DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 437–442

TRANSSYMMETRIC SPACES. NEW RESULTS

LARISSA V. SBITNEVA

Abstract. In this report some new results on perfect transsymmetric spaces are presented.

0. Introduction. The notion of transsymmetric space has been defined in [1] as a natural generalization of the notion of symmetric space of E. Cartan. Being a reductive space of a special kind such a space may be studied within the frame of homogeneous space theory. As well, due to R. Baer–L. Sabinin construction [2], [3], it may be treated within the frame of smooth loop theory.

It has been observed that symmetric spaces satisfy some quite remarkable property which is not valid in general transsymmetric spaces. This motivates the exploration of transsymmetric spaces with such a property (perfect ts-spaces).

In this report we announce some new results on perfect ts-spaces.

1. Preliminaries.

Definition 1.1. Let M be a smooth manifold and $(\sigma_x)_{x\in M}$ the family of local diffeomorphisms defined near $x \in M$ such that $ex = \sigma_x^{-1}x$ is defined.

We say that σ_x is a *transsymmetry* (ts-symmetry) at a point $x \in M$ and $(M, (\sigma_x)_{x\in M})$ is *a transsymmetric structure (manifold)*, briefly *ts-structure (manifold)*.

We introduce

$$x \cdot y \overset{\text{def}}{=} \sigma_x\, y\,, \qquad \rho_y\, x \overset{\text{def}}{=} x \cdot y\,. \tag{1}$$

Definition 1.2. A transsymmetric structure is correct if, for any x, ρ_{ex} is a local diffeomorphism near $x \in M$.

1991 *Mathematics Subject Classification.* 20N05, 53C35.
Key words and phrases. Transsymmetric spaces (ts-spaces), symmetric spaces, perfect s-spaces.
This paper is in final form and no version of it will be submitted for publication elsewhere.

Definition 1.3. A correct ts-structure $(M,(\sigma_x)_{x\in M})$ with the property

$$\sigma_x\circ\sigma_y=\sigma_{\sigma_x y}\circ\sigma_{ex}\,, \tag{2}$$

or, in another notation,

$$x\cdot(y\cdot z)=(x\cdot y)\cdot(ex\cdot z) \qquad \text{(Left F-property)} \tag{3}$$

is called *a transsymetric space (ts-space).*

Remark 1.4. One may treat a ts-space $(M,(\sigma_x)_{x\in M})$ as a smooth partial groupoid (magma) $\langle M,\cdot\rangle$. Moreover, since $a\cdot x=b$ is (locally) solvable ($x=\sigma_a^{-1}b$), a ts-space may be treated as a partial smooth left quasigroup $\langle M,\,\cdot\,,\backslash\rangle$ (where $a\backslash b\overset{\text{def}}{=}\sigma_a^{-1}b$). Henceforth we consider ts-spaces only.

Remark 1.5. If $\sigma_x\,x=x$ $(\forall\,x\in M)$ then a ts-space is called a *generalized symmetric space* (O. Kowalski [13]). In this case it is evident that $ex=x$. If, moreover, $\sigma_x\circ\sigma_x=\mathrm{id}$ then a generalized symmetric space is called *symmetric* (E. Cartan).

1.6. Let $(M,(\sigma_x)_{x\in M})=\langle M,\cdot\rangle$ be a ts-space. One can introduce

$$P(x,z,y)=P_x^z\,y=x\underset{z}{\times}y=\rho_{ez}^{-1}\,x\,\cdot\,\sigma_z^{-1}\,y\,. \tag{4}$$

Then $x\underset{z}{\times}z=x$, $z\underset{z}{\times}y=y$, and $a\underset{z}{\times}y=b$, $x\underset{z}{\times}a=b$ are uniquely solvable (locally) which means that $\langle M,\underset{z}{\times},z\rangle$ is a local smooth loop with two-sided neutral element z .

In such a way we have obtained a so-called canonical loopuscular covering (structure) of a ts-space $(M,(\sigma_x)_{x\in M})$.

One may regard now a ts-space $\langle M,\cdot\rangle$ as a smooth partial algebra $\langle M,P\rangle$ with the ternary operation P.

1.7. The formula

$$\nabla_{X_a}Y=\{\,\frac{d}{dt}\,[\,(\,P^a_{\gamma(t)}\,)^{-1}_{*,\,a}\,Y_{\gamma(t)}\,]\,\}_{t=0}\,, \tag{5}$$

Y being a vector field, $\gamma(0)=a$, $\dot\gamma(0)=X_a$, introduces an affine connection into ts-spaces $(M,(\sigma_x)_{x\in M})$ (see [5], [6]), which is called *the canonical affine connection of a transsymmetric space* $(M,(\sigma_x)_{x\in M})$. It is known that the canonical connection is reductive ($\nabla T=0$, $\nabla R=0$) . See [4].

1.8. Any affine connection ∇ uniquely defines the system of geodesic odules centered at every point $z\in M$ (see [5], [6]) with local smooth operations $x\underset{z}{*}y=L_x^z\,y=L(x,z,y)$ and $t_z\,y=\omega_t(z,y)$, where $L_x^z\,y$ means the parallel displacement of the geodesic arc $\overset{\smile}{zy}$ along the geodesic arc $\overset{\smile}{zx}$.

Thus (M, ∇) may be considered as a smooth algebraic system $\langle M, L, (\omega_t)_{t\,\in \mathbf{R}}\rangle$, a geoodular manifold with characteristic geoodular identities. See [5], [6].

The affine connection may be restored from its geoodular manifold in a unique way (see [5], [6]).

For a reductive affine connection ∇ the characteristic properties

$$t_z\, x \underset{z}{*} (u_z\, x \underset{z}{*} y) = (t+u)_z\, x \underset{z}{*} y \qquad \text{(Left monoalternativity)},$$

$$L_b^a\,(x \underset{z}{*} y) = L_b^a\, x \underset{L_b^a z}{*} L_b^a\, y \qquad \text{(Left reductivity)}$$

are valid. See [5], [6].

2. Perfect ts-spaces.

In this part we announce a number of new results (without proofs).

Definition 2.1. A ts-space $(M, (\sigma_x)_{x\,\in M})$ is called *perfect* if $\sigma_x \circ \sigma_y^{-1}$ induces a parallel displacement along the geodesic arc $\{t_y\, \sigma_x\, ey\}_{t\,\in[\,0\,,\,1\,]}$ joining y and $\sigma_x\, ey$.

Proposition 2.2. *Let $(M, (\sigma_x)_{x\,\in M})$ be a perfect ts-space. Then*

$$\sigma_{\rho_{ez}^{-1} y} \circ \sigma_z^{-1} = L_y^z\,, \tag{6}$$

where $L_y^z\, w = y \underset{z}{*} w$ is the composition in geodesic loop at a point $z \in M$, $\varphi_z = \sigma_z \circ e \in \operatorname{End} \langle Q, \underset{z}{*}, z\rangle$.

Corollary 2.3. *If $(M, (\sigma_x)_{x\,\in M})$ is a perfect ts-space, then its canonical loopuscular structure coincides with its tangent geoodular structure and, consequently, is leftmonoalternative.*

Conversely, if a canonical loopuscular structure is leftmonoalternative (after adding the canonical unary operations), see [9]*, then $(M, (\sigma_x)_{x\,\in M})$ is a perfect ts-space.*

2.4. Since $\langle M, P, (\omega_t)_{t\,\in \mathbf{R}}\rangle$ is reductive, the whole structure is determined by a geodesic odule $\langle M, \underset{\varepsilon}{*}, (t_\varepsilon)_{t\,\in \mathbf{R}}\rangle$ at an arbitrarily fixed point $\varepsilon \in M$.

With the help of (3) we may obtain the characteristic identities of such an odule (see [7], [8]) in the form (for the sake of simplicity we set $x * y$, tx instead of $x \underset{\varepsilon}{*} y$, $t_\varepsilon\, y$, etc.):

$$x * [\, y * (\varphi x \backslash z)\,] = [\, x * (\, y * [\varphi x]^{-1})\,] * z\,, \quad (\varphi = \varphi_\varepsilon = \sigma_\varepsilon \circ e)\,, \tag{7}$$

$$\varphi(x * y) = \varphi x * \varphi y\,. \tag{8}$$

(7) is the so called *M-identity.* See [10], [11].

Due to the leftmonoalternativity we get also

$$x * [\,(y * [\varphi x]^{-1}) * z\,] = [\, x * (\, y * [\varphi x]^{-1})\,] * z\,, \quad \text{(Left half Bol identity)}. \tag{9}$$

Note that due to invertibility of ρ_{ez} we get (differentiating $\gamma(t) * e(\gamma(t)) = \gamma(t)$ by t at $t = 0$, $\gamma(0) = \varepsilon$) that $(\varphi_\varepsilon)_{*,\,\varepsilon} - \mathrm{Id}_{\,\varepsilon} = (\rho_{e\varepsilon})_{*,\,\varepsilon}$ is invertible.

We may present (9) equivalently as

$$L_x \circ L_y \circ L_{(\varphi x)^{-1}} = L_{x * (y * (\varphi x)^{-1})}\,, \quad (L_x\, y \stackrel{\mathrm{def}}{=} x * y)\,. \tag{10}$$

Combining the leftmonoalternativity with (10) we get further

$$L_{\lambda x} \circ L_w \circ L_{\lambda x} = L_{\lambda x * (w * \lambda x)}, \quad (\forall\, w)\,, \tag{11}$$

where $\lambda x = x * (\varphi x)^{-1}$.

But $\lambda\colon x \mapsto \lambda x = x * (\varphi x)^{-1}$ is locally invertible near $\varepsilon \in M$, since $(\lambda)_{*,\,\varepsilon} = \mathrm{Id}_{\,\varepsilon} - \varphi_{*,\,\varepsilon}$ (which is invertible).

Thus, near ε

$$L_y \circ L_w \circ L_y = L_{y * (w * y)}\,, \quad (\forall\, y, w) \qquad \text{(Left Bol Property)}. \tag{12}$$

As a result we obtain

Proposition 2.5. *Any geodesic loop of a perfect ts-space is a left Bol loop.*

Proposition 2.6. *For any geodesic loop of a perfect ts-space,* $(\varphi x)^{-1} * x^{-1} \in N_l = \{w;\ w * (a * b) = (w * a) * b\,,\ \forall\, a, b\}$*—the left nucleus of* $\langle M, *, \varepsilon\rangle$.

Remark 2.7. It is known that for a left Bol loop the left nucleus is a normal subloop. See [10], [11].

Proposition 2.8. *If for a loop*

$$L_x \circ L_y \circ L_x = L_{x * (y * x)}\,, \quad \textit{(Left Bol property)},$$

and

$$L_{((\varphi x)^{-1} * x^{-1})} \circ L_z = L_{((\varphi x)^{-1} * x^{-1}) * z}\,,$$

then

$$L_{(\varphi x)^{-1}} \circ L_y \circ L_x = L_{(\varphi x)^{-1} * (y * x)}\,.$$

As a result, due to the above considerations, we have reduced the problem to investigate a perfect ts-space to the exploration of a smooth local loop $\langle M, *\,, \varepsilon\rangle$ with certain properties. We summarize it in the following proposition.

2.9. Proposition. *Any geodesic loop* $\langle M, *, \varepsilon\rangle$ *of a perfect ts-space* $(M, (\sigma_x)_{x\in M})$ *satisfies the identities*

(1) $L_x \circ L_y \circ L_x = L_{x * (y * x)}$ *(Left Bol property),*
(which implies, as known, leftmonoalternativity, see [5], [6]*),*

(2) $L_{\varphi x * x} \circ L_y = L_{(\varphi x * x) * y}$,
where $\sigma_\varepsilon \circ e = \varphi \in \mathrm{End}\,\langle M, *\,, \varepsilon\rangle$, $\varphi_{*,\,\varepsilon} - \mathrm{id}_{*,\,\varepsilon}$ *is invertible (meaning that* $\varphi x * x \in N_l$ *(left nucleus) which is a normal subloop; moreover it is a group),*

(3) $l\,(a\,, b) = L^{-1}_{a * b} \circ L_a \circ L_b \in \mathrm{Aut}\,\langle M, *\,, \varepsilon\rangle\,,$ *(Left A-property).*
(This last property is due to the reductivity of $(M, (\sigma_x)_{x \in M})$, *see* [1].*)*

Proposition 2.10. *If $(M, (\sigma_x)_{x \in M})$ is a perfect transsymmetric space, then it is reductive, $M = G/H$ (at least locally). Moreover if $\mathfrak{g}$, $\mathfrak{h}$ are the Lie algebras for G and H and $\mathfrak{m}$ is a reductive complement to $\mathfrak{h}$ in $\mathfrak{g}$ $(\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}\,,\ [\mathfrak{m}\,, \mathfrak{h}] \subset \mathfrak{m})$, then $\mathfrak{m}$ contains an ideal $\mathfrak{n}$ of $\mathfrak{g}$ such that $(\mathfrak{g}/\mathfrak{n})/(\mathfrak{h}/\mathfrak{n})$ is a symmetric pair.*

Proposition 2.11. *With notations and conditions as above, if*

$$\tilde{\varphi} = \begin{cases} \varphi_{*,\varepsilon} & \text{on } \mathfrak{m}\,, \\ \mathrm{Id} & \text{on } \mathfrak{h}\,, \end{cases}$$

then $\tilde{\varphi} \in \operatorname{End} \mathfrak{g}$, $\tilde{\varphi}\mathfrak{m} = \mathfrak{m}$, $\tilde{\varphi}\zeta = \zeta \Longleftrightarrow \zeta \in \mathfrak{h}$, $\tilde{\varphi}\mathfrak{n} \subset \mathfrak{n}$. Moreover $[\tilde{\varphi}\zeta, \zeta] \subset \mathfrak{n}$ for any $\zeta \in \mathfrak{m}$.

Finally, we get the following infinitesimal description of a perfect transsymmetric space.

Proposition 2.12. *Any perfect transsymmetric space M is reductive, that is $M = G/H$ (at least locally).*

If $(\mathfrak{g}\,, \mathfrak{h})$ are the Lie algebras for G and H, respectively, and $\mathfrak{m}$ is a reductive complement to $\mathfrak{h}$ in $\mathfrak{g}$ $(\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}\,,\ [\mathfrak{m}\,, \mathfrak{h}] \subset \mathfrak{m})$ then there exist an endomorphism $\tilde{\varphi}$ of $\mathfrak{g}$ such that $\tilde{\varphi}\mathfrak{m} = \mathfrak{m}$, $\tilde{\varphi}\zeta = \zeta \Longleftrightarrow \zeta \in \mathfrak{h}$, and an ideal $\mathfrak{n}$ of $\mathfrak{g}$ such that $\mathfrak{n} \subset \mathfrak{m}$, $(\tilde{\varphi} + \mathrm{Id})\mathfrak{m} \subset \mathfrak{n}$, which means in particular, that $(\mathfrak{g}/\mathfrak{n})/(\mathfrak{h}/\mathfrak{n})$ is a symmetric pair with the defining symmetry $\tilde{\varphi}/\mathfrak{n} = s$.

The above said means that a perfect transsymmetric space is "glued" in a nontrivial way by means of some normal subgroup and a symmetric space.

In particular, if G is simple, then the corresponding perfect transsymmetric space is either a group space, or a symmetric space.

Remark 2.13. The theory of perfect spaces appeared for the first time in [12] for s-spaces (a particular subcase of Transsymmetric spaces).

References

[1] Sabinin, L. V., Sabinina, L. L., *On the geometry of transsymmetric spaces*, Webs and Quasigroups (1990), 64–68. (Russian)

[2] Sabinin, L. V., *On the equivalence of categories of loops and homogeneous spaces*, Soviet. Math. Dokl. **13** no. 4 (1972), 970–974.

[3] Sabinin, L. V., *The geometry of loops*, Mathematical Notes **12** no. 5 (1972), 799-805.

[4] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry* (Interscience-Willey, N.Y., eds.), vol. 1, 1993.

[5] Sabinin, L. V., *Methods of nonassociative Algebra in Differential Geometry*, In Kobayashi S. and Nomizu K., Foundations of Differential Geometry, vol. 1, Nauka press, Moscow, 1981, pp. 293–339. (Russian)

[6] Miheev, P. O., Sabinin, L. V., *Quasigroups and Differential Geometry*, Quasigroups and Loops: Theory and Applications (O. Chein, H. Plugfelder. J.D.H. Smith, eds.), 1990.

[7] Sabinina, L. L., *On the Geometry of Smooth F-Quasigroups*, Ph.D. Dissertation, Friendship of Nations University, Moscow, 1992. (Russian)

[8] Sabinina, L. L., *On geodesic Loops of Transsymmetric spaces*, Algebras, Groups and Geometries **12** (1995), 119–125.

[9] Sabinin, L.V., *Analytic Quasigroups and Geometry*, Monograph, Friendship of Nations University Press, Moscow, 1991. (Russian)
[10] Belousov, V.D., *Foundations of the Theory of Quasigroups and Loops*, Nauka Press, Moscow, 1968. (Russian)
[11] Belousov, V.D., *Elements of Quasigroups Theory*, Kishinev University Press, Kishinev, 1981. (Russian)
[12] Sbitneva, L.V., *Perfect s-spaces*, Ph.D. Dissertation, Friendship of Nations University, Moscow, 1984. (Russian)
[13] Kowalski, O., *Generalized Symmetric Spaces*, Lecture Notes in Mathematics 805, Springer--Verlag, Berlin-Heidelberg-New York, 1980.

Quintana Roo University, Department of Mathematics
77019 Chetumal, Quintana Roo, MEXICO
E-mail: lsbitneva@balam.cuc.uqroo.mx