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THE FLUX HOMOMORPHISM IN THE FOLIATED CASE

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Abstract. By using the cotangent bundle of a foliation and the foliated cohomology we extend some results concerning the flux homomorphism. In analogy to symplectomorphisms one defines the flux for the automorphism group of a regular Poisson manifold. Next the flux extends to the case of 'foliated' loops. In view of the perfectness theorem the transgression homomorphism is an obstacle to further extension.

1. We will work in the category of (regularly) foliated smooth manifolds $(M, \mathcal{F})$ and leaf preserving diffeomorphisms.

Theorem 1 [8]. *There is a contravariant functor from this category to the category of $\mathbb{Z}$-graded associative algebras with continuous homomorphisms which assigns to $(M, \mathcal{F})$ the graded algebra $H^*(\mathcal{F})$. Here $H^k(\mathcal{F})$ denotes the k-th foliated (or tangent) cohomology group.*

Let us recall briefly some facts connected with the foliated cohomology. A *foliated* (or *tangential*) k-form at x_0 is a skew-symmetric k-form on the tangential space $T_{x_0}\mathcal{F}$ at x_0. If a foliated form assumes the form

$$\bar{\omega} = \sum_I a_I(x_1, \dots, x_p, y)\bar{\mathrm{d}}x_{i_1} \wedge \dots \wedge \bar{\mathrm{d}}x_{i_k},$$

where $y = (y_1, \dots, y_q)$, $I = (i_1, \dots, i_k)$, $p = \dim \mathcal{F}$, $q = \operatorname{codim} \mathcal{F}$, then its exterior derivative is

$$\bar{\mathrm{d}}\bar{\omega} = \sum_{I,j} \frac{\partial a_I}{\partial x_j}(x_1, \dots, x_p, y)\bar{\mathrm{d}}x_j \wedge \bar{\mathrm{d}}x_{i_1} \wedge \dots \wedge \bar{\mathrm{d}}x_{i_k}.$$

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More precisely, the complex of foliated forms can be defined as follows. We have the subcomplex $\Omega^*(M,\mathcal{F})$ of the de Rham complex $(\Omega^*(M), \mathrm{d})$: $\omega \in \Omega^r(M,\mathcal{F})$ if and only if $\omega \in \Omega^r(M)$ and $\omega(X_1,\dots,X_r)=0$ for all $X_1,\dots,X_r$ tangent to $\mathcal{F}$. We set

$$\Omega^*(\mathcal{F}) = \Omega^*(M)/\Omega^*(M,\mathcal{F}),$$

and by $\bar\omega \in \Omega^r(\mathcal{F})$ we will denote the class of $\omega \in \Omega^r(M)$.

It is visible that the exterior product $\wedge$ in $\Omega^*(M)$ descends to $\Omega^*(\mathcal{F})$. Next, it is easily checked that $\iota(X)\bar\omega = \overline{\iota(X)\omega}$ and $f^*\bar\omega = \overline{f^*\omega}$ are correct definitions whenever X is tangent to $\mathcal{F}$ and f is leaf preserving. For such an X we have the Lie derivative $\mathcal{L}_X\bar\omega = \iota(X)\bar d\bar\omega + \bar d\iota(X)\bar\omega$.

Finally, if $\bar\omega_1 = \bar\omega_2$ then $\overline{\mathrm{d}\,\omega_1} = \overline{\mathrm{d}\,\omega_2}$. In particular, $\bar\omega = \bar{\mathrm{d}}\bar\lambda$ whenever $\omega = \mathrm{d}\,\lambda$. Consequently, $\bar{\mathrm{d}}$ is a well-defined differential on $\Omega^*(\mathcal{F})$ By $H^*(\mathcal{F})$ we denote the cohomology of $\Omega^*(\mathcal{F})$, and by $[\bar\omega]$ the cohomology class of $\bar\omega$. Clearly $H^r(\mathcal{F})=0$ if $r>p=\dim\mathcal{F}$.

Likewise $H^*_c(\mathcal{F})$ will stand for the foliated cohomology with compact support, that is with $\Omega^*(M)$ replaced by $\Omega^*_c(M)$ and so on.

The foliated cohomology can be also introduced by means of $\mathcal{R}_t$, the sheaf of germs of tangentially locally constant smooth functions. Namely, one has

$$0 \to \mathcal{R}_t \to \underline{\Lambda}^0(T^*\mathcal{F}) \xrightarrow{\bar{\mathrm{d}}} \underline{\Lambda}^1(T^*\mathcal{F}) \to \cdots \to \underline{\Lambda}^p(T^*\mathcal{F}) \to 0,$$

the exact sequence of sheaves. Then $\Omega^k(\mathcal{F})$ is identified with the space of global sections of $\underline{\Lambda}^k(T^*\mathcal{F})$ and we have $H^k(M;\mathcal{R}_t) = H^k(\mathcal{F})$.

Each $H^k(\mathcal{F})$ is a topological vector space, but possibly infinite dimensional (contrary to de Rham), and possibly non-Hausdorff as, in general, the image of $\bar{\mathrm{d}}$ fail to be closed. So one may consider, if necessary,

$$\bar H^k(\mathcal{F}) := \ker(\bar{\mathrm{d}}^k)\,/\,\overline{\mathrm{im}(\bar{\mathrm{d}}^{k-1})},$$

the Hausdorff foliated cohomology. For instance, if $\mathcal{F}$ is a linear foliation with a diophantine irrational slope on the 2-dimensional torus then $H^1(\mathcal{F})$ is infinite dimensional while $\bar H^1(\mathcal{F}) = \mathbb{R}$ (cf. [8]).

Throughout we will assume that $\dim M = n, \dim\mathcal{F} = p$.

2. The concept of the flux homomorphism was initiated by E. Calabi [2] and developed by A. Banyaga and others. It is a basic invariant of diffeomorphism groups, cf. [1], [6].

For any connected and locally connected by arcs topological group G its universal cover group $\tilde G$ is the set of all pairs $(g,\{g_t\})$, where $\{g_t\}$ is the homotopy rel. endpoints class of the path g_t in G such that $g_0=e$ and $g_1=g$, endowed with the pointwise multiplication. By G_0 we denote the component of e in G. Clearly $\tilde G_0/\pi(G_0) = G_0$, where π is the first homotopy group.

Given a foliation $\mathcal{F}$ we will consider the group of all leaf preserving diffeomorphisms with compact support, $\mathrm{Diff}_c(M,\mathcal{F})$. Let ϕ_t be an isotopy in $\mathrm{Diff}_c(M,\mathcal{F})$ ($\phi_0 = id$). Then the associated time-dependent vector field $\dot{\phi}_t := \frac{d}{dt}\phi_t \circ \phi_t^{-1}$ is tangent to $\mathcal{F}$. Let $\bar{\alpha} \in \Omega^{k+1}(\mathcal{F})$ be any $\bar{\mathrm{d}}$-closed $k+1$-form. As in [1] we put

$$I_{\phi_t}(\bar{\alpha}) := \int_0^1 \phi_t^* \iota(\dot{\phi}_t)\bar{\alpha} dt \,.$$

Under the assumption $L_{\dot{\phi}_t}\bar{\alpha} = 0$, $\bar{\mathrm{d}} I_{\phi_t}(\bar{\alpha}) = 0$ and we get the foliated cohomology class

$$\left[I_{\phi_t}(\bar{\alpha})\right] \;=\; \Big[\int_0^1 \iota(\dot{\phi}_t)\bar{\alpha} dt\Big] \;\in H_c^k(\mathcal{F})\,.$$

Theorem 2. *Let $G(M,\mathcal{F},\bar{\alpha}) := \{f \in \mathrm{Diff}_c(M,\mathcal{F}) \,:\, f^*\bar{\alpha} = \bar{\alpha}\}$. There is a continuous homomorphism*

$$\tilde{\mathcal{S}}_{\bar{\alpha}} : \widetilde{G(M,\mathcal{F},\bar{\alpha})_0} \ni \{\phi_t\} \mapsto [I_{\phi_t}(\bar{\alpha})] \in H_c^k(\mathcal{F})$$

called the foliated flux homomorphism associated with $\bar{\alpha}$.

Specifically, if $\bar{\alpha}$ is a 2-form (i.e. $k = 1$) and is 'symplectic' (cf. [3]) then we get the flux in the (regular) Poisson case. The Weinstein chart for $G(M,\mathcal{F},\bar{\alpha})_0$ was constructed in [12].
Proof. For an isotopy $\phi = (\phi_t)$ in $\mathrm{Diff}_c(M,\mathcal{F})_0$ we have

$$\dot{\phi}_t = \delta^r \phi(\partial_t|_t)$$

where δ^r is the right logarithmic derivative, cf. [6]. Therefore for a smooth homotopy $(s,t) \mapsto \psi(s,t)$ with $\psi(i,t) = \phi_t^i$, $i = 0,1$, it follows from the left Maurer-Cartan equality $\mathrm{d}\,\delta^r\psi - \frac{1}{2}[\delta^r\psi, \delta^r\psi] = 0$ that

$$\partial_s \delta^r \psi(\partial_t) = \partial_t \delta^r \psi(\partial_s) + [\delta^r\psi(\partial_s), \delta^r\psi(\partial_t)]\,.$$

Hence by a standard argument

$$\begin{aligned}
&\int_0^1 \iota(\dot{\phi}_t^1)\bar{\alpha} dt - \int_0^1 \iota(\dot{\phi}_t^0)\bar{\alpha} dt \\
&\quad = \int_0^1 \iota(\delta^r\psi(\partial_t|_{(1,t)}))\bar{\alpha} dt \;-\; \int_0^1 \iota(\delta^r\psi(\partial_t|_{(0,t)}))\bar{\alpha} dt \\
&\quad = \bar{\mathrm{d}} \int_0^1\int_0^1 \iota(\delta^r\psi(\partial_s))\big(\iota(\delta^r\psi(\partial_t))\bar{\alpha}\big) ds\, dt\,.
\end{aligned}$$

This means that $\tilde{\mathcal{S}}_{\bar{\alpha}}$ is well-defined. That $\tilde{\mathcal{S}}_{\bar{\alpha}}$ is a homomorphism follows from the fact that for any topological group G the pointwise multiplication of paths and the juxtaposition of paths coincide on the level of homotopy preserving endpoints.

Questions. Denote $\Gamma_{\bar{\alpha}} := \mathrm{im}(\pi(G(M,\mathcal{F},\bar{\alpha})_0))$ where π is the first homotopy group. Our first question is whether $\Gamma_{\bar{\alpha}}$ is countable. In view of some results from R. Palais [10] and by possible extensions of some difficult methods from H. Omori [9] we conjecture the affirmative answer. The second question is whether $\Gamma_{\bar{\alpha}}$ is discrete. Observe that it is still an open problem in the symplectic case, cf. [7].

Another question is when $\tilde{\mathcal{S}}_{\bar{\alpha}}$ admits global sections. This is the case of compact symplectic manifolds by an argument involving the Hodge theory ([6]).

As in the transitive case we have

Proposition 1. *If $\bar\alpha$ is exact, i.e. $\bar{\mathrm d}\bar\beta = \bar\alpha$, then $\tilde{\mathcal S}_{\bar\alpha}(\{\phi_t\}) = [\phi_1^*\bar\beta - \bar\beta]$. In particular, $\Gamma_{\bar\alpha} = 0$.*

We see also that the flux is an obstacle to the fragmentation of isotopies. Namely, if an isotopy ϕ_t in $G(M, \mathcal F, \bar\alpha)$ can be fragmented into isotopies ϕ_t^i supported in contractible U_i then $\{\phi_t\}$ is in the kernel of $\tilde{\mathcal S}_{\bar\alpha}$.

3. Now our purpose is to extend the flux for the group of leaf preserving isotopies (i.e. to get rid of $\bar\alpha$). It will occur that this may be accomplished for foliated loops only.

Let $\pi : E \to M$ be a vector bundle. By $\widehat{\mathrm{Diff}}(E)$ we denote the group of all fiber preserving diffemorphisms f of E such that $\underline{f}$, the induced diffeomorphism of M, is compactly supported. For any global section $s \in \Gamma(E)$ we define $\phi_s \in \widehat{\mathrm{Diff}}(E)$ by

$$\phi_s(e) := (x, v - s(x)) \quad \text{for} \quad e = (x, v) \in E\,,$$

and we have the monomorphism $\Phi : \Gamma(E) \ni s \mapsto \phi_s \in \widehat{\mathrm{Diff}}(E)$.

Now let $(M, \mathcal F)$ be a foliated manifold. We set

$$E^k_{\mathcal F} := \bigcup_{x\in M} \Lambda^k(T^*_x L_x)$$

where L_x is the leaf through x. Clearly $\Gamma(E^k_{\mathcal F}) = \Omega^k(\mathcal F)$. The space $E^k_{\mathcal F}$ possesses a canonical foliated k-form $\bar\lambda^k_{\mathcal F}$ with respect to the foliation $\tau^*\mathcal F$ where $\tau : E^k_{\mathcal F} \to M$ is the canonical projection. Namely, for $U \in (E^k_{\mathcal F})_x = \Lambda^k(T^*_x L_x)$, $V_1, \dots, V_k \in T_U(\tau^*\mathcal F)$ we put

$$\bar\lambda^k_{\mathcal F}(V_1, \dots, V_k) := U(\tau_{L*}V_1, \dots, \tau_{L*}V_k)\,,$$

$\tau_{L_x} : T^*L_x \to L_x$ being the restricted projection. Let $\bar\theta \in \Omega^{k+1}(\mathcal F)$ be any $\bar{\mathrm d}$-closed form. We define

$$\omega^{k+1}_{\bar\theta} := \tau^*\bar\theta + \bar{\mathrm d}\bar\lambda^k_{\mathcal F} \in Z^{k+1}(\tau^*\mathcal F).$$

In particular, $\bar\omega^{k+1} = \bar\omega^{k+1}_0 = \bar{\mathrm d}\bar\lambda^k_{\mathcal F}$ is said to be a canonical $(k+1)$-form. Next we set

$$G(E^k_{\mathcal F}) = G(E^k_{\mathcal F}, \omega^{k+1}_{\bar\theta}) = \{f \in \widehat{\mathrm{Diff}}(E^k_{\mathcal F}) : f^*\omega^{k+1}_{\bar\theta} = \omega^{k+1}_{\bar\theta},\ \underline{f} \in \mathrm{Diff}_c(M, \mathcal F)\}\,.$$

Proposition 2. *Let $\bar\alpha \in \Omega^k(\mathcal F) = \Gamma(E^k_{\mathcal F})$. Then $\bar{\mathrm d}\bar\alpha = 0$ iff $\phi_{\bar\alpha} \in G(E^k_{\mathcal F})$. Moreover, the sequence*

$$(*) \qquad 0 \to Z^k_c(\mathcal F) \xrightarrow{\Phi} G(E^k_{\mathcal F})_0 \xrightarrow{\tau_-} \mathrm{Diff}_c(M, \mathcal F)_0 \to 1\,,$$

where τ_- is induced by τ, is exact.

Indeed, this follows from the equalities (cf. [1])

$$\phi^*_{\bar\alpha}\bar\lambda^k_{\mathcal F} = \bar\lambda^k_{\mathcal F} - \tau^*\bar\alpha$$

and

$$\phi^*_{\bar\alpha}\omega^{k+1}_{\bar\theta} = \omega^{k+1}_{\bar\theta} - \tau^*\bar{\mathrm d}\bar\alpha$$

which are still fulfilled in the foliated case. Therefore, as in the transitive case we get

Corollary 1. $\pi(G(E^k_{\mathcal{F}})_0) = \pi(\mathrm{Diff}_c(M,\mathcal{F})_0)$.

Now we are in a position to define the flux homomorphism on loops. For any $\bar\theta \in Z^{k+1}_c(\mathcal{F})$ we have already the homomorphism

$$\tilde{\mathcal{S}}_{\omega^{k+1}_{\bar\theta}} : \widetilde{G(E^k_{\mathcal{F}})} \ni \{\phi_t\} \mapsto [I_{\phi_t}(\omega^{k+1}_{\bar\theta})] \in H^k_c(\tau^*\mathcal{F}) .$$

(Here $H^k_c(\tau^*\mathcal{F})$ denotes the foliated cohomology group compactly supported in the horizontal direction.) By using Corollary 1 and taking the restriction of $\tilde{\mathcal{S}}_{\omega^{k+1}_{\bar\theta}}$ we get the *flux homomorphism on foliated loops*

$$\tilde{S}_{\bar\theta} : \pi(\mathrm{Diff}_c(M,\mathcal{F})_0) \to H^k_c(\mathcal{F}) .$$

Specifically for $\bar\theta = 0$ this homomorphism is defined canonically.

4. For any topological group G we have its central extension (cf. [5])

$$0 \to \pi(G) \xrightarrow{i} \tilde{G} \xrightarrow{p} G \to 1 .$$

Let $U_G \in H^2(G, \pi(G))$ be its characteristic class. Let A be an abelian group on which G acts trivially. Recall that

$$t : \mathrm{Hom}(\pi(G), A) \to H^2(G, A), \quad \text{where} \quad t(\psi) = \psi \circ U_G ,$$

is called the transgression homomorphism. The following fact is well-known

Theorem 3 [5]. *The sequence*

$$0 \to \mathrm{Hom}(G,A) \xrightarrow{p^*} \mathrm{Hom}(\tilde{G},A) \xrightarrow{i^*} \mathrm{Hom}(\pi(G),A) \xrightarrow{t} H^2(G,A) \xrightarrow{p^*} H^2(\tilde{G},A)$$

is exact and natural.

Corollary 2. *If $\tilde{G}$ is perfect then t is bijective, and t is an obstruction for extending* $\mathrm{Hom}(\pi(G),A)$ *to* $\mathrm{Hom}(\tilde{G},A)$.

Now we use an analogue of a difficult theorem of Thurston on the simplicity of the diffeomorphism group.

Theorem 4 [11]. $\mathrm{Diff}_c(M,\mathcal{F})_0$ *and its universal covering are perfect.*

Observe that Theorem 4 holds as well for generalized foliations with no leaves of dimension 0, cf. [4]. In this case Theorems 1 and 2 still holds unlike to the rest of considerations.

Corollary 3. *The transgression homomorphism*

$$t : \mathrm{Hom}(\pi(\mathrm{Diff}_c(M,\mathcal{F})_0), H^k_c(\mathcal{F})) \to H^2(\mathrm{Diff}_c(M,\mathcal{F})_0, H^k_c(\mathcal{F}))$$

is bijective, and there is in general no extension of $\tilde{S}_{\bar\theta}$ to $\widetilde{\mathrm{Diff}_c(M,\mathcal{F})}_0$.

For any $\bar\theta \in Z^{k+1}_c(\mathcal{F})$ we have constructed the central extension $(*)$. Let us denote by $\hat{U}_{\bar\theta} \in H^2(\mathrm{Diff}_c(M,\mathcal{F})_0, H^k_c(\mathcal{F}))$ the image of its characteristic class under the projection $Z^k_c(\mathcal{F}) \to H^k_c(\mathcal{F})$. Simultaneously we have the k-flux on loops $\tilde{S}_{\bar\theta} : \pi(\mathrm{Diff}_c(M,\mathcal{F})_0) \to H^k_c(\mathcal{F})$. The following theorem relates these concepts similarly as in the transitive case, cf. [1].

Theorem 5. *Under the above notation and assumptions one has* $t(\tilde{S}_{\bar{\theta}}) = \hat{U}_{\bar{\theta}}$, *i.e. the transgression of the foliated flux is the image of the characteristic class of* $(*)$. *Moreover,* $\tilde{S}_{\bar{\theta}}$ *depends only on* $[\bar{\theta}]$ *and we have a group homomorphism*

$$H_c^{k+1}(\mathcal{F}) \ni [\bar{\theta}] \mapsto t(\tilde{\mathcal{S}}_{\bar{\theta}}) \in H^2(\mathrm{Diff}_c(M, \mathcal{F})_0, H_c^k(\mathcal{F}))\,.$$

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