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HANTZSCHE–WENDT MANIFOLDS OF DIMENSION 7

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Abstract. We classify Hantzsche-Wendt manifolds in dimension 7, proving that there are exactly 62 diffeomorphism classes. We consider three different notions of isospectrality, showing that all three are equivalent to each other. We also show that a finite part of the spectrum determines the full spectrum. We list the isospectral classes in dimension 7 and give a formula to compute the multiplicities of the eigenvalues.

Introduction

Hantzsche-Wendt manifolds (or HW manifolds) are compact flat Riemannian manifolds which are natural generalizations, to any odd dimension n, of the classical 3-dimensional Hantzsche-Wendt manifold. These manifolds have holonomy group $\mathbb{Z}_2^{n-1}$ and are rational homology spheres. In [MR], by using certain graphs associated to any such manifold, we proved that $h(n)$, the number of diffeomorphism classes of HW manifolds, grows (at least) exponentially with n, showing also that $h(5) = 2$. Furthermore we proved that for any $n \geq 7$, there exist pairs of isospectral manifolds non homeomorphic to each other.

In this paper we shall study in some detail Hantzsche-Wendt manifolds of dimension 7, using the results in [MR]. We shall compare isospectrality and $\mathcal{F}$-isospectrality, a notion having to do with a finite part of the spectrum. We will prove that in dimensions 7 and 9, these two notions are equivalent to each other. We shall also see that if $n = 7$ or 9, a finite part of the spectrum, namely the multiplicities of those eigenvalues λ such that $\frac{\lambda}{4\pi^2} \leq \frac{n^2-7}{2}$, determines the full

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spectrum. Also, if $n = 7$, we shall see that these two notions are equivalent to Sunada-isospectrality.

At the end we shall give a complete system of representatives of HW manifolds in dimension 7, showing that there are exactly 62 diffeomorphism classes. This classification has been obtained by means of a computer program. We will also include explicit realizations of each manifold, computing the numbers $m_{k,l}$ (see Definition 2.1), which determine the multiplicities of the eigenvalues, and also the Sunada numbers $c_{s,t}$ (see §2).

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1. Preliminaries

In this section we recall some known facts on Hantzsche-Wendt Bieberbach groups. We refer to [MR] for the proofs.

A *Bieberbach group* is a discrete cocompact subgroup Γ of $I(\mathbb{R}^n)$ which is torsion-free. Then $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ is a compact flat Riemannian manifold with fundamental group Γ. Any element $\gamma \in I(\mathbb{R}^n)$ decomposes uniquely $\gamma = uL_v$, with $u \in O(n)$ and $v \in \mathbb{R}^n$. The translations in Γ form a normal abelian subgroup L_Λ of Γ, Λ a lattice in $\mathbb{R}^n$. The point group $F = \Lambda\backslash\Gamma$ coincides with the holonomy group of M_Γ and acts on Λ by conjugation. This action defines an integral representation of F, called the holonomy representation.

A *Hantzsche-Wendt group* (or *HW group*) Γ is a Bieberbach group with holonomy group $F \simeq \mathbb{Z}_2^{n-1}$ and such that the action of every $x \in F$ diagonalizes on a $\mathbb{Z}$-basis of Λ with $\det(x) = 1$.

By conjugating by an element of $GL_n(\mathbb{R})$, if necessary, we may assume that $\Lambda = \sum_1^n \mathbb{Z}e_i$, the canonical lattice in $\mathbb{R}^n$; hence F can be identified to the diagonal subgroup $\{B : Be_i = \pm e_i,\ 1 \leq i \leq n,\ \det B = 1\}$. Since Γ is torsion-free, then $-I \notin F$, hence n must be odd.

If Γ is an HW group, the corresponding manifold $M_\Gamma := \Gamma\backslash\mathbb{R}^n$ will be called a *Hantzsche-Wendt* (or *HW*) *manifold.*

We choose $B_i \in F$ such that B_i fixes e_i and $B_i e_j = -e_j$ (if $j \neq i$), for each $1 \leq i \leq n$. Clearly, F is generated by $B_1, B_2, \ldots, B_{n-1}$. Furthermore $B_n = B_1 B_2 \ldots B_{n-1}$ (since n is odd).

Thus, if Γ is a Hantzsche-Wendt group, then $\Gamma = \langle B_1 L_{b_1}, \ldots, B_{n-1} L_{b_{n-1}}, L_\lambda : \lambda \in \Lambda\rangle$, for some $b_i \in \mathbb{R}^n$, $1 \leq i \leq n-1$. Now, if $I = (i_1, \ldots, i_s)$ with $1 \leq i_1 < i_2 < \cdots < i_s \leq n$, one can write $B_{i_1} L_{b_{i_1}} \ldots B_{i_s} L_{b_{i_s}} = B_I L_{b(I)}$ with

$$(1.1)\quad b(I) = b(i_1, \ldots, i_s) = B_{i_s} \ldots B_{i_2} b_{i_1} + B_{i_s} \ldots B_{i_3} b_{i_2} + \cdots + B_{i_s} b_{i_{s-1}} + b_{i_s}\,.$$

and $B_I = B_{i_1} \ldots B_{i_s}$. A complete system of representatives for $\Lambda\backslash\Gamma$ is

$$\{B_{i_1} \ldots B_{i_s} L_{b(i_1,\ldots,i_s)} \,:\, 1 \leq i_1 < i_2 < \cdots < i_s \leq n-1\}\,.$$

It is shown in [MR] that, by replacing Γ by an isomorphic group, one may assume that the components of b_i, b_{ij}, lie in $\{0, \frac{1}{2}\}$, $1 \leq i, j \leq n$.

Thus any $\gamma \in \Gamma$ can be written *uniquely*

$$\gamma = B_{i_1} \dots B_{i_r} L_{b_o(I)} L_\lambda \tag{1.2}$$

where r is *odd*, $1 \leq i_1 < \cdots < i_r \leq n$, $\lambda \in \Lambda$ and $b_o(I)_j \in \{0, \frac{1}{2}\}$, for $1 \leq j \leq n$.

In [DM], Proposition 2.1, it is shown that the condition for Γ to be torsion-free is as follows:

$$(B_{i_1} \dots B_{i_s} + I) b(i_1, \dots, i_s) \in \Lambda - (B_{i_1} \dots B_{i_s} + I)\Lambda \tag{1.3}$$

for any $1 \leq i_1 < i_2 < \cdots < i_s \leq n-1$. This condition is equivalent to the condition that the class defined by Γ in $H^2(\mathbb{Z}_2^{n-1}, \Lambda)$ be a special class (see [Ch], Theorem 2.1, p. 79).

We can rephrase (1.3) in terms of the entries of the matrix $A := [b_{ij}]$ associated to Γ as follows (see [MR], Prop. 1.2): Γ is torsion free with translation lattice Λ if and only if for any $I \subsetneq \{1, 2, \dots, n\}$, there exists $j \in I$, if $|I|$ is odd (respectively $j \notin I$, if $|I|$ is even) such that

$$\text{card}\left\{i \in I : b_{ij} = \tfrac{1}{2}\right\} \text{ is odd.} \tag{1.4}$$

In particular, (1.4) implies that for each fixed j, $b_{jj} = \frac{1}{2}$, and card$\{i : 1 \leq i \leq n$, $b_{ij} = \frac{1}{2}\}$ is even.

In [MR] we saw that to show that Γ is torsion free, it is sufficient to verify that for each j, card $\left\{i : 1 \leq i \leq n,\ b_{ij} = \frac{1}{2}\right\}$ is even and that (1.4) holds for all I with $|I|$ odd (or else, for all I with $|I| \leq \frac{n-1}{2}$).

Concerning conditions for the isomorphism of two HW groups, we show in [MR] that if Γ, Γ' are HW groups with associated matrices $A = [b_{ij}]$, $A' = [b'_{ij}]$ respectively, and b_{ij}, $b'_{ij} \in \{0, \frac{1}{2}\}$ for all i, j, then $\Gamma \simeq \Gamma'$ if and only if

$$A' = P_\sigma A_c {P_\sigma}^{-1}, \tag{1.5}$$

where $\sigma \in S_n$ and A_c is a "complementary matrix" of A with respect to a set of columns of A, in the sense that A_c is obtained from A by replacing in these columns all $\frac{1}{2}$'s (except those in the diagonal) by 0's and all 0's by $\frac{1}{2}$'s.

Condition (1.5) implies that when $n \equiv 1 \bmod 4$, (resp. $n \equiv 3 \bmod 4$), in each isomorphism class of HW groups there is a representative such that the associated matrix has at most $\frac{n-1}{2}$ (resp. $\frac{n+1}{2}$) non zero entries on each column. If $n \equiv 1 \bmod 4$ and we restrict ourselves to such representatives, the above condition for isomorphism reduces to $A' = P_\sigma A {P_\sigma}^{-1}$ for some $\sigma \in S_n$.

Let Γ be an HW group with associated matrix A. If A has m_i nonzero entries on the i-th column for $i = 1, \dots, n$, we shall say that Γ has *column pattern* $(m_1, \dots, m_n)$. The column pattern is an invariant of the isomorphism class of Γ.

When $n = 5$ there are exactly 2 isomorphism classes, both having column pattern $(2, 2, 2, 2, 2)$. More generally, for any odd n there are exactly two HW manifolds having column pattern $(2, \dots, 2)$ ([MR], Thm 2.1).

In the case when $n = 7$ (and also for $n = 9$), we need only consider $n \times n$ matrices A having either 2 or 4 non-zero entries on each column. The possible column patterns are of the type $(4, \dots, 4, 2, \dots, 2)$ where 4 and 2 are repeated h and $n - h$ times, respectively. We shall see in Section 3, that if $n = 7$, necessarily $0 \leq h \leq 5$, giving a full list of representatives for each value of h.

2. Isospectrality of Hantzsche-Wendt manifolds

The purpose of this section is to study the spectrum of HW manifolds. Given Γ an HW group, we wish to describe the spectrum of the Laplacian on $L^2(\Gamma\backslash\mathbb{R}^n)$. We first consider the spectrum of $\Lambda\backslash\mathbb{R}^n$.

If $v \in \Lambda^*$, the dual lattice of Λ, the function $f_v(x) = e^{2\pi i\, v.x}$ on $\mathbb{R}^n$ is Λ-invariant and $-\Delta f_v = 4\pi^2\|v\|^2 f_v$, ($\Delta$ the Laplacian on $\mathbb{R}^n$). Furthermore, if $v, v' \in \Lambda^*$, then $\langle f_v, f_{v'}\rangle = \delta_{v,v'}$ and it easily follows from the Stone-Weierstrass theorem that $\{f_v : v \in \Lambda^*\}$ is a complete orthonormal system of $L^2(\Lambda\backslash\mathbb{R}^n)$. Thus, for each $\mu \in \mathbb{N}$, the eigenspace of $-\Delta$ with eigenvalue $4\pi^2\mu$, $\mathcal{H}_\mu := L^2(\Lambda\backslash\mathbb{R}^n)_{4\pi^2\mu} = \operatorname{span}\{f_v : v \in \Lambda^*_\mu\}$, where $\Lambda^*_\mu = \{v \in \Lambda^* : \|v\|^2 = \mu\}$. The spectrum of $-\Delta$ in $\Lambda\backslash\mathbb{R}^n$ is thus determined by the cardinality of the sets Λ^*_μ, $\mu \in \mathbb{N}$.

If we now look at $M_\Gamma = \Gamma\backslash\mathbb{R}^n$, we see that for each $\gamma \in \Gamma, \mu \in \mathbb{N}, \gamma$ preserves $\mathcal{H}_\mu$. Indeed, if $\gamma = BL_b, B \in O(n), b \in \mathbb{R}^n$, then $\gamma f_v(x) = e^{-2\pi i\, v.b} f_{Bv}(x)$. It easily follows that

$$L^2(\Gamma\backslash\mathbb{R}^n) \simeq L^2(\Lambda\backslash\mathbb{R}^n)^F = \oplus \sum_\mu \mathcal{H}_\mu^F.$$

Hence

$$\operatorname{Spec}(\Gamma\backslash\mathbb{R}^n) = \{(\mu, d_{\mu,\Gamma}) : \mu = 0, 1, \ldots\},$$

where $d_{\mu,\Gamma} = \dim \mathcal{H}_\mu^F$, for each $\mu \in \mathbb{N} \cup \{0\}$.

On the other hand, for each μ we have an orthogonal projection $p_\mu : \mathcal{H}_\mu \to \mathcal{H}_\mu^F$, given by $p_\mu = |F|^{-1} \sum_{\gamma\in\Lambda\backslash\Gamma} \gamma_{|\mathcal{H}_\mu}$. Thus we get:

$$d_{\mu,\Gamma} = |F|^{-1} \left(\sum_{\gamma\in\Lambda\backslash\Gamma} \operatorname{tr} \gamma_{|\mathcal{H}_\mu} \right). \tag{2.1}$$

If Γ is an HW group, then $\Lambda = \Lambda^*$, the canonical lattice. Furthermore, any $\gamma \in \Gamma$ can be written (see (1.2)), $\gamma = B_I L_{b_o(I)} L_\lambda$, $|I|$ odd, $b_o(I)_j \in \{0, \frac{1}{2}\}$, for all $1 \le j \le n$ and $\lambda \in \Lambda$. Hence $\gamma f_v(x) = e^{-2\pi i b_o(I).v} f_{B_I v}(x)$. For each $v \in \Lambda$ we set

$$I_v = \{j : v.e_j \neq 0\}, \quad I_v^{odd} = \{j : v.e_j \text{ is odd }\}.$$

It follows that $\langle \gamma f_v, f_v\rangle \neq 0$ if and only if $B_I v = v$, that is, iff $I \supset I_v$, by the definition of B_I, since $|I|$ is odd. If this holds then we get

$$\langle \gamma f_v, f_v\rangle = (-1)^{|I_{b_o(I)} \cap I_v^{odd}|}.$$

Therefore, for each $\mu \in \mathbb{N}$ we have

$$d_{\mu,\Gamma} = |F|^{-1} \sum_{v\in\Lambda_\mu} \sum_{I\supset I_v} (-1)^{|I_{b_o(I)} \cap I_v^{odd}|}. \tag{2.2}$$

For $v, v' \in \Lambda_\mu$, we say that $v \sim v'$ if v' is obtained from v by a permutation, followed by a sign change of some of the coordinates. We fix for each μ, $\widetilde{\Lambda}_\mu$, a complete system of representatives of Λ_μ. Also, we set for each $v \in \Lambda$,

$$m_{v,\Gamma} = |F|^{-1} \sum_{v' \sim v} \sum_{I \supset I_{v'}} (-1)^{|I_{b_o(I)} \cap I_{v'}^{odd}|}\,. \tag{2.3}$$

We note that $m_{v,\Gamma}$ is the dimension of the space of F-invariant vectors of the subspace of $\mathcal{H}_\mu$, generated by $f_{v'}$, $v' \sim v$, hence $m_{v,\Gamma} \geq 0$.

Thus, for each μ we have $d_{\mu,\Gamma} = \sum_{v \in \widetilde{\Lambda}_\mu} m_{v,\Gamma}$, by (2.2) and (2.3).

Now, given $v \in \Lambda_\mu$, denote by $\bar{v} \in \Lambda$ the only vector with the following properties: $\bar{v}.e_j \in \{0, 1, 2\}$ for any j, $|I_v| = |I_{\bar{v}}|$, $|I_v^{odd}| = |I_{\bar{v}}^{odd}|$ and the components of $\bar{v}$ are arranged in non-increasing order.

We have that, for any v, $m_{v,\Gamma} = \alpha_v m_{\bar{v},\Gamma}$, with α_v a natural number. Actually, if v has e_v (resp. o_v) distinct nonzero even (resp. odd) components with $e_1, \dots, e_r$ repetitions up to sign, $\sum_{j=1}^r e_j = e_v$ (resp. $o_1, \dots, o_s$ repetitions up to sign, $\sum_{j=1}^s o_j = o_v$), then it is not hard to verify that $\alpha_v = \binom{e_v}{e_1, \dots, e_r}\binom{o_v}{o_1, \dots, o_s}$. For instance, if $v = (8, -9, 3, 0, -1, 1, -4, 0, 8, 0, 0)$, then $\bar{v} = (2, 2, 2, 1, 1, 1, 1, 0, 0, 0, 0)$ and we have $m_v = \binom{3}{2,1}\binom{4}{1,1,2} m_{\bar{v}} = 3\frac{4!}{2!} m_{\bar{v}}$.

For simplicity, in what follows we shall often omit the group Γ from the notation.

Definition 2.1. For any k, l, with $0 \leq l \leq k \leq n$, let $m_{k,l} = m_{\bar{v}}$ where v is any vector with $|I_v| = k$ and $|I_v^{odd}| = l$. We shall say that Γ and Γ' are *$\mathcal{F}$-isospectral* if

$$m_{k,l,\Gamma} = m_{k,l,\Gamma'}, \text{ for any } 1 \leq l \leq k \leq n\,. \tag{$\mathcal{F}$}$$

Summing up we have obtained:

Proposition 2.2. *For any $\mu \in \mathbb{N}$ we have:*

$$d_\mu = \sum_{v \in \widetilde{\Lambda}_\mu} \alpha_v m_{k_v, l_v} \tag{2.4}$$

where $k_v = |I_v|, l_v = |I_v^{odd}|$ and $\alpha_v \in \mathbb{N}$ is as above. Hence, if Γ and Γ' are HW groups and ($\mathcal{F}$) holds, then M_Γ and $M_{\Gamma'}$ are isospectral.

We shall now consider the converse question, that is, whether isospectrality implies condition ($\mathcal{F}$). By (2.4), the problem reduces to showing that one can invert a system of linear equations. We shall see that this can be done for $n = 7, 9$ and we believe this should be the case for general n, but at this point we do not know how to handle the linear system in general.

We shall now show that if $n = 7, 9$, the system in Proposition 2.2 can be inverted, allowing to determine the $m_{k,l}$'s in terms of the d_μ's.

The number of variables $m_{k,l}$ to be determined is $\sum_{j=0}^n (j+1) = \binom{n+2}{2}$, that is, there are 36 variables if $n = 7$ and 55 if $n = 9$.

We note that if $k = n$ or $n-1$, the only element contributing to the sum is the identity element ($I = \{1, \dots, n\}$, $I_{b_o(I)} = \emptyset$), and one finds that

$$m_{k,l} = 2^{k-n+1}\binom{n}{k}\binom{k}{l}, \tag{2.5}$$

so these variables are determined independently of Γ. On the other hand, if $l = 0$, then $I_v^{odd} = \emptyset$, hence one computes that

$$m_{k,0} = \frac{2^k}{2^{n-1}} \sum_{i=0}^{[\frac{n-k}{2}]} \binom{n}{n-2i}\binom{n-2i}{k} = 2^{k-n+1}\binom{n}{k} \sum_{i=0}^{[\frac{n-k}{2}]} \binom{n-k}{2i} = \binom{n}{k} \tag{2.6}$$

for all Γ. We are thus left with the $m_{k,l}$ corresponding to $1 \le l \le k \le n-2$, a total of $\binom{n-1}{2}$ unknowns. Finally we observe that, for all n, if $1 \le \mu \le 4$, $m_{\mu,\mu}$ is immediately determined by d_μ. We are thus left with 11 unknowns to determine, if $n = 7$ and 24, if $n = 9$. Assume first $n = 7$. If we let successively $\mu = 1, 2, \dots, 17, 21$ and use the partitions of μ as a sum of squares, we get the equations:

$$\begin{aligned}
d_5 &= m_{5,5} + m_{2,1}, & d_6 &= m_{6,6} + m_{3,2},\\
d_7 &= m_{7,7} + m_{4,3}, & d_8 &= m_{5,4} + m_{2,0},\\
d_9 &= m_{1,1} + m_{6,5} + m_{3,1}, & d_{10} &= 2m_{2,2} + m_{7,6} + m_{4,2},\\
d_{11} &= 3m_{3,3} + m_{5,3}, & d_{12} &= 4m_{4,4} + m_{6,4} + m_{3,0},\\
d_{13} &= 5m_{5,5} + m_{2,1} + m_{7,5} + m_{4,1}, & d_{14} &= 6m_{6,6} + 2m_{3,2} + m_{5,2},\\
d_{15} &= 7m_{7,7} + 3m_{4,3} + m_{6,3}, & d_{16} &= m_{1,0} + 4m_{5,4} + m_{7,4} + m_{4,0},\\
d_{17} &= m_{2,1} + 5m_{6,5} + m_{3,1} + m_{5,1}, & &\\
d_{21} &= 10m_{5,5} + m_{6,5} + 2m_{3,1} + 5m_{7,5} + m_{4,1} + m_{6,1}\,. & &
\end{aligned}$$

Now we note that $m_{3,2}$ can be obtained from d_6, since $m_{6,6}$ is known. Similarly, $m_{4,3}$ and $m_{5,4}$ are obtained from d_7, d_8 respectively, since $m_{7,7}$ and $m_{2,0}$ are known. By this method we can determine successively all unknowns except for $m_{5,5}, m_{2,1}, m_{4,1}$ and $m_{5,1}$, which are connected by the equations corresponding to d_5, d_{13}, d_{17} and d_{21}. Once the known variables are disregarded, the linear system resulting has the following matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 10 & 0 & 1 & 0 \end{bmatrix}$$

which is invertible, hence we can solve for the $m_{k,l}$'s in terms of the d_μ's. This concludes the proof in the case $n = 7$.

For $n = 9$ the previous equations remain valid, with equality replaced by congruence modulo terms involving variables of the type $m_{k,l}$ with $k = 8$ or 9, which are already known. We now need many more equations, namely, those corresponding to d_μ with $5 \le \mu \le 33$ and $\mu = 37$ (actually not all of them will be used). Ommitting the variables of the type $m_{k,l}$ with $k = 8$ or 9, $m_{k,0}$ and $m_{l,l}$, $1 \le l \le 4$, which have already been determined (see (2.5), (2.6)), the resulting equations are:

$$
\begin{aligned}
d_{18} &\equiv m_{3,2} + 6m_{7,6} + 2m_{4,2} + m_{6,2},\\
d_{19} &\equiv m_{4,3} + 3m_{5,3} + m_{7,3},\\
d_{20} &\equiv m_{5,4} + 4m_{6,4},\\
d_{22} &\equiv 15m_{6,6} + m_{3,2} + m_{7,6} + 2m_{4,2} + 2m_{5,2} + m_{7,2},\\
d_{23} &\equiv 21m_{7,7} + 3m_{4,3} + 2m_{5,3} + 3m_{6,3},\\
d_{24} &\equiv 6m_{5,4} + 2m_{6,4} + 4m_{7,4},\\
d_{25} &\equiv m_{2,1} + 10m_{6,5} + 2m_{7,5} + 3m_{4,1} + m_{5,1} + m_{7,1},\\
d_{26} &\equiv 2m_{3,2} + 15m_{7,6} + m_{4,2} + 3m_{5,2} + 2m_{6,2},\\
d_{27} &\equiv 3m_{4,3} + 3m_{5,3} + 3m_{6,3} + 3m_{7,3},\\
d_{28} &\equiv 4m_{5,4} + 6m_{6,4} + 3m_{7,4},\\
d_{29} &\equiv 15m_{5,5} + m_{2,1} + 5m_{6,5} + 2m_{3,1} + 10m_{7,5} + 4m_{5,1} + m_{6,1},\\
d_{30} &\equiv 26m_{6,6} + 2m_{3,2} + 6m_{7,6} + 4m_{4,2} + m_{5,2} + 4m_{6,2} + 2m_{7,2},\\
d_{31} &\equiv 42m_{7,7} + 4m_{4,3} + 6m_{5,3} + 3m_{6,3} + 4m_{7,3},\\
d_{32} &\equiv 8m_{5,4} + 8m_{6,4} + 6m_{7,4},\\
d_{33} &\equiv 15m_{6,5} + 2m_{3,1} + 10m_{7,5} + 3m_{4,1} + 5m_{6,1} + m_{7,1},\\
d_{37} &\equiv 25m_{5,5} + m_{2,1} + 10m_{6,5} + 16m_{7,5} + 4m_{4,1} + 4m_{5,1} + 6m_{7,1}.
\end{aligned}
$$

Now we observe that the system can be split into 4 subsystems according to congruence mod 4. Indeed, for each k, l, the unknown $m_{k,l}$ appears only in the equations corresponding to d_μ with $\mu \equiv l \pmod 4$.

The subsystem corresponding to $\mu \equiv 0 \pmod 4$ can be easily solved. We are thus left with three linear systems: 9×9, if $\mu \equiv 1$, 7×7, if $\mu \equiv 2$, and 5×5, if $\mu \equiv 3$ mod 4. In order to write down the respective matrices of coefficients, it will be convenient to choose the $m_{k,l}$'s in the lexicographic ordering in $(k - l, l)$, namely $m_{5,5}, m_{6,6}, m_{7,7}, m_{2,1}, \ldots, m_{7,6}, m_{3,1}, \ldots, m_{7,1}$. The resulting matrices are:

$$
\begin{bmatrix}
1 & 1 \\
0 & 0 & 1 & 1 \\
5 & 1 & 0 & 0 & 1 & 1 \\
0 & 2 & 5 & 1 & 0 & 1 & 1 \\
10 & 0 & 1 & 2 & 5 & 1 & 0 & 1 \\
0 & 1 & 10 & 0 & 2 & 3 & 1 & 0 & 1 \\
15 & 1 & 5 & 2 & 10 & 0 & 1 & 1 & 0 \\
0 & 0 & 15 & 2 & 10 & 3 & 0 & 5 & 1 \\
25 & 1 & 10 & 0 & 16 & 4 & 4 & 0 & 6
\end{bmatrix};
\qquad
\begin{bmatrix}
1 & 1 \\
0 & 0 & 1 & 1 \\
6 & 0 & 0 & 2 & 1 \\
0 & 1 & 6 & 2 & 0 & 1 \\
15 & 1 & 1 & 2 & 2 & 0 & 1 \\
0 & 2 & 15 & 1 & 3 & 2 & 0 \\
26 & 2 & 6 & 4 & 1 & 4 & 2
\end{bmatrix};
\qquad
\begin{bmatrix}
1 & 1 \\
0 & 0 & 1 \\
7 & 3 & 0 & 1 \\
0 & 1 & 3 & 0 & 1 \\
21 & 3 & 2 & 3 & 0
\end{bmatrix}.
$$

all of which are invertible. This completes the proof if $n = 9$.

Remark 2.3. In [MR], we applied a method due to T. Sunada (see [Su]) to produce many examples of isospectral HW manifolds. Given an HW group, we defined for each fixed $s < n$, s odd, and each $t \leq s$, the Sunada number $c_{s,t}$ of Γ, by

$$c_{s,t}(\Gamma) = \mathrm{card}\left\{ I \subset \{1, \ldots, n\} : |I| = s \text{ and } 2\sum_{i \in I} b_o(I)_i = t \right\}.$$

We have shown in [MR], Proposition 3.3 that if $c_{s,t}(\Gamma) = c_{s,t}(\Gamma')$, for each s, t as above, then M_Γ and $M_{\Gamma'}$ are Sunada-isospectral, hence isospectral. From the explicit values of the numbers $m_{k,l}$ and $c_{s,t}$, for $n = 7$ in Table 3.1, we shall see conversely that if $n = 7$, isospectrality implies Sunada-isospectrality.

3. Hantzsche-Wendt manifolds of dimension 7

In this section we shall give the associated matrices A (see §1) for a full set of representatives of the diffeomorphism classes of Hantzsche-Wendt manifolds in dimension $n = 7$. For convenience, we shall write 1's in place of $\frac{1}{2}$'s, in each matrix. The matrices have been obtained by using the results stated in §1, with the help of computer programs.

We have considered separately the 8 possible column patterns. Each program generates first all the possible matrices with the prescribed column pattern. Then it determines those satisfying the torsion-free condition (1.4) and for each one of them it selects a canonical representative, ordered as follows. The first condition is that the columns having 4 nonzero entries should come first. Secondly, for all the matrices in each given equivalence class (according to (1.5)), the program forms a binary number of n^2 digits, obtained by putting first the first column of A, then the second column, and so on, up to the n^{th} column, and chooses the one with the greatest such binary number. Finally, each time a new matrix is obtained and reduced to its canonical form, it is compared with those existing so far and added to the list when different from all of them.

As a result, one obtains exactly 2 diffeomorphism classes of type $(2, \ldots, 2)$, 8 classes of type $(4, 2, \ldots, 2)$, 19 classes of type $(4, 4, 2, \ldots, 2)$, 22 classes of type $(4, 4, 4, 2, \ldots, 2)$, 10 classes of type $(4, 4, 4, 4, 2, 2, 2)$, 1 class of type $(4, 4, 4, 4, 4, 2, 2)$ and none of type either $(4, 4, 4, 4, 4, 4, 2)$ or $(4, 4, 4, 4, 4, 4, 4)$. The resulting 62 matrices, listed in each column pattern in decreasing order, according to the ordering above, are as follows:

$$\begin{bmatrix} 1&0&0&0&0&0&1 \\ 1&1&0&0&0&0&0 \\ 0&1&1&0&0&0&0 \\ 0&0&1&1&0&0&0 \\ 0&0&0&1&1&0&0 \\ 0&0&0&0&1&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&0&0&0&0&0&0 \\ 1&1&0&0&0&0&1 \\ 0&1&1&0&0&0&0 \\ 0&0&1&1&0&0&0 \\ 0&0&0&1&1&0&0 \\ 0&0&0&0&1&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$$

$$\begin{bmatrix} 1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0 \\ 1&0&1&0&0&1&1 \\ 1&0&0&1&0&0&0 \\ 0&0&0&1&1&0&0 \\ 0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0 \\ 1&0&1&0&0&1&0 \\ 1&0&0&1&0&0&1 \\ 0&0&0&1&1&0&0 \\ 0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0 \\ 1&0&1&0&0&1&0 \\ 1&0&0&1&0&0&0 \\ 0&0&0&1&1&0&1 \\ 0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0 \\ 1&0&1&0&0&1&0 \\ 1&0&0&1&0&0&0 \\ 0&0&0&1&1&0&0 \\ 0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&0&0&0&0&0 \\ 1&1&1&0&0&0&0 \\ 1&0&1&0&0&0&1 \\ 1&0&0&1&0&0&0 \\ 0&0&0&1&1&0&0 \\ 0&0&0&0&1&1&0 \\ 0&0&0&0&0&1&1 \end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0&0&0&0\\1&1&0&0&0&0&1\\1&1&1&0&0&0&0\\1&0&1&1&0&0&0\\0&0&0&1&1&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&1&1\end{bmatrix} \quad \begin{bmatrix}1&0&0&0&0&0&0\\1&1&0&0&0&0&1\\1&1&1&0&0&0&0\\1&0&0&1&1&0&0\\0&0&1&0&1&0&0\\0&0&0&1&0&1&0\\0&0&0&0&0&1&1\end{bmatrix} \quad \begin{bmatrix}1&0&0&0&0&0&0\\1&1&0&0&0&0&1\\1&0&1&0&1&0&0\\1&0&0&1&0&1&0\\0&1&0&0&1&0&0\\0&0&1&0&0&1&0\\0&0&0&1&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&0&0&0\\1&1&1&1&0&0&0\\1&0&0&1&0&1&1\\0&1&0&0&1&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&0&0&0\\1&1&1&1&0&0&0\\1&0&0&1&0&1&0\\0&1&0&0&1&0&1\\0&0&0&0&1&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&0&0&0\\1&1&1&1&0&0&0\\1&0&0&1&0&1&0\\0&1&0&0&1&0&0\\0&0&0&0&1&1&1\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&0&0&0\\1&1&1&1&0&0&0\\1&0&0&1&0&0&1\\0&1&0&0&1&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&0&1&0&0&0\\1&1&0&0&0&0&0\\1&1&1&0&0&1&1\\1&0&1&1&0&0&0\\0&1&0&0&1&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&1&0&1&0&0&0\\1&1&0&0&0&0&0\\1&1&1&0&0&1&0\\1&0&1&1&0&0&0\\0&1&0&0&1&0&1\\0&0&0&0&1&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&0&1&0&0&0\\1&1&0&0&0&0&0\\1&1&1&0&0&1&0\\1&0&1&1&0&0&0\\0&1&0&0&1&0&0\\0&0&0&0&1&1&1\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&0&1&0&0&0\\1&1&0&0&0&0&0\\1&1&1&0&0&0&1\\1&0&1&1&0&0&0\\0&1&0&0&1&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&1&0&0\\1&1&0&0&0&0&0\\1&1&1&0&0&0&1\\1&0&1&1&0&0&0\\0&1&0&0&1&1&0\\0&0&0&1&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&1&0&0\\1&1&0&1&0&0&0\\1&1&1&0&0&0&0\\1&0&0&1&0&1&0\\0&1&0&0&1&0&1\\0&0&1&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&1&0&0&0&0&0\\1&1&0&1&0&0&0\\1&1&1&0&1&0&0\\1&0&0&1&0&1&0\\0&1&0&0&1&0&1\\0&0&1&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&1&0&0\\1&0&1&1&0&0&0\\1&0&0&1&0&1&0\\0&1&0&0&1&0&0\\0&1&0&0&0&1&1\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&0&0&0\\1&0&1&1&0&0&0\\1&0&0&1&1&1&0\\0&1&0&0&1&0&1\\0&1&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&0&0&0\\1&0&1&1&0&0&0\\1&0&0&1&1&0&0\\0&1&0&0&1&1&1\\0&1&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&0&0&0\\1&0&1&1&0&0&0\\1&0&0&1&1&0&0\\0&1&0&0&1&1&0\\0&1&0&0&0&1&1\\0&0&0&0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&0&0&0\\1&0&1&1&0&0&0\\1&0&0&1&1&0&0\\0&1&0&0&1&0&1\\0&1&0&0&0&1&0\\0&0&0&0&0&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&1&0&0\\1&1&1&0&0&0&0\\1&0&1&1&0&0&0\\1&0&0&1&0&1&0\\0&1&0&0&1&0&1\\0&1&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0&0\\1&1&1&0&0&0&0\\1&0&1&1&0&0&0\\1&0&0&1&1&0&0\\0&1&0&0&1&1&0\\0&1&0&0&0&1&1\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0&0\\1&1&1&0&0&0&0\\1&0&1&0&0&0&1\\1&0&0&1&0&0&0\\0&1&0&1&1&0&0\\0&1&0&0&1&1&0\\0&0&0&0&0&1&1\end{bmatrix}$$

$$\begin{bmatrix}1&1&1&0&1&0&0\\1&1&0&1&0&0&0\\1&1&1&0&0&0&0\\1&0&0&1&0&1&0\\0&1&1&0&1&0&1\\0&0&1&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&1&0&0\\1&1&0&1&0&0&0\\1&1&1&0&0&0&0\\1&0&0&1&0&0&1\\0&1&1&0&1&1&0\\0&0&1&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&1&0&0&0\\1&1&1&0&1&0&0\\1&0&0&1&0&0&1\\0&1&1&0&1&1&0\\0&0&1&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&0&0&0\\1&1&1&1&0&0&0\\1&0&0&1&0&0&1\\0&1&1&0&1&1&0\\0&0&1&0&0&1&0\\0&0&0&0&1&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&1&0&0\\1&0&1&0&0&1&0\\1&1&1&1&0&0&1\\0&1&0&1&1&0&0\\0&0&1&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&1&0&0\\1&0&1&0&0&0&0\\1&1&1&1&0&1&1\\0&1&0&1&1&0&0\\0&0&1&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&1&0&0\\1&0&1&0&0&0&0\\1&1&1&1&0&1&0\\0&1&0&1&1&0&0\\0&0&1&0&0&1&1\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&1&0&0\\1&0&1&0&0&0&0\\1&1&1&1&0&0&1\\0&1&0&1&1&0&0\\0&0&1&0&0&1&0\\0&0&0&0&0&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&1&0&0&0\\1&0&1&0&0&1&0\\1&1&0&1&1&0&0\\0&1&1&0&1&0&1\\0&0&1&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&1&0&0&0\\1&0&1&0&0&0&0\\1&1&0&1&1&0&0\\0&1&1&0&1&1&1\\0&0&1&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&1&0&0&0\\1&0&1&0&0&0&0\\1&1&0&1&1&0&0\\0&1&1&0&1&1&0\\0&0&1&0&0&1&1\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&1&0&0&0\\1&0&1&0&0&0&0\\1&1&0&1&1&0&0\\0&1&1&0&1&0&1\\0&0&1&0&0&1&0\\0&0&0&0&0&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&1&0&0\\1&1&0&0&0&0&0\\1&0&1&1&0&0&0\\1&1&0&1&0&0&1\\0&1&1&0&1&1&0\\0&0&1&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&0&0&0\\1&0&1&1&0&0&0\\1&1&0&1&0&0&1\\0&1&1&0&1&1&0\\0&0&1&0&0&1&0\\0&0&0&0&1&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&0&0&0&0\\1&1&0&0&0&0&0\\1&0&1&0&0&1&0\\1&1&0&1&0&0&1\\0&1&1&1&1&0&0\\0&0&1&0&1&1&0\\0&0&0&0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix} 1&1&1&0&0&0&0\\ 1&1&0&0&1&0&0\\ 1&0&1&0&0&0&1\\ 1&1&0&1&0&0&0\\ 0&1&0&0&1&0&0\\ 0&0&1&1&0&1&0\\ 0&0&1&0&0&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&0&0&0&0\\ 1&1&0&0&0&0&0\\ 1&0&1&0&0&0&1\\ 1&1&0&1&1&0&0\\ 0&1&0&0&1&0&0\\ 0&0&1&1&0&1&0\\ 0&0&1&0&0&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&0&0&0&1&0\\ 1&0&1&0&0&0&1\\ 1&0&0&1&0&0&0\\ 0&1&1&0&1&0&0\\ 0&1&0&0&1&1&0\\ 0&0&1&0&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&0\\ 1&0&0&1&0&0&1\\ 0&1&1&0&1&1&0\\ 0&1&0&0&0&1&0\\ 0&0&1&0&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&0&0&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&0\\ 1&0&0&1&0&0&1\\ 0&1&1&0&1&1&0\\ 0&1&0&0&0&1&0\\ 0&0&1&0&1&0&1 \end{bmatrix}$$

$$\begin{bmatrix} 1&1&1&0&0&0&0\\ 1&1&0&0&0&1&0\\ 1&0&1&0&0&0&1\\ 1&0&0&1&0&0&0\\ 0&1&1&1&1&0&0\\ 0&1&0&0&1&1&0\\ 0&0&1&0&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&0&0&0&0\\ 1&1&0&0&0&1&0\\ 1&0&1&0&0&0&0\\ 1&0&0&1&0&0&1\\ 0&1&1&1&1&0&0\\ 0&1&0&0&1&1&0\\ 0&0&1&0&0&0&1 \end{bmatrix}$$

$$\begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&1&0&0&0&0\\ 1&1&1&0&0&1&0\\ 1&0&0&1&0&0&1\\ 0&1&0&1&1&0&0\\ 0&0&1&0&0&1&0\\ 0&0&0&1&1&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&0&0&1&0&0\\ 1&1&1&0&0&0&0\\ 1&0&1&1&0&1&0\\ 0&1&0&0&1&0&0\\ 0&0&1&1&0&1&1\\ 0&0&0&1&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&0&0&1&0&0\\ 1&1&1&0&0&0&0\\ 1&0&0&1&0&0&1\\ 0&1&1&0&1&0&0\\ 0&0&1&1&0&1&0\\ 0&0&0&1&0&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&0&0&0&0&0\\ 1&1&1&0&1&0&0\\ 1&0&0&1&0&0&1\\ 0&1&1&0&1&1&0\\ 0&0&1&1&0&1&0\\ 0&0&0&1&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&0&0&0&0&0\\ 1&1&1&0&1&0&0\\ 1&0&0&1&0&0&1\\ 0&1&1&0&1&0&0\\ 0&0&1&1&0&1&0\\ 0&0&0&1&0&1&1 \end{bmatrix}$$

$$\begin{bmatrix} 1&1&1&1&1&0&0\\ 1&1&0&0&0&0&0\\ 1&0&1&0&0&1&0\\ 0&1&1&1&0&0&1\\ 1&1&0&1&1&0&0\\ 0&0&1&1&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&0&0&0&0&0\\ 1&0&1&1&0&1&0\\ 0&1&0&1&0&0&1\\ 1&1&1&0&1&0&0\\ 0&0&1&1&1&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&0&0&0&0&0\\ 1&0&1&1&0&1&0\\ 0&1&0&1&0&0&0\\ 1&1&1&0&1&0&1\\ 0&0&1&1&1&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&0&0&0&0&0\\ 1&0&1&1&1&0&0\\ 0&1&0&1&0&0&1\\ 1&0&1&1&1&1&0\\ 0&1&1&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&0&0&0&0&0\\ 1&0&1&1&1&0&0\\ 0&1&0&1&0&0&0\\ 1&0&1&1&1&1&0\\ 0&1&1&0&0&1&1\\ 0&0&0&0&0&0&1 \end{bmatrix}$$

$$\begin{bmatrix} 1&1&1&1&1&0&0\\ 1&1&1&0&0&0&0\\ 1&1&1&0&1&1&0\\ 1&0&0&1&0&0&1\\ 0&1&0&1&1&0&0\\ 0&0&1&0&1&1&0\\ 0&0&0&1&0&0&1 \end{bmatrix}$$

By using the information in §2 and Table 3.1, given $\mu \in \mathbb{N}$, it is a simple matter to compute d_μ, once known the different partitions of μ as a sum of squares. As an example, we will find explicitly the multiplicities of some small eigenvalues in the case of the largest isospectral family (42=43=45=51) in Table 3.1. We shall use the equations in §2 relating the d_μ's with the $m_{k,l}$'s, formulas (2.5) and (2.6) and the information in Table 3.1. We have that:

$$\begin{gathered} d_0 = 1; \quad d_1 = m_{1,1} = 0; \quad d_2 = m_{2,2} = 0; \quad d_3 = m_{3,3} = 2; \\ d_4 = m_{4,4} + m_{2,0} = 5 + 7 = 12; \quad d_5 = m_{5,5} + m_{2,1} = 10 + 4 = 14; \\ d_6 = m_{6,6} + m_{3,2} = 7 + 9 = 16; \quad d_7 = m_{7,7} + m_{4,3} = 2 + 30 = 32; \\ d_{17} = m_{2,1} + 5m_{6,5} + m_{3,1} + m_{5,1} = 4 + 10\binom{7}{5} + 20 + 102 = 336. \end{gathered}$$

We next give a table which lists, for each one of the 62 matrices in the given order, the values of the 15 variables $m_{k,l}$, given in the same ordering as before. We also include the numbers $c_{s,t}$, for $(s,t) = (3,1); (3,2); (3,3); (5,1); \ldots; (5,5)$.

Table 3.1. Spectral Information.

Matrix	$m_{k,l}$ $1 \le l \le k \le 5$															$c_{s,t}$ $1 \le t \le s = 3, 5$							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	0	0	0	7	14	7	7	14	49	35	42	84	70	98	70	7	21	7	7	14	0	0	0
2	0	1	1	4	8	7	12	20	43	35	51	90	70	104	70	8	19	8	10	8	3	0	0
3	0	1	1	4	8	6	12	24	43	30	51	94	62	104	66	12	15	8	9	7	4	1	0
4	0	0	1	7	10	6	7	22	49	30	42	92	62	98	66	11	17	7	7	10	4	0	0
5	0	1	1	3	10	6	13	22	41	30	54	92	62	106	66	11	17	7	9	8	2	2	0
6	0	1	2	3	6	6	13	26	41	30	54	96	62	106	66	11	17	7	10	5	5	1	0
7	0	0	2	6	8	6	8	24	47	30	45	94	62	100	66	10	19	6	8	8	5	0	0
8	0	0	4	4	4	6	10	28	43	30	51	98	62	104	66	8	23	4	10	4	7	0	0
9	0	0	2	6	8	6	8	24	47	30	45	94	62	100	66	10	19	6	8	8	5	0	0
10	0	0	0	8	12	6	6	20	51	30	39	90	62	96	66	12	15	8	6	12	3	0	0
11	0	1	2	2	8	5	14	28	39	25	57	98	54	108	62	14	15	6	9	5	4	3	0
12	0	0	2	5	10	5	9	26	45	25	48	96	54	102	62	13	17	5	7	8	4	2	0
13	0	1	1	5	6	5	15	22	45	25	60	92	54	110	62	13	17	5	9	5	5	1	1
14	0	0	3	4	8	5	10	28	43	25	51	98	54	104	62	12	19	4	8	6	5	2	0
15	0	1	1	5	6	5	11	30	45	25	48	100	54	102	62	17	9	9	8	5	7	1	0
16	0	0	1	7	10	5	7	26	49	25	42	96	54	98	62	15	13	7	6	9	5	1	0
17	0	1	1	3	10	5	13	26	41	25	54	96	54	106	62	15	13	7	8	7	3	3	0
18	0	0	2	6	8	5	8	28	47	25	45	98	54	100	62	14	15	6	7	7	6	1	0
19	0	0	1	6	12	5	8	24	47	25	45	94	54	100	62	14	15	6	6	10	3	2	0
20	0	0	1	7	10	5	7	26	49	25	42	96	54	98	62	15	13	7	6	9	5	1	0
21	0	0	2	6	8	5	8	28	47	25	45	98	54	100	62	14	15	6	7	7	6	1	0
22	0	0	1	7	10	5	7	26	49	25	42	96	54	98	62	15	13	7	6	9	5	1	0
23	0	1	1	3	10	5	13	26	41	25	54	96	54	106	62	15	13	7	8	7	3	3	0
24	0	1	1	5	6	5	15	22	45	25	60	92	54	110	62	13	17	5	9	5	5	1	1
25	0	0	2	4	12	5	10	24	43	25	51	94	54	104	62	12	19	4	7	9	2	3	0
26	0	0	3	5	6	5	9	30	45	25	48	100	54	102	62	13	17	5	8	5	7	1	0
27	0	0	2	5	10	5	9	26	45	25	48	96	54	102	62	13	17	5	7	8	4	2	0
28	0	0	3	3	10	5	11	26	41	25	54	96	54	106	62	11	21	3	8	7	3	3	0
29	0	0	3	4	8	5	10	28	43	25	51	98	54	104	62	12	19	4	8	6	5	2	0
30	0	0	0	8	12	4	10	20	51	20	51	90	46	104	58	16	15	4	5	10	3	2	1
31	0	0	2	6	8	4	12	24	47	20	57	94	46	108	58	14	19	2	7	6	5	2	1
32	0	0	2	7	6	4	11	26	49	20	54	96	46	106	58	15	17	3	7	5	7	1	1
33	0	0	2	7	6	4	11	26	49	20	54	96	46	106	58	15	17	3	7	5	7	1	1
34	0	0	0	9	10	4	5	30	53	20	36	100	46	94	58	21	5	9	4	9	7	1	0
35	0	1	0	5	10	4	11	30	45	20	48	100	46	102	58	21	5	9	6	7	5	3	0
36	0	0	1	6	12	4	8	28	47	20	45	98	46	100	58	18	11	6	5	9	4	3	0
37	0	0	2	7	6	4	7	34	49	20	42	104	46	98	58	19	9	7	6	5	9	1	0
38	0	0	1	6	12	4	8	28	47	20	45	98	46	100	58	18	11	6	5	9	4	3	0

Matrix	$m_{k,l} \quad 1 \leq l \leq k \leq 5$															$c_{s,t} \quad 1 \leq t \leq s = 3, 5$							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
39	0	1	0	5	10	4	15	22	45	20	60	92	46	110	58	17	13	5	7	7	3	3	1
40	0	0	1	7	10	4	11	22	49	20	54	92	46	106	58	15	17	3	6	8	4	2	1
41	0	0	3	4	8	4	10	32	43	20	51	102	46	104	58	16	15	4	7	5	6	3	0
42	0	0	2	5	10	4	9	30	45	20	48	100	46	102	58	17	13	5	6	7	5	3	0
43	0	0	2	5	10	4	9	30	45	20	48	100	46	102	58	17	13	5	6	7	5	3	0
44	0	0	0	9	10	4	9	22	53	20	48	92	46	102	58	17	13	5	5	9	5	1	1
45	0	0	2	5	10	4	9	30	45	20	48	100	46	102	58	17	13	5	6	7	5	3	0
46	0	0	2	7	6	4	11	26	49	20	54	96	46	106	58	15	17	3	7	5	7	1	1
47	0	0	2	7	6	4	7	34	49	20	42	104	46	98	58	19	9	7	6	5	9	1	0
48	0	0	1	7	10	4	11	22	49	20	54	92	46	106	58	15	17	3	6	8	4	2	1
49	0	0	2	6	8	4	12	24	47	20	57	94	46	108	58	14	19	2	7	6	5	2	1
50	0	0	1	6	12	4	8	28	47	20	45	98	46	100	58	18	11	6	5	9	4	3	0
51	0	0	2	5	10	4	9	30	45	20	48	100	46	102	58	17	13	5	6	7	5	3	0
52	0	0	1	9	6	3	9	30	53	15	48	100	38	102	54	21	9	5	5	5	9	1	1
53	0	0	1	7	10	3	11	26	49	15	54	96	38	106	54	19	13	3	5	7	5	3	1
54	0	0	3	3	10	3	11	34	41	15	54	104	38	106	54	19	13	3	6	5	5	5	0
55	0	0	1	7	10	3	11	26	49	15	54	96	38	106	54	19	13	3	5	7	5	3	1
56	0	0	2	6	8	3	12	28	47	15	57	98	38	108	54	18	15	2	6	5	6	3	1
57	0	0	1	7	10	3	7	34	49	15	42	104	38	98	54	23	5	7	4	7	7	3	0
58	0	0	1	7	10	3	7	34	49	15	42	104	38	98	54	23	5	7	4	7	7	3	0
59	0	0	1	7	10	3	7	34	49	15	42	104	38	98	54	23	5	7	4	7	7	3	0
60	0	0	0	8	12	3	10	24	51	15	51	94	38	104	54	20	11	4	4	9	4	3	1
61	0	0	1	7	10	3	11	26	49	15	54	96	38	106	54	19	13	3	5	7	5	3	1
62	0	0	0	9	10	2	9	30	53	10	48	100	30	102	50	25	5	5	3	7	7	3	1

The table shows that manifolds with different column patterns are not isospectral to each other. Furthermore, we have isospectral families, for each column pattern (according to the numbering in the table) as follows: 7=9; 12=27; 13=24; 14=29; 17=23; 18=21; 31=49; 37=47; 40=48; 16=20=22; 32=33=46; 36=38=50; 53=55=61; 57=58=59; 42=43=45=51. Summing up, there are 9 pairs, 5 triples, 1 family of 4 manifolds, and 25 isolated manifolds.

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