DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 361–370

ON LOCAL CONVEXIFIABILITY OF TYPE FOUR DOMAINS IN $\mathbf{C}^2$

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ABSTRACT. Results of Kohn-Nirenberg and Fornaess lead to the conjecture that any pseudoconvex domain in $\mathbb{C}^2$ with real analytic boundary is convexifiable in a neighbourhood of a point of finite type four. We prove this under an additional assumption on the model domain at the point. Techniques from the construction of Moser normal form are used as a main tool.

Introduction

Let $\Omega \subseteq \mathbf{C}^2$ be a smooth, pseudoconvex domain. One approach to the study of holomorphic functions on Ω, solutions to $\bar\partial u = 0$, is to study the inhomogeneous Cauchy-Riemann equation

$$\bar\partial u = \alpha \tag{0.1}$$

and the $\bar\partial$ complex

$$C^\infty(\Omega) \to \Lambda^{0,1}(\Omega) \to \Lambda^{0,2}(\Omega) \to \dots$$

The boundary behaviour of holomorphic functions on Ω is closely related to regularity properties of $\bar\partial$ on the boundary $b\Omega$, and both depend on the geometry of $b\Omega$.

There are two contrasting results which relate local geometry of $b\Omega$ to the boundary regularity of $\bar\partial$. If Ω is strongly pseudoconvex and $\alpha = \sum_{i=1}^{n} \alpha_i d\bar z_i$ is a $(0,1)$-form with coefficients smooth up to the boundary, which satisfies the necessary compatibility condition $\bar\partial\alpha = 0$, then the unique solution to (0.1) which is

1991 *Mathematics Subject Classification.* 32F15.
Key words and phrases. Finite type, weakly pseudoconvex domains, Kohn-Nirenberg example, convexity and convexifiability, type four.
This paper is in final form and no version of it will be submitted for publication elsewhere.

orthogonal to the null space of $\bar{\partial}$ is also smooth up to the boundary. This result is a consequence of the solution to the $\bar{\partial}$ - Neumann problem (see ref [FK]). On the other hand, if the boundary is flat near a point p, and so there are complex curves in $b\Omega$ passing through p, then singularities of α can propagate, and local regularity fails (see [K]).

Seeking geometric conditions equivalent to boundary regularity of $\bar{\partial}$, Kohn started to study the intermediate case, when the Levi form degenerates, but does not vanish identically. In [K] he introduced the concept of finite type, a higher order condition which measures the maximal order of contact of complex curves with the boundary. If the domain is pseudoconvex, then the type of a boundary point is an even integer. Type two coresponds to strongly pseudoconvex points (see Definition 1.1).

One of the basic properties of strongly pseudoconvex domains is local convexifiability, i.e., the fact that in a neighbourhood of any boundary point one can introduce local holomorphic coordinates such that the boundary is convex in the induced Euclidean space. A similar situation occurs also when the Levi form vanishes identically near a point, then the domain is locally biholomorphically equivalent to a convex, flat domain.

An example discovered in [KN] shows that local geometry of pseudoconvex domains of finite type is substantially different from the strongly pseudconvex case. The Kohn-Nirenberg example is a pseudoconvex domain which has no complex supporting surface at a point of finite type eight. As a consequence, this domain is not locally convexifiable. By a result of Fornaess in [F], the same phenomenon exists for type six, and all higher types.

In this paper we consider pseudoconvex domains of finite type four in $\mathbf{C}^2$. A result of Fornaess in [FS] shows that such domains always admit a complex supporting surface. This leads to the following well known conjecture.

Conjecture. *Pseudoconvex domains of finite type 4 in $\mathbf{C}^2$ with real analytic boundary are locally convexifiable.*

Our aim in this paper is to show that the conjecture is true in a certain generic case. An important fact about finite type domains in $\mathbf{C}^2$ is the existence of model domains, which substitute to some extent the role of the unit ball for strongly pseudoconvex domains. Model domains of type four, up to biholomorphic equivalence, form a one parameter family. The generic case is the assumption that the model domain takes a generic value of the parameter. For the limit value, the conjecture is proved in [Ko2] in the so called rigid case, when the defining equation of (2.1) is independent of u.

Let us remark that the conjecture is false if real analytic is replaced by C^∞.

1. Domains of finite type 4

There are several equivalent definitions of finite type. The most convenient for our purpose is to use local coordinates.

Definition 1.1. *Let $\Omega \subseteq \mathbf{C}^2$ be a pseudoconvex domain with real analytic boundary. A point $p \in b\Omega$ is a point of finite type k if in suitable local holomorphic coordinates (z, w) around p the boundary is described by*

$$Re\ w = P(z, \bar{z}) + o(|z|^k, Im\ w)\,, \tag{1.1}$$

where P is a real valued homogeneous polynomial of an even degree k

$$P(z, \bar{z}) = |z|^k + \sum_{j=2,4,\ldots,k-2} |z|^{k-j} Re(a_j z^j) \tag{1.2}$$

for some $a_j \in \mathbf{C}$.

The number k is uniquely determined, while P is unique up to a rotation in the z variable. For the original definition of finite type in $\mathbf{C}^2$ see [K].

The model domain at p is the domain

$$\{(z, w) \in \mathbf{C}^2 |\ Re\ w > P(z, \bar{z})\}\,.$$

Sufficient conditions for the existence of a complex supporting surface on a domain given by (1.1) and for convexity of (1.2) are proved in [Ko1]. These results are obtained as a consequence of exact conditions which are computable for domains of Kohn-Nirenberg type. Such model domains, building blocks for the general case, are of the form

$$M_a^{k,l} = \{(z, w) \in C^2 \ \ |\ \ Re\ w > P_a^{k,l}(z, \bar{z})\}$$

where

$$P_a^{k,l}(z, \bar{z}) = |z|^k + a|z|^{k-l} Re\ z^l\,,$$

and $a \geq 0$. The following result is proved in [Ko1].

Proposition 1.2. *If $l^2 \geq 3k - 2$, then $M_a^{k,l}$ is convex for*

$$a \leq \frac{k}{l^2 - k}.$$

If $l^2 < 3k - 2$, then $M_a^{k,l}$ is convex for

$$a^2 \leq \frac{(4k - l^2 - 4)k^2}{(4k - 4)(k^2 - l^2)}.$$

Moreover, if l is not a divisor of k, then these conditions are equivalent to convexifiability of $M_a^{k,l}$.

When $k = 4$, from (1.2) we get a one parameter family of mutually nonequivalent model domains. We will denote

$$P_a(z, \bar{z}) = P_a^{4,2}(z, \bar{z}) = |z|^4 + a\ Re\ z^2 |z|^2\,,$$

where $a \geq 0$. The corresponding model domain M_a is pseudoconvex when P_a is subharmonic, namely for $a \leq \frac{4}{3}$. By Proposition 1.2, M_a is convex for $a \leq \frac{2\sqrt{2}}{3}$. But in fact, every M_a with $a \leq \frac{4}{3}$ is equivalent to a convex model domain. To see that, we take $\tilde{P}(z,\bar{z}) = (\dfrac{z+\bar{z}}{2})^4 = x^4$. It is easy to verify that $\tilde{P}$ is equivalent to $P_{\frac{4}{3}}$, and that every P_a with $0 < a < \frac{4}{3}$ is therefore equivalent to a combination of $|z|^4$ and x^4 with positive coefficients.

We will denote by $\tilde{P}_a$ this convex version of P_a, namely

$$\tilde{P}_a(z,\bar{z}) = (1 - \frac{3}{4}a)|z|^4 + 2ax^4 = |z|^4 + a|z|^2 Re z^2 + \frac{a}{4} Re z^4 \,. \tag{1.3}$$

We shall prove the following

Theorem. *Let $\Omega \subseteq \mathbf{C}^2$ be a psedoconvex domain with real analytic boundary and $p \in b\Omega$. If the model domain at p is M_a and $a < \frac{4}{3}$, then $b\Omega$ is convexifiable near p.*

2. Changes of coordinates

We will assume that Ω is a pseudoconvex domain with real analytic boundary and $p \in b\Omega$ is a point of type 4.

We will use local holomorphic coordinates (z,w), where $z = x+iy$, $w = u+iv$, centered at p such that the hyperplane $v = 0$ is tangent to $b\Omega$ at 0 and the positive v-axis points inside of Ω. Near p, $b\Omega$ is given as a graph of a function

$$v = F(z,\bar{z},u) \tag{2.1}$$

where F is real analytic and has convergent Taylor expansion starting with 2-nd order terms

$$F(z,\bar{z},u) = \sum_{i+j+k\geq 2} a_{ijk} z^i \bar{z}^j u^k \,, \tag{2.2}$$

with $a_{ijk} = \overline{a_{jik}}$.

We will consider holomorphic transformations

$$z^* = z + g(z,w) \qquad w^* = w + f(z,w) \tag{2.3}$$

where f and g are represented by power series

$$f(z,w) = \sum_{i,j} f_{ij} z^i w^j \qquad g(z,w) = \sum_{i,j} g_{ij} z^i w^j .$$

In the new coordinates, the hyperplane $v^* = 0$ is tangent to $b\Omega$ at p and the positive v^*-axis points inside, if

$$f = 0, \quad g = 0, \quad f_z = 0, \quad Im\ f_w = 0, \quad Re(1 + f_w) > 0 \qquad \text{at} \quad z = w = 0 \,.$$

We will use only transformations of this kind, which preserve (2.1), (2.2). Let F^* be the function describing $b\Omega$ in new coordinates. Substituting (2.3) into $v^* = F^*(z^*, \bar{z}^*, u^*)$, and restricting variables z, w to $b\Omega$, we get a change of variables formula

$$\begin{aligned} &F^*(z+g(z,u+iF(z,\bar z,u)),\overline{z+g(\dots)},u+Re\ f(z,u+iF(z,\bar z,u)))= \\ &\qquad\qquad = F(z,\bar z,u)+Im\ f(z,u+iF(z,\bar z,u))\,. \end{aligned} \tag{2.4}$$

It can be viewed as an equality of two power series in $z, \bar{z}, u$, which allows us to obtain relations between coefficients of F^* and F, f, g by equating coefficients of various term.

We will also need a coordinate expression for the Levi form. In $\mathbf{C}^2$ it is a scalar and for a fixed defining function r and fixed coordinates we will think of L as a real valued functon on $b\Omega$. It is given by the formula

$$L = |r_w|^2 r_{z\bar z} + |r_z|^2 r_{w\bar w} - 2Re(r_{z\bar w} r_{\bar z} r_w)\,.$$

In our case we take $r = F - v$ and obtain

$$L = F_{z\bar z}(1+|F_u|^2) + F_{uu}|F_z|^2 + 2Re F_{zu}(F_u+i)F_{\bar z}\,. \tag{2.5}$$

If $a < \frac{4}{3}$ then $\tilde{P}_a$ is strongly convex in directions x, y away from $x = y = 0$. By (1.3), the value of its Hessian at z on a unit vector ξ satisfies

$$D^2\tilde P_a(z,\xi) \geq \epsilon|z|^2\,, \tag{2.6}$$

for some $\epsilon > 0$.

By a standard argument (see[FS]), we find coordinates in which

$$F(z,\bar z,u) = \tilde P_a(z,\bar z) + \tilde F(z,\bar z,u) \tag{2.7}$$

where $\tilde{F}$ is $O(|z|^5, u)$. We will restrict ourselves to coordinates of this kind. In the following we will use partial Taylor expansion of $\tilde{F}$.

$$\tilde F = \sum_{i,j=0}^{\infty} Z_{ij}(u) z^i \bar z^j \tag{2.8}$$

where

$$Z_{ij}(u) = \sum_k a_{ijk} u^k$$

when $i + j \neq 4$. First we consider convexity in the complex tangential directions x, y. From (2.6) it follows that in a small neighbourhood of 0 only those Z_{ij} in $\tilde{F}$ with $i + j \leq 3$ are relevant for the Hessian of F in the two directions. Our aim is to make Z_{0i}, $0 \le i \le 3$ and Z_{12} vanish to a sufficiently high order, since pseudoconvexity then implies $Z_{11} \ge 0$, and convexity in the two directions follows.

An essential tool for dealing with the change of variables formula is the use of weigths associated to the variables in (2.4). We give weight 1 to z and $\bar{z}$ and weight 4 to u and w. A monomial $z^i \bar{z}^j u^k$ has then weight $i + j + 4k$. In terms of weights we decompose $\tilde{F}$:

$$\tilde{F} = \sum_{\nu=5}^{\infty} F_\nu ,$$

where

$$F_\nu(z, \bar{z}, u) = \sum_{i+j+4k=\nu} a_{ijk} z^i \bar{z}^j u^k .$$

We will consider polynomial transformations of the form

$$z^* = z + \sum_{\nu=2}^{n-3} g_\nu \qquad w^* = w + \sum_{\nu=5}^{n} f_\nu ,$$

wher again f_ν and g_ν denote the part of f and g respectively containing all terms of weight ν.

It follows immediately from the change of variables formula that such transformations do not change terms of weight 4, and F^* has the same decomposition as F in (2.7) and (2.8).

3. Proof of the Theorem

We will need the following

Lemma 3.1. *For every $n \ge 4$ there is a transformation (2.3) such that the coefficients a^*_{ijk} in F^* for $i + j \le 3$, $(i, j) \ne (1, 1)$, and weight $i + j + 4k \le n$ are equal to* 0.

Proof. We will use the change of variables formula to determine the coefficients of f and g. The following fact allows to use induction on weight:
The coefficients of terms of weight ν in (2.4) depend only on coefficients in $\tilde{F}^*$ and f of weight $\le \nu$ and on coefficients of weight $\le \nu - 3$ in g. They depend linearly on the coefficients of weight ν in f, $\tilde{F}^*$ and $\nu - 3$ in g.
We prove that in the following steps.
(i) Since u has weight 4 and F has weight ≥ 4, the terms of weight ν in $g(z, \bar{z}, u + iF(z, \bar{z}, u))$ depend only on terms of weight $\le \nu$ in $g(z, w)$. More precisely, if $g_{ij} z^i w^j$ has weight ν, then it contributes to terms of weight ν in $g(z, \bar{z}, u + iF(z, \bar{z}, u))$ by

$$g_{ij} z^i \left[\sum_{l=0}^{j} (\sqrt{-1})^l \binom{j}{l} (\tilde{P}_a)^l u^{j-l} \right] . \tag{3.1}$$

The same holds if g is replaced by $Im\ f$ and $Re\ f$.
(ii) Consider a monomial $f_{ij} z^i w^j$ of weight ν in f. Using (i) and the fact that F^*

has weight ≥ 2, it only contributes to terms of weight $> \nu$ on the left hand side of (2.4). On the right hand side, by (i), it contributes a term of form (3.1):

$$Im\ f_{ij} z^i (u^j + O(\tilde{P}_a))\,. \tag{3.2}$$

(Note that for our purpose we can disregard $O(\tilde{P}_a)$, since it contributes only to Z_{ij} with $i + j \geq 4$.)
(iii) Let $g_{ij} z^i w^j$ be a term of weight $\nu - 3$ in g. Since F^* has weight ≥ 4, and using (i), it contributes only to terms of weight $\geq \nu$. Since all terms of weight 4 are in $\tilde{P}_a$, it contributes to terms of weight ν by

$$Re\left[(g_{ij})\left(4\bar{z}^2 z^{i+1} + 3az^{i+2}\bar{z} + az^i\bar{z}^3 + a\ z^{i+3}\right)\right]\left(u^j + O(\tilde{P}_a)\right)\,. \tag{3.3}$$

(Note that only for $i = 0$ this expression contributes to $Z_{ij},\ i + j \leq 3$.)
(iv) Consider a term $a^*_{ijk}\ (z^*)^i(\bar{z}^*)^j(u^*)^k$ in $\tilde{F}^*$ of weight ν. Since g starts with weight 2 and f with weight 5, it contributes to terms of weight ν in (2.4) only by $a^*_{ijk}\ z^i\bar{z}^j u^k$.
Now we proceed by induction. For $n = 4$ we take the identity transformation. Let now n be an integer > 4 and suppose that the coefficients of $f_\nu,\ \nu < n$ and $g_\nu,\ \nu < n - 3$ are already determined so that the corresponding polynomial transformation satisfies the conclusion for $n - 1$.

Denote $g_i = g_{ij}$ and $f_i = f_{ij}$ if $i + 4j = n$. Similarly, let $a^*_{ij} = a^*_{ijk}$ if $i + j + 4k = n$. From (3.2) and (3.3) we get

$$\begin{aligned}
a^*_{0i} &= f_i + \ldots \qquad\qquad i = 1, 2\\
a^*_{00} &= Im\ f_0 + \ldots\\
a^*_{03} &= f_3 + a\ \bar{g}_0\\
a^*_{12} &= 2g_0 + 3a\ \bar{g}_0 + \ldots
\end{aligned}$$

where dots denote already determined numbers.
The first two equations determine f_1, f_2 and $Im\ f_0$. The fourth has a solution for g_0, since we assume $a \neq \frac{4}{3}$. Finally, the third equation gives f_3. All other coefficients of terms of weight n can be arbitrary.

For a real analytic function $f : U \to R$ defined in a neighbourhood of $0 \in R$, let $\mu(f)$ denote the order of vanishing of f at 0.

For a parametrized real analytic curve γ in $b\Omega$ passing through p and transversal to the plane $\{w = 0\}$ at p consider the number $\mu(L \circ \gamma)$. By tranformation properties of L, it does not depend on coordinates in which L is evaluated, and is independent of parametrization. If such numbers form a bounded set of integers, we denote by s the maximum of $\mu(L \circ \gamma)$ over all γ. Otherwise we define $s = \infty$. Clearly s is even and $s \geq 2$.

Proposition 3.2. *If $s < \infty$, then there are local holomorphic coordinates such that*

$$F(z, \bar{z}, u) = \tilde{P}_a(z, \bar{z}) + |z|^2 u^s + o(|z|^4, u^s)\,. \tag{3.4}$$

Proof. We take $n = 2 + 4s$ and use the coordinates given by the previous lemma. Let $Z_{11}(u) = a_{11l} u^l + o(u^l)$ for some $l \geq 2$ and $a_{11l} \neq 0$. From (2.5) we have

$$L(0, u) = a_{11l} u^l + o(u^s + u^l)\,,$$

since in our coordinates all other terms than $F_{z\bar{z}}$ in (2.5) are $O(z)$ or $o(u^s)$. So $l \leq s$ by the definition of s. Also l is even and $a_{11l} > 0$. Using (2.6) we have $L(z, u) \geq \epsilon(|z|^2 + u^l)$ for some $\epsilon > 0$, and so $l = s$.
To normalize the coefficient of $|z|^2 u^s$ we apply a dilation

$$(z^*, w^*) = (\delta z, \delta^4 w)$$

for a suitable $\delta > 0$.

Proposition 3.3. *If $s = \infty$, then there are local holomorphic coordinates in which*

$$F(z, \bar{z}, u) = \tilde{P}_a(z, \bar{z}) + o(|z|^4)\,. \tag{3.5}$$

Proof. First we show that if $s = \infty$ then $b\Omega$ contains a curve of weakly pseudoconvex points passing through p and transversal to the plane $\{w = 0\}$ at p. In fixed coordinates we can view L as a real analytic function of three variables. If there is no such curve, then by the curve selection lemma, p is an isolated weakly pseudoconvex point. By the Lojasiewicz inequality ,

$$L(z, u) > (|z| + |u|)^m$$

for some integer m, and so $s < \infty$.
By Lemma 3.2 in [CM], for any (unparametrized) real analytic curve γ in $b\Omega$ transversal to the plane $\{w = 0\}$ at p we can find local holomorphic coordinates (z, w) around p such that γ is the u axis and $Z_{0j} = 0,\ \ j = 0, 1, \ldots$ (this proposition holds on any real analytic domain). We take γ to be the curve of weakly pseudoconvex points. Then in such coordinates $Z_{11} = 0$, since $L(0, u) = Z_{11}(u)$. Since $L \geq 0$, its first derivatives on the u axis also vanish, so $Z_{12} = 0$. That gives (3.5).

The purpose of the following lemma is to add a quadratic term in the remaining direction u.

Lemma 3.4. *There exist local holomorphic coordinates in which F satisfies (3.4) or (3.5), and*

$$Z_{00}(u) = u^2 + o(u^2)\,. \tag{3.6}$$

Proof. Let (z, w) be local holomorphic coordinates in which (3.4) holds, and put

$$w^* = w + iw^2, \qquad z^* = z\,.$$

We claim that (3.4) and (3.6) are satisfied in the new coordinates (after deleting stars). From (2.4) we get

$$F^*(z, \bar{z}, u - 2uF(z, \bar{z}, u)) = F(z, \bar{z}, u) + u^2 - F(z, \bar{z}, u)^2\,.$$

Since Z_{ij}, $i+j \le 3$ in F vanish up to order $s-1$, the same terms in the expansion of F^2 vanish up to order at least $s+1$. Therefore F^* has form (3.4). Since $Z_{00} = o(u^2)$ in the original coordinates, (3.6) is also satisfied. The same argument applies if F has form (3.5).

The following Proposition completes the proof of the Theorem.

Proposition 3.5. *In the coordinates given by Lemma 3.4, F is a convex function.*

Proof. We view F as a function of three real variables x, y, u and consider its Hessian in a neighbourhood of p. First we prove that the 2x2 submatrix of the Hessian formed by the derivatives with respect to x and y is positive semidefinite. From (3.4) and (3.5) we have

$$F(z, \bar{z}, u) = \tilde{P}_a(z, \bar{z}) + Z_{00}(u) + |z|^2 u^s + o(|z|^4, u^s) \tag{3.7}$$

and

$$F(z, \bar{z}, u) = \tilde{P}_a(z, \bar{z}) + Z_{00}(u) + o(|z|^4) \tag{3.8}$$

respectively. First let F satisfy (3.7). If ξ is a linear combination of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ and $|\xi| = 1$, then by (2.6) there exists an ϵ such that

$$D^2F((z, u), \xi) \ge \epsilon(|z|^2 + u^s)$$

holds in a sufficiently small neighbourhood of 0.
To finish the proof, we need to show that the determinant of D^2F is nonnegative. From (3.7) we have $(F_{ux})^2 = o(|z|^2, u^s)$, so $(F_{ux})^2 = o(F_{xx})$ and similarly $(F_{uy})^2 = o(F_{yy})$. The remaining diagonal entry is $F_{uu} = 1 + o(1)$. Simple algebra implies that the determinant of $D^2 f$ is nonnegative in a neighbourhood of 0.

For (3.8) the proof is obtained from the previous by leaving out u^s whenever it appears.

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