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# VARIATIONAL SEQUENCES AND VARIATIONAL BICOMPLEXES

DEMETER KRUPKA

ABSTRACT. We compare basic features of the variational sequence, and the variational bicomplex theories on a fibered manifold. We show that these two theories, based on different underlying geometric structures, give different results.

## 1. Introduction

The idea that there should exist a close relationship between the exterior derivative operator on one side, and the Euler-Lagrange expressions on the other side, first appeared in the works of Lepage and Dedecker. Later, this idea led to introducing Lepage forms in field theory (Krupka [10], [12]). Let $\pi : Y \to X$ be a fibered manifold over an $n$-dimensional base $X$, and let $J^rY$ be the $r$-jet prolongation of $Y$. The concept of a Lepage form allowed to construct a commutative diagram

$$\begin{array}{ccc} Lep & \xrightarrow{d} & \Omega^{2r-1}_{n+1}Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \tilde{h}} \\ Lagr & \xrightarrow{E} & \Omega^{2r}_{n+1}Y \end{array}$$

in which $Lagr$ stands for lagrangians of order $r$ on $Y$ (i.e., for *horizontal* $n$-forms on $J^rY$), $Lep$ denotes *Lepage equivalents* of lagrangians of order $r, \Omega^{2r-1}_{n+1}Y$ (resp. $\Omega^{2r}_{n+1}Y$) denotes the module of $(n+1)$-forms on $J^{2r-1}Y$ (resp. $J^{2r}Y$), and $h$ and $\tilde{h}$ are *horizontalizations* of differential forms, induced by the fibered structure of $Y$; $E$ is the *Euler-Lagrange mapping*, assigning to a lagrangian the corresponding Euler-Lagrange form.

To the end of the seventieth, Anderson and Duchamp [2], Dedecker and Tulczyjew [4], Takens [18], Tulczyjew [19], and Vinogradov [20], observed, on the basis of

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different assumptions, that one can include the Euler-Lagrange operator into an exact sequence, constructed by means of a variational bicomplex. To this purpose, they used an invariant decomposition of the exterior derivative operator on the infinite prolongation $J^\infty Y$ into its horizontal and contact components.

A different modification of similar ideas was used by Krupka [11], [13], who considered the De Rham sequence of differential forms on $J^rY$, with *$r$ fixed*, and a subsequence of the De Rham sequence, such that the corresponding quotient sequence included the Euler-Lagrange mapping as one of its quotient mappings.

Pommaret [15] and Vitolo [21] compared and discussed the relationship of these two theories.

In this short note, we briefly recall basic concepts of the variational sequence theory, and on the differences between this theory and the variational bicomplex theory. For the variational bicomplex theory, the reader is referred to [1], [2], [3], [4], [6], [8], [16], [18], [19], [20].

## 2. The variational sequence

Let $\pi : Y \to X$ be a fibered manifold, $n = \dim X$, $n + m = \dim Y$. We denote by $J^rY$ the *$r$-jet prolongation* of $Y$, and by $\pi^r : J^rY \to X$, $\pi^{r,s} : J^rY \to J^sY$ the canonical jet projections. $J^r\gamma : X \to J^rY$ denotes the *$r$-jet prolongation* of a section $\gamma : X \to Y$.

An $n$-form $\lambda$ on $J^rY$, *horizontal* with respect to the projection $\pi^r$, is called a *lagrangian* of order $r$ for $Y$. If $(V, \psi)$, $\psi = (x^i, y^\sigma)$, is a fibered chart on $Y$, and $(V^r, \psi^r)$, $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \dots, y^\sigma_{j_1j_2\dots j_r})$, is the associated chart on $J^rY$, then

$$\lambda = \mathcal{L}(x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1j_2}, \dots, y^\sigma_{j_1j_2\dots j_r})\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n.$$

Given $\lambda$, we get for any compact, $n$-dimensional submanifold $\Omega \subset X$ with boundary, a real-valued function

$$\gamma \to \int_\Omega J^r\gamma^*\lambda \tag{1}$$

defined on sections of $Y$ (the *action function* associated with $\lambda$ and $\Omega$). The *first variation formula* for (1) can be derived in an intrinsic way by means of the Lepage forms, the Lie derivative, and the exterior derivative $d$. This involves introducing a global *Euler-Lagrange form* associated with $\lambda$,

$$E_\lambda = E_\sigma(\mathcal{L})\, dy^\sigma \wedge dx^1 \wedge dx^2 \wedge \dots \wedge dx^n,$$

where

$$E_\sigma(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial y^\sigma} + \sum_{k=1}^{r} (-1)^k d_{i_1} d_{i_2} \dots d_{i_k} \frac{\partial \mathcal{L}}{\partial y^\sigma_{i_1 i_2 \dots i_k}}$$

are the *Euler-Lagrange expressions*. The mapping $\lambda \to E_\lambda$ is called the *Euler-Lagrange mapping*. The *kernel* of this mapping describes *trivial*, or *null* lagrangians, its image are *variational $(n+1)$-forms*.

If $(V,\psi)$, $\psi = (x^i, y^\sigma)$, is a fibered chart on $Y$, we define

$$\omega^\sigma_{j_1 j_2 \dots j_k} = dy^\sigma_{j_1 j_2 \dots j_k} - y^\sigma_{j_1 j_2 \dots j_k i} dx^i, \quad 1 \le k \le r. \tag{2}$$

These forms define a basis of 1-forms on $J^{r+1}Y$, $dx^i$, $\omega^\sigma$, $\omega^\sigma_{j_1}, \dots, \omega^\sigma_{j_1 j_2 \dots j_r}$, $dy^\sigma_{j_1 j_2 \dots j_{r+1}}$. If $\rho$ is any $k$-form on $J^rY$, the forms (2) define a decomposition

$$(\pi^{r+1,r})^* \rho = h\rho + p_1\rho + p_2\rho + \dots + p_k\rho$$

as follows: We take for the chart expression of $h\rho$, the terms in $(\pi^{r+1,r})^*\rho$ which contain no factor (2), and for $p_k\rho$ the terms containing exactly $k$ factors (2). Then $h$ is a homomorphism of exterior algebras called the *horizontalization*. We say that $\rho$ is *contact* if $h\rho = 0$. If $k \ge n+1$, $\rho$ is always contact; we say that $\rho$ is *strongly contact* if $p_{k-n}\rho = 0$. In a fibered chart,

$$hf = (\pi^{r+1,r})^* f, \quad hdx^i = dx^i, \quad hdy^\sigma_{j_1 j_2 \dots j_k} = y^\sigma_{j_1 j_2 \dots j_k i} dx^i.$$

Now we introduce some sheaves of forms on $J^rY$. Let $\Omega^r_k$ be the sheaf of smooth $k$-forms over $J^rY$. If $1 \le k \le n$ (resp. $k \ge n+1$), denote by $\Omega^r_{k,c}$, the subsheaf of contact (resp. strongly contact) forms ($\Omega^r_{0,c} = \{0\}$). Define

$$\Theta^r_k = \Omega^r_{k,c} + d\Omega^r_{k-1,c},$$

where $k \ge 1$, and $d\Omega^r_{k-1,c}$ is the *image sheaf* of $\Omega^r_{k-1,c}$ by $d$. We get a subsequence $0 \to \Theta^r_1 \to \Theta^r_2 \to \Theta^r_3 \to \cdots$ of the DeRham sequence

$$0 \to \mathbf{R}_Y \to \Omega^r_1 \to \Omega^r_2 \to \Omega^r_3 \to \cdots$$

and the quotient sequence

$$0 \to \mathbf{R}_Y \to \Omega^r_1/\Theta^r_1 \to \Omega^r_2/\Theta^r_2 \to \Omega^r_3/\Theta^r_3 \to \cdots \tag{3}$$

which is called the *variational sequence*. The *quotient mapping*, denoted by $E$, is defined by $E([\rho]) = [d\rho]$, where $[\rho]$ is the class of a form $\rho$. It can be shown that the variational sequence is an acyclic resolution of the constant sheaf $\mathbf{R}_Y$. Consequently, by the abstract DeRham theorem, the complex of global sections of the variational sequence has the same cohomology as the manifold $Y$. Denoting the sequence (3) by $0 \to \mathbf{R}_Y \to \mathcal{V}^r$, and denoting the corresponding cochain complex of global sections by

$$0 \to \Gamma(Y, \mathbf{R}_Y) \to \Gamma(Y, \Omega^r_0) \to \Gamma(Y, \Omega^r_1/\Theta^r_1) \to \Gamma(Y, \Omega^r_2/\Theta^r_2) \to \cdots$$

or simply by $\Gamma(Y, \mathcal{V}^r)$, we have $H^k(\Gamma(\mathbf{R}_Y, \mathcal{V}^r)) = H^k(Y, \mathbf{R})$.

It should be pointed out that each term in the variational sequence, which is, as a quotient, formed by *classes* of forms, is determined up to an isomorphism. This means, however, that we may ask whether there exists a representation of a term by means of a module of *forms*. A basic observation connecting the variational sequence with the calculus of variations, comes from an analysis of the $(n-1), n, (n+1)$- terms in (3) (the *variational terms*). Describing the sheaf $\Omega^r_n/\Theta^r_n$ (resp. $\Omega^r_{n+1}/\Theta^r_{n+1}$) as a certain subsheaf of the sheaf of forms $\Omega^{r+1}_k$ (resp. $\Omega^{2r-1}_k$), one can easily see that the corresponding representation of the qoutient mapping $E : \Omega^r_n/\Theta^r_n \to \Omega^r_{n+1}/\Theta^r_{n+1}$ coincides with the Euler-Lagrange mapping $\lambda \to E_\lambda$.

Representations of variational sequences by forms in higher order mechanics and in the first order field theory, as well as different computational techniques, are discussed by Kašparová [7], Krbek and Musilová [9], Musilová [14], Francaviglia, Palese and Vitolo [5], and Štefánek [17].

## 3. The variational bicomplex

Now we briefly recall some concepts of the variational bicomplex theory. There are several approaches to the subject; for the discussion in this paper, we choose [16]. As before, $\pi : Y \to X$ denotes a fibered manifold, $n = \dim X$, $n+m = \dim Y$.

The *infinite* jet prolongation of $Y$, needed in the variational bicomplex theory, is denoted by $J^\infty Y$; $J^\infty Y$ is introduced by various authors in slightly different ways, with different (topological, algebraic, smooth) structures understood on the *set* $J^\infty Y$ (compare with Dedecker and Tulczyjew, Saunders, Takens, Vinogradov). The module of $(p+q)$-forms on $J^\infty Y$, constructed by the inverse limit operation with respect to the jet projections $\pi^{\infty,r} : J^\infty Y \to J^r Y$ can canonically be decomposed into the direct sum of submodules containing $p$ *contact*, and $q$ *horizontal* factors. These submodules are denoted by $\Phi^p_q$.

Correspondingly, the exterior derivative $d$ admits a decomposition into its horizontal and vertical components as well. We write for this decomposition

$$d = d_h + d_v.$$

We have the following commutative diagram with *locally exact* rows and columns:

$$\begin{array}{cccccccccccc}
 & & 0 & & 0 & & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & & \\
 & & \mathbf{R} & & \mathbf{R} & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \Lambda^0 X & \longrightarrow & \Phi^0_0 & \xrightarrow{d_\nu} & \Phi^0_1 & \cdots & \Phi^0_s & \xrightarrow{d_\nu} & \Phi^0_{s+1} & \cdots \\
 & & \downarrow & & {\scriptstyle d_h}\downarrow & & {\scriptstyle -d_h}\downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \Lambda^1 X & \longrightarrow & \Phi^1_0 & \xrightarrow{d_\nu} & \Phi^1_1 & \cdots & \Phi^1_s & \xrightarrow{d_\nu} & \Phi^1_{s+1} & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
0 & \longrightarrow & \Lambda^{n-1} X & \longrightarrow & \Phi^{n-1}_0 & \xrightarrow{d_\nu} & \Phi^{n-1}_1 & \cdots & \Phi^{n-1}_s & \xrightarrow{d_\nu} & \Phi^{n-1}_{s+1} & \cdots \\
 & & \downarrow & & {\scriptstyle d_h}\downarrow & & {\scriptstyle -d_h}\downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \Lambda^n X & \longrightarrow & \Phi^n_0 & \xrightarrow{d_\nu} & \Phi^n_1 & \cdots & \Phi^n_s & \xrightarrow{d_\nu} & \Phi^n_{s+1} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & \longrightarrow & \Xi_0 & \longrightarrow & \Xi_1 & \cdots & \Xi_s & \longrightarrow & \Xi_{s+1} & \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

In this diagram, the first two horizontal arrows in each row are natural inclusions, and the firs column consists of exterior derivatives. $\Xi_k$ stands for the corresponding quotient space, and the mappings in the last row stand for the quotient morphisms. In order to ensure commutativity, the horizontal component $d_h$ of $d$ is taken with the minus sign in each odd column. The *Euler-Lagrange mapping* in this context is the mapping $E : \Xi_n \to \Xi_{n+1}$.

## 4. Remarks

The reader can easily see that the theory of finite order variational sequences *differs* from the variational bicomplex theory in many respects. Now we discuss some points where differences appear.

Differences in *formulation* of the theories are apparent, and consist in the choice of basic structures serving as domains of definition of the exterior derivative operator $d$. Both theories are based on an analysis of $d$. In the first case, $d$ acts on the modules $\Omega^r_p W$ of differential forms on jet prolongations $J^r Y$, while the basic geometric structures for the variational bicomplex are $J^\infty Y$, and the corresponding spaces $\Phi^p_q$ of forms on $J^\infty Y$, constructed by the inverse limit operation.

The *results* differ in several points.

(a) The main difference is that the finite order theory keeps the order of all sequence operations fixed. This gives us a deeper understanding of the role of the *order* and of the *analytic structure* of basic variational notions: order increasing in operators as the Euler-Lagrange operator appears due to a *different representation* of the same class of forms.

Namely, if $[\rho]$ is an element of $\Omega^r_k/\Theta^r_k$ in (3), then its image $E([\rho])$ is again of order $r$.

As we have pointed out, classes of forms in the quotient $\Omega^r_k/\Theta^r_k$ may also be expressed in equivalent ways, in particular, as differential forms defined, formally, on jet prolongations of higher order than $r$. For the discussion of basic ideas, as well as for concrete formulas, we refer to the simplest cases $n = 1$, $r = 1$ (*first order mechanics* [9], [13], [17]), and $n = 1$, $r$ arbitrary (*first order field theory* [7]).

(b) Classes of forms in (3) have a certain *algebraic structure*, which determines the meaning of the cohomology of the sequence of global sections associated with (3): the cohomology concerns *structured* objects. For example, if $\varepsilon \in \Omega^r_{n+1}/\Theta^r_{n+1}$ is a locally variational form, and $H^{n+1}Y = 0$, then there exists a lagrangian $\lambda$ for $\varepsilon$ *in* $\Omega^r_n/\Theta^r_n$, i.e., a lagrangian of a *specific* form defined by the structure of $\Omega^r_n/\Theta^r_n$. Therefore, the cohomology conditions have a different meaning than in the bicomplex theory, where only existence of a lagrangian for $\varepsilon$ is guaranteed, without specifying its *order*, eventually its further properties.

(c) Now assume that $n = 1$, $r = 1$; this choice refers to the (first order) mechanics. Recall that a form $\alpha \in \Omega^1_2 Y$ is called *locally variational*, if $d\alpha \in \Theta^1_3 Y$ [13].

To a locally variational form $\alpha \in \Omega^1_2 Y$ one can associate two distributions. The *associated distribution* of $\alpha$, denoted by $\mathcal{D}_\alpha$, is generated by 1-forms $i_\xi \alpha$ with $\xi$ running through all vector fields on $J^1 Y$. The distribution, locally generated by 1-forms $i_\xi \alpha$ with $\xi$ running through all $\pi^1$-*vertical* vector fields, is called the *Euler-Lagrange distribution* associated with $\alpha$, and is denoted by $\mathcal{E}_\alpha$. $\mathcal{D}_\alpha$ is a subdistribution of $\mathcal{E}_\alpha$.

Integral sections of these distributions have a close relation to the solutions of the Euler-Lagrange equations. In particular, *holonomic* integral sections of the Euler-Lagrange distribution $\mathcal{E}_\alpha$ coincide with the solutions of the Euler-Lagrange equations defined by $\alpha \in \Omega_2^1 Y$.

(d) Since the lagrangians and the Euler-Lagrange forms arising on a fixed order jet bundles are *a priory* of a certain algebraic structure, differences also appear in the form of the theorems giving necessary and sufficient local variationality conditions.

As mentioned in (b), fixing order in the variational sequence requires appearance of classes of forms, represented by forms of a certain algebraic structure. These structured expressions imply that the *variational triviality conditions*, and the *Helmholtz-Sonin variationality conditions*, have a different form, and a different meaning in the variational sequence, and the variational bicomplex theories.

(e) An important difference consists in possibilities of a generalization of the variational sequence theory. The idea of constructing the Euler-Lagrange mapping as a quotient mapping, can be generalized to *any* subsequence of the DeRham sequence.

(f) Another difference consists in the fact that no *horizontal* Poincaré lemma is needed in the variational sequence theory. In fact, the need of the horizontal Poincaré lemma in the bicomplex theory is replaced by the exactness of the variational sequence.

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Department of Mathematics , Silesian University at Opava
74601 Opava, CZECH REPUBLIC
*E-mail*: demeter.krupka@slu.fpf.cz