega^r_n$ then the class $[\lambda] \in \Omega^r_{n+1}/\theta^r_{n+1}$ is called the *Euler-Lagrange class* of λ. If $T \in \Omega^r_{n+1}$ then $[T] \in \Omega^r_{n+2}/\theta^r_{n+2}$ is called the *Helmholtz-Sonin class* of T.

One can give characterizations of $\mathcal{V}^r_q, \quad q = n, n+1$ using Euler-Lagrange and Helmholtz-Sonin forms.

3. Variationally Trivial Lagrangians

A *variationally trivial Lagrange form of order r* is any $\lambda \in \Omega^r_{n,X}$ such that $E(\lambda) = 0$. In other words, the corresponding Euler-Lagrange expressions are identically zero, or the Euler-Lagrange equations are identities for any section γ. Here we give the general form of such a Lagrange form. We follow, basically, the line of argument from [13]; as before, we simplify considerably and complete the proofs from this reference using the techniques developed in [8]. The main reward is the possibility to obtain a very explicit expression for a locally trivial Lagrangian.

First, we remind that $\rho \in \Omega^{r+1}_q$ is called $\pi^{r+1,r}$*-projectable iff* there exists $\rho' \in \Omega^r_q$ such that

$$\rho = (\pi^{r+1,r})^* \rho'\,. \tag{3.1}$$

One can easily see that locally this amounts to the condition that if ρ is written as a polynomial in dx^i and $dy^\sigma_J \quad (|J| \leq r+1)$ then, the differentials $dy^\sigma_J \quad (|J| = r+1)$ must be absent and moreover, the coefficient functions must not depend on $y^\sigma_J \quad (|J| = r+1)$.

We start with the following result [12]:

Proposition 3.1. *Let $\eta \in \Omega^r_q \quad (q = 1, ..., n-1)$ such that $hd\eta$ is a $\pi^{r+1,r}$-projectable $(q+1)$-form. Then one can write η as follows:*

$$\eta = \nu + d\phi + \psi \tag{3.2}$$

where $\nu \in \Omega^r_{q,X}$ is a basic q-form and $\psi \in \Omega^r_{q(c)}$ is a contact q-form.

Proof. Let (V,ψ) be a chart on Y. Then in the associated chart (V^r,ψ^r) one can write η in the standard form

$$\eta = \eta_0 + \eta_1 \tag{3.3}$$

where in η_0 we collect all terms containing at least one factor $\omega^\sigma_J \quad (|J| \leq r-1)$ and η_1 is a polynomial only in dx^i and $dy^\sigma_J \quad (|J| = r+1)$:

(3.4)

$$\eta_1 = \sum_{s=0}^{q} \sum_{|I_1|,\dots,|I_s|=r} A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q} dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_q},$$

where the coefficients $A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}$ have antisymmetry properties of type (2.3).

In particular η_0 is a contact form so we have $hd\eta = hd\eta_1$ i.e. the form $hd\eta_1$ is, by hypothesis, $\pi^{r+1,r}$-projectable. One first computes in general:

$$\begin{aligned} d\eta_1 = \sum_{s=0}^{q} \sum_{|I_1|,\dots,|I_s|=r} \sum_{|J|\leq r-1} &\left(\partial^J_\nu A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}\right) \\ &\omega^\nu_J \wedge dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_q} \\ &+ \sum_{s=0}^{q+1} \sum_{|I_1|,\dots,|I_s|=r} \tilde{A}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}} \\ &dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_{q+1}}, \end{aligned}$$

where

$$\begin{aligned} \tilde{A}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}} \equiv\ & \mathcal{S}^-_{(I_1,\sigma_1),\dots,(I_s\sigma_s)} \partial^{I_1}_{\sigma_1} A^{I_2,\dots,I_s}_{\sigma_2,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}} + \\ & (-1)^s \mathcal{S}^-_{i_{s+1},\dots,i_{q+1}} d_{i_{s+1}} A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+2},\dots,i_{q+1}}; \end{aligned} \tag{3.5}$$

here $\mathcal{S}^-_{(I_1,\sigma_1),\dots,(I_s\sigma_s)}$ is the antisymmetrization projector in the corresponding couples of indices.

Then one gets

$$hd\eta = hd\eta_1 = \sum_{s=0}^{q+1} \sum_{|I_1|,\dots,|I_s|=r} \tilde{A}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}} y^{\sigma_1}_{I_1 i_1} \cdots y^{\sigma_s}_{I_s i_s} dx^{i_1} \wedge \cdots \wedge dx^{i_{q+1}}$$

which is $\pi^{r+1,r}$-projectable *iff* the coefficient functions do not depend on $y^\sigma_I \quad (|I| = r+1)$. By repeatedly applying derivative operators, one obtains that the condition of $\pi^{r+1,r}$-projectability is equivalent to

$$B_1 \cdots B_s \tilde{A}_{\sigma_1,\dots,\sigma_s} = 0, \quad (s = 1,\dots,q+1). \tag{3.6}$$

It is possible to apply lemma 2.1 and one obtains the following generic expression: $\tilde{A}_{\sigma_1,\dots,\sigma_s} = \sum_{\alpha=1}^{s} B_\alpha \tilde{A}^\alpha_{\sigma_1,\dots,\sigma_s}$ or, in index notations, this means the function $\tilde{A}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{q+1}}$ is a sum of terms containing at least a factor of the type δ^j_i where j belongs to one of the multi-indices $I_p, \quad p = 1,\dots,s$ and $i \in \{i_{s+1},\dots,i_{q+1}\}$.

Inserting back into the expression of $d\eta_1$ we get after minor prelucrations that

$$d\eta_1 = \nu_1 + d\phi + \sum_{|J|\leq r-1} \omega^\sigma_J \wedge \Phi^J_\sigma \tag{3.7}$$

with $\nu_1 \in \Omega^r_{q,X}$ a basic q-form, $\phi \in \Omega^r_{q(c)}$ and Φ^J_σ a polynomial of degree q in dx^i and $dy^\sigma_I \quad |I| = r$.

Now one continues by induction and finally obtains:

$$d\eta_1 = \nu_1 + d\phi_1 + \sum_{|J|\leq r-1} \Phi^J_{\sigma,i_1,\ldots,i_q} \omega^\sigma_J \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q} + \psi \tag{3.8}$$

where ϕ_1 (resp. ψ) is some contact (resp. 2-contact) form of degree q (resp. $q+1$.)

(iii) The last step of our proof consists in using the contact homotopy operator A (see the definition contained in the relations (2.18) - (2.20)). We apply the relation (2.21) to the form $\rho = \eta_1 - \phi_1$:

$$\begin{aligned}\eta_1 - \phi_1 = A\left(\nu_1 + \sum_{|J|\leq r-1} \Phi^J_{\sigma,i_1,\ldots,i_q} \omega^\sigma_J \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q} + \psi\right) + \\ dA(\eta_1 - \phi_1) + \eta_1(0) - \phi_1(0).\end{aligned} \tag{3.9}$$

One proves by direct computation that: $A\nu_1 = 0$; if ψ is 2-contact form, then $A\psi$ is a contact form; $A(\sum \cdots)$ and $\eta_1(0)$ are basic forms; if ϕ_1 is a contact form, then $\phi_1(0)$ is also a contact form.

Inserting this information in the preceding relation one obtains that the result from the statement of the proposition is true for the form η_1. Taking into account (3.3) we obtain the same result for the form η. □

Let us note that the converse of this statement is not true. In fact the condition of $\pi^{r+1,r}$-projectability imposes additional constraints on the basic form ν which will be analyzed in the next lemma. A complete proof of the following result appears in [2]; here we offer an alternative proof which seems to be much more simpler.

Proposition 3.2. *Let $\nu \in \Omega^r_{q,X} \quad (q = 1,...,n-1)$ Then the basic form $hd\nu$ is $\pi^{r+1,r}$-projectable iff there exists for any chart (V,ψ) a form $\tilde{\nu}_V \in \Omega^{r-1}_q$ such that we have in the associated chart (V^r,ψ^r) the equality $\nu = h\tilde{\nu}_V$.*

Proof. According to (2.1) we have in the associated chart (V^r,ψ^r):

$$\nu = A_{i_1,\ldots,i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}.$$

Then one obtains by direct computation that:

$$hd\nu = \left[\left(d_{i_0} A_{i_1,\ldots,i_q}\right) + \sum_{|I|=r} \left(\partial^I_\sigma A_{i_1,\ldots,i_q}\right) y^\sigma_{Ii_0}\right] dx^{i_0} \wedge \cdots \wedge dx^{i_q}.$$

(here d_j is the formal derivative on V^r).

This form is $\pi^{r+1,r}$-projectable *iff* the square bracket does not depend on $y^\sigma_I \quad (|I| = r+1)$, i.e.

$$\mathcal{S}^-_{i_0,\dots,i_q}\mathcal{S}^+_{Ip}\delta^p_{i_0}\left(\partial^I_\sigma A_{i_1,\dots,i_q}\right) = 0\,. \tag{3.10}$$

After some work (using some ideas from [3]) one obtains

$$A_{i_1,\dots,i_q} = \sum_{s=0}^{q}\frac{1}{s!(q-s)!}\sum_{|J_1|,\dots,|J_s|=r-1}\mathcal{S}^-_{i_1,\dots,i_q}A^{J_1,\dots,J_s}_{\sigma_1,\dots,\sigma_s,i_1,\dots,i_q}y^{\sigma_1}_{J_1i_1}\cdots y^{\sigma_s}_{J_si_s} \tag{3.11}$$

and we have the formula from the statement with:

$$\tilde{\nu}_V \equiv \sum_{s=0}^{q}\frac{1}{s!(q-s)!}\sum_{|J_1|,\dots,|J_s|=r-1}A^{J_1,\dots,J_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_q}$$
$$dy^{\sigma_1}_{J_1}\wedge\cdots\wedge dy^{\sigma_s}_{J_s}\wedge dx^{i_{s+1}}\wedge\cdots\wedge dx^{i_q}. \tag{3.12}$$ □

Now we are ready to obtain the most general form of a variationally trivial Lagrangian. First we note that we have

Theorem 3.3. *Let $\lambda \in \Omega^r_{n,X}$ be a variationally trivial Lagrange form. Then for every point $j^r_x\gamma \in J^rY$ there exists a neighborhood V of $\gamma(x) \in Y$ and a n-form ρ_V defined in the chart (V^{r-1},ψ^{r-1}) such that we have: (1) $\lambda = h\rho_V$ in the chart (V^r,ψ^r); (2) $d\rho_V = 0$. Conversely, if such a local form ρ_V exists, then the form λ is variationally trivial.*

One can now obtain the most general form of a variationally trivial local Lagrangian.

Theorem 3.4. *Any variationally trivial local Lagrangian of order r has the following form in the chart (V^r,ψ^r):*

$$L = \sum_{s=0}^{n}\frac{1}{s!(n-s)!}\sum_{|I_1|,\dots,|I_s|=r-1}\mathcal{L}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_n}\mathcal{J}^{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_n}_{I_1,\dots,I_s}\,. \tag{3.13}$$

Here we have defined (see also [4] *and* [10]*):*

$$\mathcal{J}^{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_n}_{I_1,\dots,I_s} \equiv \varepsilon^{i_1,\dots,i_n}\prod_{l=1}^{s}y^{\sigma_l}_{I_li_l} \quad (s=0,\dots,n) \tag{3.14}$$

and the function $\mathcal{L}^{\cdots}_{\cdots}$ are given by

$$\mathcal{L}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s i_{s+1},\dots,i_n} \equiv \sum_{k=1}^{s}(-1)^{k-1}\partial^{I_k}_{\sigma_k}A^{I_1,\dots,\hat{I_k},\dots,I_s}_{\sigma_1,\dots,\hat{\sigma_k},\dots,\sigma_s,i_{s+1},\dots,i_n} +$$
$$\sum_{k=s+1}^{q}(-1)^{k-1}d'_{i_k}A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,\hat{i_k},\dots,i_n}\,; \tag{3.15}$$

the expressions $A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s i_{s+1},\dots,i_{n-1}}$ are smooth functions on V^{r-1} verifying the symmetry property (2.3) and $d'_j = d^{r-1}_j$ is the corresponding formal derivative on V^{r-1}.

Proof. It is convenient to introduce the following forms:

$$\mathcal{A} \equiv \sum_{s=0}^{n-1} \frac{1}{s!(n-1-s)!} \sum_{|I_1|,\dots,|I_s|=r-1} A^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_{n-1}} dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_{n-1}}$$

and

$$\mathcal{F} \equiv \sum_{s=0}^{n} \frac{1}{s!(n-s)!} \sum_{|I_1|,\dots,|I_s|=r-1} \mathcal{L}^{I_1,\dots,I_s}_{\sigma_1,\dots,\sigma_s,i_{s+1},\dots,i_n} dy^{\sigma_1}_{I_1} \wedge \cdots \wedge dy^{\sigma_s}_{I_s} \wedge dx^{i_{s+1}} \wedge \cdots \wedge dx^{i_n}.$$

Then one finds out by direct computation that

$$\mathcal{F} = d\mathcal{A} + \text{contact terms}\,.$$

Next, one takes in theorem 3.3 $\tilde{\nu}_V = \mathcal{A}$ and it follows that $\rho_V = \mathcal{F} +$ contact terms. Finally, by direct computation one discovers that $\lambda = h\rho_V$ has the expression (3.13). □

This theorem generalizes some results obtained in particular cases (see [9] for the references.)

One can analyze in the same way the generic form of a locally variational differential equation but the results are not of the *if and only if* type as above.

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