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THE HIGHER SPIN DIRAC OPERATORS

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ABSTRACT. There is a certain family of conformally invariant first order elliptic systems which include the Dirac operator as its first and simplest member. Their general definition is given and some of their basic properties are described. A special attention is paid to the Rarita-Schwinger operator, the second simplest operator in the row. Its basic properties are described in more details. In the last part indices of discussed operators are computed.

1. Introduction

Let M be a smooth oriented compact n-dimensional manifold, endowed with a Riemannian metric and a spin structure. A lot of information has been collected concerning basic invariant differential operators on M as the Laplace (of the second order) and the Dirac operator (of the first order). These invariant operators and their properties are strongly related with geometry and topology of M.

Their behavior under conformal change of metric allowed us to study their conformal invariance and put it into general scheme of invariant operators on AHS manifolds. There is a list of conformal invariant operators of different order, all of them are defined on any Riemannian spin manifold. For this operators similar notions can be defined and properties and relations as for basic operators (Dirac and Laplace) can be studied.

Recently, a growing interest is paid to properties of more complicated invariant first order differential operators on M. A prototype of them is the Rarita-Schwinger operator (see [Fra1, Fra2, FraS, MP, Pe1, Pe2, Pe3, Pe4, Pe5, RaS, US, Wa]). It acts on sections of the bundle associated to a more complicated representation of the group $\mathrm{Spin}(n)$.

It it also included in a series of operators on M which are called higher spin Dirac operators and also in a series of operators called higher Rarita-Schwinger operators which are defined on spin valued symmetric tensor fields and has been intensively used also in Clifford analysis in connection with monogenic differential forms (see

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[DSS, So1, So2, SoS, So3]) or symmetric functions (see [Som1]). In general, there is a question which new geometrical and topological characterizations of Riemannian spin manifold can be obtained from properties of this additional invariant operators on it. To try to see an answer, it is first necessary to learn more about properties of these operators. The aim of the paper is to collect and review facts which exist mostly only in a preprint form and to add some new facts (index computations). In the contribution, the operators are defined including their normalisation and some of their properties are described. Main attention is concentrated to the Rarita-Schwinger operator. Basic questions discussed in the paper are:

1. The description of conformally invariant first order differential operators (including the Rarita-Schwinger operator).
2. The spectrum of Rarita Schwinger operator on the flat model, i.e. on spheres.
3. A complete description of polynomial solutions of the Rarita-Schwinger equation.
4. The index of Rarita Schwinger operator and higher spin Dirac operators.

The first three parts have a review character, most results there are taken from papers which are at present only in a preprint form, the index properties are new results.

2. First order conformally invariant operators

There is a scheme for a construction of conformal invariant operators let us recall it shortly from ([BuSo]).

Let M be a compact oriented spin manifold with a conformal structure. Fix a Riemannian metric g in the given conformal class then we have on M principal fibre bundles $\tilde{\mathcal{P}} \equiv \tilde{\mathcal{P}}_{\mathrm{Spin}} \to \mathcal{P}_{SO} \to M$.

Finite-dimensional irreducible representations $\mathbf{V}_\lambda$ of the group $\mathrm{Spin}(n)$ are determined by their highest weights $\lambda \in \Lambda^+$, where for $n = 2k$ even, we have

$$\Lambda^+ = \{\lambda = (\lambda_1, ..., \lambda_k); \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{k-1} \geq |\lambda_k|\}, \lambda_i \in \mathbf{Z} \cup \frac{1}{2}\mathbf{Z}$$

and for $n = 2k+1$ odd, we have

$$\Lambda^+ = \{\lambda = (\lambda_1, ..., \lambda_k); \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{k-1} \geq \lambda_k \geq 0\}, \lambda_i \in \mathbf{Z} \cup \frac{1}{2}\mathbf{Z}\,.$$

Invariant operators are acting among spaces of sections of the corresponding associated bundles $V_\lambda = \tilde{\mathcal{P}} \times_{\mathrm{Spin}(n)} \mathbf{V}_\lambda$ over M. Let us consider the Levi-Civita connection ω of the chosen Riemannian metric on $\mathcal{P}$ and let $\tilde{\omega}$ be its (unique) lift to $\tilde{\mathcal{P}}$. For any choice of $\lambda \in \Lambda^+$, we have the associated covariant derivative $\nabla_\lambda : \Gamma(V_\lambda) \to \Gamma(V_\lambda \otimes T^*(M))$. Tensor product $\mathbf{V}_\lambda \otimes \mathbf{C}_n$ can be decomposed into irreducible components

$$\mathbf{V}_\lambda \otimes \mathbf{C}_n = \oplus_{\lambda' \in A} \mathbf{V}_{\lambda'}\,,$$

where A is the set of highest weights of all irreducible components (multiplicities included). There are simple rules how to describe $A = A(\lambda)$ explicitely for any λ

(see [F]). Let $\pi_{\lambda'}$ be the projection from $\mathbf{V}_\lambda \otimes \mathbf{C}_n$ to $\mathbf{V}_{\lambda'}$. Then operators

$$D_{\lambda,\lambda'} : \Gamma(V_\lambda) \to \Gamma(V_{\lambda'}) \, , \; D_{\lambda,\lambda'} := \pi_{\lambda'} \circ \nabla^\lambda$$

are first order conformally invariant differential operators and all such operators can be constructed in this way.

Any conformally invariant first order differential operator is uniquely determined (up to a constant multiple) by a choice of allowed λ and λ' but there is no natural normalization in general. To study spectral properties, it is necessary fix a scale of the operator, to choose appropriate normalization. For the Dirac operator, the choice of normalization is given by the Clifford action. By using twisted Dirac operators, we shall extend this normalization to a wide class of first order operators (which includes our higher spin Dirac perators as well as higher Rarita-Schwinger operators).

Definition 1. [BuSo] *Let $\mathbf{S}_{\frac{1}{2}}$ (for $n = 2k+1$), resp. $\mathbf{S}_{\frac{1}{2}} = \mathbf{S}^+_{\frac{1}{2}} \oplus \mathbf{S}^-_{\frac{1}{2}}$ (for $n = 2k$), denote the basic spinor representations with highest weights $\sigma = (\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2})$, resp. $\sigma^\pm = (\frac{1}{2}, \dots, \frac{1}{2}, \pm\frac{1}{2})$.*

Let $\lambda \in \Lambda^+$, (for $n = 2k+1$), resp. $\lambda^\pm \in \Lambda^+$ (for $n = 2k$) be dominant weights with $\lambda = (\lambda_1, ..., \lambda_{k-1}, \frac{1}{2})$, resp. $\lambda^\pm = (\lambda_1, ..., \lambda_{k-1}, \pm\frac{1}{2})$. Denote further $\lambda' = \lambda - \sigma \in \Lambda^+$, resp. $\lambda' = \lambda^+ - \sigma^+ \in \Lambda^+$. In even dimensions, we shall use the notation $\mathbf{V}_\lambda = \mathbf{V}_{\lambda^+} \oplus \mathbf{V}_{\lambda^-}$.

The representation $\mathbf{V}_\lambda$ appears with multiplicity one in the decomposition of the tensor product $\mathbf{S}_{\frac{1}{2}} \otimes \mathbf{V}_{\lambda'}$ (it is the Cartan product of both representations). Hence we can write the product as $\mathbf{S}_{\frac{1}{2}} \otimes \mathbf{V}_{\lambda'} = \mathbf{V}_\lambda \oplus \mathbf{W}$ where $\mathbf{W}$ is the sum of all other irreducible components in the decomposition.

Let $D^T_{\lambda'}$ be the twisted Dirac operator on $S_{\frac{1}{2}} \otimes V_{\lambda'}$. If we write the operator $D^T_{\lambda'}$ in the block form as

$$\begin{tikzcd}
\Gamma(S_{\frac{1}{2}} \otimes V_{\lambda'}) \arrow[r, "D^T_{\lambda'}"] \arrow[d, equal] & \Gamma(S_{\frac{1}{2}} \otimes V_{\lambda'}) \arrow[d, equal] \\
\Gamma(V_\lambda) \oplus \Gamma(W) & \Gamma(V_\lambda) \oplus \Gamma(W)
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma(V_\lambda) \arrow[r, "D_\lambda"] \arrow[dr] & \Gamma(V_\lambda) \\
\Gamma(W) \arrow[r] \arrow[ur] & \Gamma(W)
\end{tikzcd}$$

we have defined four invariant operators, one of them being the operator

$$D_\lambda : \Gamma(V_\lambda) \to \Gamma(V_\lambda) \, .$$

Operators D_λ defined in such a way will be called generalized (higher spin) Dirac operators.

The case of Rarita-Schwinger operator is included at the beginning of the scheme, as follows.

A certain subclass of invariant operators discussed above is related with the following higher dimensional generalizations of holomorphic differential forms (see [DSS, So2]). Spinor valued differential forms are elements of the twisted de Rham sequence ([BuSo]):

$$\Gamma(S^{\pm}_{\frac{1}{2}}) \overset{\nabla^S}{\to} \dots \Gamma(\Omega^k_c \otimes S^{\pm}_{\frac{1}{2}}) \overset{\nabla^S}{\to} \dots \overset{\nabla^S}{\to} \Gamma(\Omega^n_c \otimes S^{\pm}_{\frac{1}{2}})$$

where ∇^S denotes the associated covariant derivative on spinor bundles extended to $S_{\frac{1}{2}}$-valued forms (see [So2, VSe]).

Every representation $\Lambda^k(\mathbf{C}_n) \otimes \mathbf{S}_{\frac{1}{2}}$ can be split into irreducible pieces. There are no multiplicities in the decomposition, so the irreducible pieces are well defined. For k forms ($k \leq [n/2]$), there are k pieces in the decomposition and the decomposition is symmetric with respect to the action of the Hodge star operator. The space of spinor valued k-forms $\Gamma(\Omega^k_c \otimes S^{\pm})$ ($k \leq [n/2]$) can be written as the sum $\oplus_{j=1}^k E^{k,j}$ and it can be checked (see [DSS, VSe, So2]) that $E^{k,j}$ is the bundle associated with the representation with the highest weight $\lambda_j = (\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \pm\frac{1}{2})$, where the number j indicates that the component $\frac{3}{2}$ appears with multiplicity equal to j. Signs $\pm$ at the last components are relevant only in even dimensions (more details can be found in [VSe]). The whole splitting can be described by the following triangle shaped diagram (in odd dimensions, there are two columns of the same length in the middle).

$$\begin{array}{ccccccccccccc}
E^{0,0} & \xrightarrow{D_0} & E^{1,0} & \xrightarrow{D_0} & \dots & \xrightarrow{D_0} & E^{k,0} & \xrightarrow{D_0} & \dots & \xrightarrow{D_0} & E^{2k-1,0} & \xrightarrow{D_0} & E^{2k,0} \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
 & & E^{1,1} & \xrightarrow{D_1} & \dots & \xrightarrow{D_1} & E^{k,1} & \xrightarrow{D_1} & \dots & \xrightarrow{D_1} & E^{2k-1,1} & & \\
 & & & & \oplus & & \oplus & & \oplus & & & & \\
 & & & & \dots & \xrightarrow{D_j} & \vdots & \xrightarrow{D_j} & \dots & & & & \\
 & & & & & & \oplus & & & & & & \\
 & & & & & & E^{k,k} & & & & & &
\end{array}$$

The general construction of invariant operators described above can be used in the special case of spinor valued forms. The covariant derivative ∇^S restricted to $E^{k,j}$ and projected to $E^{k+1,j'}$ is an example of this general construction. It can be shown that if $|j - j'| > 1$, then the corresponding invariant operator is trivial. We shall be mainly interested in 'horizontal arrows', i.e. in operators D_j given by restriction to $E^{k,j}$ and projection to $E^{k+1,j}$. They are indicated in the above scheme.

The simplest cases among them are well known. The operator D_0 is (a multiple of) the Dirac operator.

The operator D_1 is (an elliptic version of) the operator called Rarita-Schwinger operator by physicists (see [RaS, Wa]).

All of them are elliptic operators (see [So1, Bra]).

Note that all operators D_j on the same row in the scheme above cannot be identified without further comments. To compare them, it is necessary first to choose an equivariant isomorphism among corresponding bundles. Then they coincide up to a constant multiple.

To compare the operators D_j in the above scheme with the higher spin Dirac operators (see Def.1), we shall choose a certain identification of the corresponding source and target bundles. We shall do it for the first operator D_j in the row.

Let us characterize an algebraic operator $Y : \Gamma(\Omega_c^{k+1} \otimes S_{\frac{1}{2}}) \to \Gamma(\Omega_c^k \otimes S_{\frac{1}{2}})$ by a local formula

$$Y(\omega \otimes s) = -\sum_i \iota(e_i)\omega \otimes e_i \cdot s\,,$$

where $\{e_i\}$ is a (local) orthonormal basis of TM and ι denotes the contraction of a differential form by a vector. As shown in [VSe], the map $Y : E^{k+1,j} \to E^{k,j}, j < k < [n/2]$ is an isomorphism.

The twisted Dirac operator D^T maps the space $\Gamma(\Omega_c^k \otimes S_{\frac{1}{2}})$ to itself. In [VSe], it was proved that we have a relation $\nabla \circ Y + Y \circ \nabla = -D^T$. Let us denote the projection from $\Omega_c^k \otimes S$ onto $E^{k,j}$ by $\pi_{k,j}$. Symbols $\tilde{D}_j$, $0 \leq j < [n/2]$ will denote operators

$$\tilde{D}_j = Y \circ D_j = \pi_{j,j} \circ Y \circ \nabla^S|_{E^{j,j}}\,,$$

mapping the space of sections of $E^{j,j}$ to itself. Then $Y|_{E^{j,j}} = 0$ implies that

$$\tilde{D}_j = \pi_{j,j} \circ Y \circ \nabla^S|_{E^{j,j}} = -\pi_{j,j} \circ D^T|_{E^{j,j}} = -D_{\lambda_j}\,,$$

where D_{λ_j} is the generalized higher spin Dirac operator corresponding to the bundle V_{λ_j}, $\lambda_j = (\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ (component $\frac{3}{2}$ appearing j times). More precisely, there are no signs in odd dimensions, while in even dimension, $V_{\lambda_j} = V_{\lambda_j^+} \oplus V_{\lambda_j^-}$.

3. Definition and general properties of Rarita-Schwinger operator

In this section, the attention is concentrated to the simplest special case of the general definition given in above, i.e. to the Rarita-Schwinger operator. This operator is mentioned in [RaS] and was studied in more details in the paper by M.Y.Wang ([Wa]) and the dissertation of U. Semmelman ([US]) and also in the physical context e.g in [Fra1, E, Pe1, Pe2, Pe3]. It is also related to the Stein-Weiss operators studied by Branson ([Bra]).

In the paper of Wang, it is studied mainly on compact Einstein spin manifolds admitting nonzero Killing spinors and in the context of deformation of Einstein metrics on manifolds.

In the Dissertation of U. Semmelmann, the results of Wang are extended and also some eigenvalue problems as the relations between eigenvalues of the Dirac operator D, the twisted Dirac operator D_T and Rarita-Schwinger operator $D_{\frac{3}{2}}$ are added.

Let us first describe (a slight modification of) the Wang definition and his decription of the basic properties of operators involved ([Wa]).

We shall use in the following section the notation

$$\mathbf{S}_{\frac{k}{2}} = V_\lambda \quad \text{for } \lambda = (\frac{k}{2}, \frac{1}{2}, ..., \frac{1}{2}) \,.$$

Let $\mathbf{e} = \{e_1, ..., e_n\}$ be an orthogonal frame on some open set on M and $\{\epsilon^1, ..., \epsilon^n\}$ the dual coframe.

Remark 1. The classical Dirac operator on M is the operator

$$D : \Gamma(S_{\frac{1}{2}}) \to \Gamma(S_{\frac{1}{2}})$$

and has with respect to frame $\mathbf{e}$ the form

$$D(\sigma) = \sum_i e_i \, \nabla^S_{e_i} \sigma \,.$$

There is a fundamental diagram, which presents the relation among operators in discussion.

From the theory of representation of the group $\mathrm{Spin}(n)$ we have a decomposition of the tensor product

$$\mathbf{S}_{\frac{1}{2}} \otimes \mathbf{C}^{n*} = \mathbf{S}'_{\frac{1}{2}} \oplus \mathbf{S}_{\frac{3}{2}}$$

into irreducible $\mathrm{Spin}(n)$-modules, where $\mathbf{S}'_{\frac{1}{2}}$ is isomorphic to $\mathbf{S}_{\frac{1}{2}}$.

We shall use the following canonical identifications

$$\iota : \mathbf{S}_{\frac{1}{2}} \to \mathbf{S}_{\frac{1}{2}} \otimes \mathbf{C}^{n*}$$

$$\iota(\sigma) := -\frac{1}{n} \sum_i e_i \; \sigma \otimes \epsilon^i$$

which is the isomorphisms onto $\mathbf{S}'_{\frac{1}{2}}$ and $\mathbf{S}_{\frac{3}{2}} \equiv \mathrm{Ker}(\mu)$ where

$$\mu : \mathbf{S}_{\frac{3}{2}} \otimes \mathbf{C}^{n*} \to \mathbf{S}_{\frac{1}{2}}$$

is the Clifford multiplication

$$\mu(\sum_i \psi_i \otimes \epsilon^i) = \sum_i e_i \; \psi_i \,.$$

Moreover we have the projections $\pi_{\frac{1}{2}}, \pi_{\frac{3}{2}}$ from $\mathbf{S}_{\frac{3}{2}} \otimes \mathbf{C}^{n*}$ onto the individual irreducible components given by

$$\pi_{\frac{1}{2}}(\sum_i \psi_i \otimes \epsilon^i) = -\frac{1}{n} \sum_j e_j \; (\sum_i e_i \; \psi_i) \otimes \epsilon^j$$

and

$$\pi_{\frac{3}{2}}(\sum_i \psi_i \otimes \epsilon^i) = \sum_j (\psi_j + \frac{1}{n} e_j \; \sum_i e_i \; \psi_i) \otimes \epsilon^j$$

For the corresponding operators we get

$$\begin{tikzcd}
\Gamma(S_{\frac{1}{2}}) \arrow[r, "\nabla"] \arrow[ddr, "\mathcal{D}"] \arrow[ddddr, "\mathcal{T}"'] & \Gamma(S_{\frac{1}{2}} \otimes T_C^*) \arrow[r, "D_T"] & \Gamma(S_{\frac{1}{2}} \otimes T_C^*) \\
& \| & \| \\
& \Gamma(S_{\frac{1}{2}}) \arrow[r, "\mathcal{D}'"] \arrow[ddr, "\mathcal{T}'"] & \Gamma(S_{\frac{1}{2}}) \\
& \oplus & \oplus \\
& \Gamma(S_{\frac{3}{2}}) \arrow[r, "\mathcal{R}"] \arrow[uur, "\mathcal{T}^*"] & \Gamma(S_{\frac{3}{2}})
\end{tikzcd}$$

with

$$\nabla(\sigma) = \sum_i \nabla^S_{e_i} \sigma \otimes \epsilon^i$$

and the (transported) Dirac operator

$$\mathcal{D}(\sigma) := \pi_{\frac{1}{2}}(\nabla^S \sigma) = -\frac{1}{n} \sum_i e_i \ D\sigma \otimes \epsilon^i = \iota(D(\sigma))$$

the Twistor operator

$$\mathcal{T}(\sigma) := \pi_{\frac{3}{2}}(\nabla^S \sigma) = \sum_j (\nabla^S_{e_j} \sigma + \frac{1}{n} e_j \ D\sigma) \otimes \epsilon^j$$

and another operators which with respect to the above identifications are

$$\mathcal{D}' = -\frac{n-2}{n} \iota \circ D \circ \iota^{-1}$$

$$\mathcal{T}' = \frac{2}{n} \mathcal{T} \circ \iota^{-1}$$

and if we denote the operator

$$\delta : \Gamma(S_{\frac{3}{2}}) \to \Gamma(S_{\frac{1}{2}})$$

by

$$\delta(\sum_i \psi_i \otimes \epsilon^i) = -\sum_i \nabla_{e_i} \psi_i$$

then we get

$$\mathcal{T}^*(\psi) = 2.\iota(\delta(\psi))\,.$$

Theorem 1. ([Wa]) *The operator D_T with respect to the decomposition*

$$\Gamma(S_{\frac{1}{2}} \otimes T^*_C) = \Gamma(S_{\frac{1}{2}}) \oplus \Gamma(S_{\frac{3}{2}})$$

has the following form

$$\begin{pmatrix} \frac{2-n}{n}\iota \circ D \circ \iota^{-1} & 2.\iota \circ \delta \\ \frac{2}{n}(\mathcal{T} \circ \iota^{-1}) & \mathcal{R} \end{pmatrix}$$

4. Eigenvalues on spheres

The spectrum of the Dirac operator on the sphere is well known for some time already (see [Baer]). The spectra of operators $\tilde{D}_j$ on sphere were computed in ([BuSo]), the case of a general elliptic first order operators can be found in [Br1].

Lemma 1. *The eigenvalues of the Dirac operator on the sphere S_n with standard metric are*

$$\mu_l = \pm\left(\frac{n}{2} + l\right); \quad l = 0, 1, 2, \dots.$$

with multiplicity

$$2^{[\frac{n}{2}]}\binom{l+n-1}{l}.$$

The main result of the paper [BuSo] is given in the following theorem.

Theorem 2. *Let $D_{\lambda_j} = -\tilde{D}_j$, $0 < j < n/2$, be the higher spin Dirac operators defined above, considered on the sphere S_n with the standard metric. Then their eigenvalues are:*

$$\mu^1_l = \pm\left(\frac{n}{2} + l\right); \quad l = 1, 2, \dots.$$

with multiplicity

$$2^{[\frac{n}{2}]}\binom{n+1}{j+1}\binom{l+n}{l-1}\frac{(n-2j)(j+1)}{(l+j)(l+n-j)}$$

and

$$\mu^2_l = \pm\left[\frac{n-2j}{n-2j+2}\left(\frac{n}{2} + l\right)\right]; \quad l = 1, 2, \dots.$$

with multiplicity

$$2^{[\frac{n}{2}]}\binom{n+1}{j}\binom{l+n}{l-1}\frac{(n-2j+2)j}{(l+j-1)(l+n-j+1)}.$$

It is proved using modification of results of Branson et others in [BOO].

The Rarita-Schwinger operator is the operator $\tilde{D}_1$ and its eigenvalues are:

$$\mu^1_l = \pm\left(\frac{n}{2} + l\right); \quad l = 1, 2, \dots.$$

with multiplicity

$$2^{[\frac{n}{2}]}\binom{n+1}{2}\binom{l+n}{l-1}\frac{2(n-2)}{(l+1)(l+n-1)}$$

and

$$\mu_l^2 = \pm\left[\frac{n-2}{n}\left(\frac{n}{2}+l\right)\right]\ ;\ l=1,2,\ldots .$$

with multiplicity

$$2^{[\frac{n}{2}]}(n+1)\binom{l+n}{l-1}\frac{(n)}{(l)(l+n)} .$$

The second series of eigenspaces on sphere comes out as the image by Twistor operator of the eigenspaces of the Dirac operator on sphere.

5. Homogeneous solutions of Rarita-Schwinger equation on $\mathbf{R_0^{n+1}}$

The solutions of the Dirac equation on the flat space are studied traditionally in Clifford analysis. Special attention is paid to the solutions on $\mathbf{R_0^{n+1}} := \mathbf{R}^{n+1} - \{0\}$ which are polynomial (homogeneous) of some degree k, namely to the functions

$$\phi : \mathbf{R_0^{n+1}} \to \mathbf{S}_{\frac{1}{2}}$$

which satisfy

(1) $D\phi = 0$

(2) $\phi(\lambda x) = \lambda^k \phi(x)$ for all $x \in \mathbf{R_0^{n+1}}$.

Such solutions are strongly related with the eigenfunctions of the Dirac operator on the unit sphere $S^n \subset \mathbf{R_0^{n+1}}$, the relation follows simply from the restriction of solution to the sphere. Such functions are also called spherical monogenics (of degree k).

We make a simple generalization of the problem, instead of the Dirac operator we shall study the Rarita-Schwinger operator, which is define on some special spinor-valued one forms on $\mathbf{R_0^{n+1}}$and is the next operator in the serie of first order elliptic invariant operators on Riemannian spin manifolds.

We would like again to study homogenic solutions of the Rarita-Schwinger operator on $\mathbf{R_0^{n+1}}$and their relations with the eigenfunctions of the corresponding operators (Dirac and the Rarita-Schwinger) on the sphere S^n. Of course the situation is more complicated here.

There is a possibility to study the same problem for other higher spin operators on spinor valued forms or higher spin Rarita-Schwinger operators.

Let me mention only the main results, the full description of results and another related topics will be presented in fortcoming paper [Bu], see also [BuLN].

We shall use on $\mathbf{R_0^{n+1}}$fixed cartesian coordinates $x = (x_1, ..., x_{n+1})$ and also:

$$e_i = \frac{\partial}{\partial x_i}\ ,\ x = \sum_i x_i e_i \in \mathbf{R}^n\ ,$$

and we have

$$\epsilon^i = dx^i\ ,\ \nabla_{e_i} = \frac{\partial}{\partial x_i} .$$

Let us take

$$\psi = \sum_{1}^{n+1} \psi_i \otimes dx^i \in \mathbf{S}_{\frac{1}{2}} \otimes \mathbf{R}^{n*}$$

and suppose that

$$\psi \in \Gamma(S_{\frac{3}{2}}) (\iff \mu(\psi) = \sum_i e_i \ \psi_i = 0) .$$

Then the Rarita-Schwinger operator $\mathcal{R}$ on ψ has the form

$$\mathcal{R}\psi = \sum_i (D\psi_i + \frac{1}{n+1} e_i \ (\sum_k e_k \ D\psi_k)) \otimes dx^i .$$

Remark 2. The spinor valued 1-form ψ satisfying the condition $\mu(\psi) = 0$ on M is solution of Rarita-Schwinger equation iff there is a spinor field ϕ on M such that

$$D_T \psi = \iota(\phi) .$$

Let us denote by $\mathcal{P}_k(1)$ the space of all polynomial k-homogeneous solutions of Rarita-Schwinger equation on and by $\mathcal{P}_k(0)$ the space of polynomial k-homogeneous solutions of the Dirac equation on $\mathbf{R_0^{n+1}}$.

We would like to find a good description of the space $\mathcal{P}_k(1)$, namely its decomposition into some well defined and natural pieces.

Let us define the map

$$\mathcal{L} : \Gamma(S_{\frac{3}{2}}) \to \Gamma(S_{\frac{1}{2}})$$

by

$$\mathcal{L}(\psi) := \sum_i x_i \psi_i$$

Then the map is homomorphism of the corresponding bundles, namely

$$\mathcal{L}(\psi + \phi) = \mathcal{L}(\phi) + \mathcal{L}(\phi), \ \mathcal{L}(f.\psi) = f.\mathcal{L}(\psi) \ \text{for} f \in C^\infty .$$

Lemma 2. *Suppose* $\psi \in \mathcal{P}_k(1)$, *then*

$$D^3(\mathcal{L}(\psi)) = 0 .$$

Let ψ_0 be a (k-1) homogeneous solution of the Dirac equation on $\mathbf{R_0^{n+1}}$, then k-homogeneous solution ψ of the Rarita-Schwinger equation satisfying

$$D_T \psi = \sum_i e_i \ \psi_0 \otimes dx^i \qquad \text{(eq)}$$

can be constructed in the following way. Let

$$\Xi : \Gamma(S_{\frac{1}{2}}) \to \Gamma(S_{\frac{1}{2}} \otimes \Lambda^1)$$

be the map defined by:

$$\Xi(\psi_0) = \frac{1}{2.(n+k)} (\|x\|^2 \mathcal{T}(\psi_0) + \sum_j x_j.\psi_0 \otimes dx^j + \sum_j e_j \ (x.\psi_0) \otimes dx^j)$$

Lemma 3. $\Xi(\psi_0)$ *is a k-homogeneous solution of the equation (eq).*

Theorem 3. *Let ψ be a k-homogeneous solution of the Rarita-Schwinger equation. Then there is a unique decomposition of ψ into (k-homogeneous) pieces*

$$\psi = \psi_1 + \psi_2 + \psi_3$$

with $\psi_j \in \mathcal{M}^j$ where

$$\mathcal{M}^1 = \{\psi \in \Gamma(\mathbf{R_0^{n+1}}, S_{\frac{3}{2}}) \mid \mathcal{L}(\psi) = 0\}$$

$$\mathcal{M}^2 = \mathcal{T}'(KerD) \subset \{\psi \in \Gamma(\mathbf{R_0^{n+1}}, S_{\frac{3}{2}}) \mid D\mathcal{L}(\psi) = 0\}$$

$$\mathcal{M}^3 = \Xi(\mathcal{P}_{k-1}(0)) \subset \{\psi \in \Gamma(\mathbf{R_0^{n+1}}, S_{\frac{3}{2}}) \mid D\mathcal{L}(\psi) \neq 0 \; for \; \psi \neq 0\} \, .$$

We can also describe the space $\mathcal{P}_k(1)$ of k-homogeneous solutions of Rarita-Schwinger equation from representation point of view. Any space $\mathcal{M}^j$ is an irreducible representation of the group Spin(n) and its type is determined by its highest weight λ. We shall speak about the space with highest weight λ as about the space of representation type λ.

Recall that the Rarita-Schwinger equation for $\psi \in \Gamma(\mathbf{R_0^{n+1}}, S_{\frac{3}{2}})$ has a form:

$$D_T \psi = \iota(\phi) \, ,$$

with

$$D\phi = 0 \, .$$

The space $\mathcal{P}_k(1)$ is a direct sum of three spaces (representation types) of solutions, namely

$\mathcal{M}^1$:

The space is characterized by the condition $\mathcal{L}(\psi) = 0$. It corresponds (by restrictions of fields) to the eigenspace of induced Rarita-Schwinger operator on unit sphere of type $(\frac{2k+1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2})$.

$\mathcal{M}^2$:

The space is characterized by the condition $\mathcal{L}(\psi) \neq 0$ for $\psi \neq 0$, $D_0\mathcal{L}(\psi) = 0$, The corresponding space $\mathcal{P}_k^{B1}(1)$ is constructed from the space $\{\sigma \in \mathcal{P}_{k+1}(0)\}$ using the Twistor operator and is of the type $(\frac{2k+3}{2}, \frac{1}{2}, ..., \frac{1}{2})$.

Both preceding types are characterized by condition $\phi = 0$ and consists of solutions non only of RS-equation but of the whole twisted Dirac equation.

$\mathcal{M}^3$:

The space is characterized by the condition $D_0(\psi) \neq 0$ for $\psi \neq 0$, and can be constructed from $\mathcal{P}_{k-1}(0)$ the space of (k-1)-homogenic solutions of Dirac equation, and is of the type $(\frac{2k-1}{2}, \frac{1}{2}, ..., \frac{1}{2})$.

Remark 3. All types of solutions can be uniquely determined by its restriction to unit sphere and can be described by pure spherical data (see [BuLN]).

Remark 4. The classification of polynomial solutions of Rarita-Schwinger equation can be done also for polynomial solutions of the general higher spin Dirac equation as well as for higher Rarita-Schwinger equations. There are several papers in preparation for publication by the authors P. Van Lancker (Gent), F. Sommen (Gent), V. Souček (Prague) and myself.

6. Index of elliptic differential operator

Let $E \to M$ and $F \to M$ be complex vector bundles over a compact m-dimensional manifold M. Denote $\Gamma(E)$ the space of smooth sections of E.

Let

$$D : \Gamma(E) \to \Gamma(F)$$

be an elliptic differential operator between E and F, and

$$D^* : \Gamma(F) \to \Gamma(E)$$

its adjoint. Both operators D, D^* have finite dimensional kernels.

The index of the operator D is defined as

$$\mathrm{IndD} = \dim \mathrm{KerD} - \dim \mathrm{KerD}^* .$$

There are formulas, characterizing the index of an operator in terms of topological invariants of the manifold M and the corresponding bundles.

In the formulas appear some characteristic classes of complex vector bundle E on M, which can be expressed through closed differential forms representing Chern classes $c_i(E)$ namely :
Todd genus

$$td(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_2(E) + c_1(E)^2) + ...$$

and
Chern character

$$Ch(E) =$$

$$\dim(E) + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E) + \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 12c_3(E)) + ...$$

Remark 5. Chern classes $c_i(E)$ are represented by closed differential forms of degree 2i, computable from curvature form of any connection on E.

From the index theorem it follows, that index of the operator does not depend on operator itself but only on bundles E and F.

In the following section, suppose that the dimension of M is $m = 2n$ even, even if will be not especially mentioned, because odd dimensional cases are trivial.

For the index of Dirac operator and its twisted version there are well-known index formula, containing the invariant $\hat{A}$-genus.

Theorem 4. *(Atiyah-Singer,* [BGV]*). Let M be an oriented spin manifold of dimension m = 2n. Let $\mathcal{S}$ be the spin bundle and let $\mathcal{W}$ be an arbitrary vector bundle over M, If D_W is the twisted Dirac operator on $\Gamma(M, \mathcal{W} \otimes \mathcal{S})$ then*

$$\mathrm{ind} D_W = (2\pi \mathbf{i})^{-n} \int_M \hat{A}(M) ch(\mathcal{W}) .$$

We shall use normalization of forms representing characteristic classes as is given in [BGV].

The invariant $\hat{A}(M)$ is the Hirzebruch's $\hat{A}$ genus, given by the formula

$$\hat{A}(M) = 1 - \frac{1}{24}p_1(M) + (\frac{7}{5760}p_1(M)^2 - \frac{1}{1440}p_2(M)) + ...$$

where

$$p_i(M) = (-1)^i c_{2i}(T_{\mathbf{C}}(M))$$

are the Pontrjagin forms.

Remark 6. Let us remark, that index of the classical Dirac (called also Atiyah-Singer [LM]) operator

$$D^+_{\frac{1}{2}} : \Gamma(S^+) \to \Gamma(S^-)$$

is

$$\mathrm{Ind}D^+_{\frac{1}{2}} := \hat{A}_{\frac{1}{2}}[M] = (2\pi\mathbf{i})^{-n}\int_M \hat{A}(M)\,.$$

The Chern character $Ch(E)$ of the bundle E satisfies the conditions

$$Ch(E \oplus F) = Ch(E) + Ch(F)$$

$$Ch(E \otimes F) = Ch(E).Ch(F)$$

Now we shall try to use general theory for computation of the index of our operators D_j in discussion, namely

$$\mathrm{ind}D^+_j \;=\; \dim\ \ker D^+_j \;-\; \dim\ \ker D^-_j$$

First of all we have from the Th.1 :

Corollary 1. *The index of the twisted Dirac operator on M, dim M = 2n,*

$$D^+_T : \Gamma(T^*_C(M) \otimes S^+) \to \Gamma(T^*_C(M) \otimes S^-)$$

is given by the formula

$$\mathrm{IndD^+_T} = \int_{\mathrm{M}} \mathrm{Ch}(\mathrm{T^*_C(M)})\hat{\mathrm{A}}(\mathrm{M}) =$$

$$(2\pi\mathbf{i})^{-n}\int_M (2n + p_1(M) + \frac{1}{12}(p_1(M)^2 - 2p_2(M)) + .)$$
$$(1 - \frac{1}{24}p_1(M) + \frac{1}{5760}(7p_1(M)^2 - 4p_2(M)) + .)\,.$$

6.1. Index of the Rarita-Schwinger operator. Because of equality

$$S^{\pm}_{\frac{1}{2}} \otimes T^*(M) = S^{\mp}_{\frac{1}{2}} \oplus S^{\pm}_{\frac{3}{2}}$$

we have

$$Ch(S^+_{\frac{3}{2}}) - Ch(S^-_{\frac{3}{2}}) = (Ch(T^*_{\mathbf{C}}(M) + 1)(Ch(S^+_{\frac{1}{2}}) - Ch(S^-_{\frac{1}{2}}))\,.$$

We can simply compute

$$(Ch(T^*_{\mathbf{C}}(M)) + 1 = 2n + 1 + p_1(M) + \frac{1}{12}(p_1(M)^2 - 2p_2(M)) + \ldots.$$

and finally we get

$$\mathrm{Ind} D_{\frac{3}{2}} := \hat{A}_{\frac{3}{2}}[M] = (2\pi\mathbf{i})^{-n}\int_M (Ch(T^*_c(M)) + 1)\hat{A}(M).$$

Especially for dimension $2n = 4$ we have

$$\mathrm{Ind} D_{\frac{3}{2}} =$$

$$(2\pi\mathbf{i})^{-2}\int_M (5 + p_1(M) + \ldots)(1 - \frac{1}{24}p_1(M) + \ldots)$$

$$= (2\pi\mathbf{i})^{-n}\int_M (5 + \frac{19}{24}p_1(M)) = -19(2\pi\mathbf{i})^{-2}\int_M \hat{A}(M).$$

So we have the relation between the index of Dirac operator and Rarita-Schwinger operator by

$$\mathrm{Ind} D_{\frac{3}{2}} = -19.\mathrm{Ind} D_{\frac{1}{2}}.$$

Next nontrivial case is dimension $m = 2n = 8$, we have:

$$\begin{aligned}\mathrm{Ind} D_{\frac{3}{2}} &= \\ &= (2\pi\mathbf{i})^{-4}\int_M (9 + p_1(M) + \frac{1}{12}(p_1(M)^2 - 2p_2(M)) + .)\cdot \\ &\cdot(1 - \frac{1}{24}p_1(M) + \frac{1}{5760}(7p_1(M)^2 - 4p_2(M)) + .) = \\ &= (2\pi\mathbf{i})^{-4}\int_M (9 + \frac{19}{24}p_1(M) + \frac{1}{5760}(543p_1(M)^2 - 996p_2(M)) + .)\,.\end{aligned}$$

So we do not have the simple relation between the index of Dirac operator and Rarita-Schwinger operator as above, only we have

$$\mathrm{Ind} D_{\frac{3}{2}} = 249\mathrm{Ind}\ D_{\frac{1}{2}} - \frac{21}{144}(2\pi\mathbf{i})^{-4}\int_M p_1(M)^2.$$

There is a possibility to find a manifold of dimension 8 without harmonic spinors, but with nontrivial kernel of Rarita-Schwinger operator.

6.2. Index of the Dirac higher spin operators. For the computation of the index for operator D_j^+ for $2 \leq j < n$ there is an induction procedure.

We have

$$S^{\pm} \otimes \Lambda^j T_C^*(M) \simeq V_0^{\mp} \oplus V_1^{\pm} \oplus ... \oplus V_j^{\mp} \quad \text{j} \quad \text{even}$$
$$\simeq V_0^{\pm} \oplus V_1^{\mp} \oplus ... \oplus V_j^{\pm} \quad \text{j} \quad \text{odd.}$$

and

$$Ch(S^{\pm}).Ch(\Lambda^j T_C^*(M)) = \sum_{k=0}^{j} Ch(V_k^{\mp \mathrm{sgn(k)}})$$

for j even,

$$Ch(S^{\pm}).Ch(\Lambda^j T_C^*(M)) = \sum_{k=0}^{j} Ch(V_k^{\pm \mathrm{sgn(k)}})$$

for j odd, with $\mathrm{sgn}(k) = \pm$ if k is even or odd.

Together we get

$$Ch(V_j^+) - Ch(V_j^-) = (-1)^{j+1}(Ch(\Lambda^j T_C^*) + Ch(\Lambda^{j-1} T_C^*)).(Ch(S^+) - Ch(S^-)).$$

Theorem 5. *The index of the operator D_j is :*

$$\mathrm{indD_j} := \hat{\mathrm{A}}_{\frac{\mathrm{j}}{2}}[\mathrm{M}] = (2\pi\mathbf{i})^{-\mathrm{n}}.\int_{\mathrm{M}} (\mathrm{Ch}(\Lambda^{\mathrm{j}-1}) - \mathrm{Ch}(\Lambda^{\mathrm{j}}))\hat{\mathrm{A}}(\mathrm{M}).$$

Using the theorem we can study problem of existence of solutions of equation $\tilde{D}_j \psi = 0$.

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