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CLIFFORD BUNDLES OVER MAPPING SPACES

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Abstract. Let X be compact spin manifold, E an Hermitian vector bundle over X, D a non-degenerate 1-st order self-adjoint elliptic pseudo-differential operator. Fixing the Sobolev metric on $W^k(X)$, the k-Sobolev space of section of E, by D, we have developped calculus of differential forms, including $(\infty - p)$–forms, on an open set of $W^k(X)$ ([3], [4]) and constructed Clifford algebra over $W^{-k}(X)$ with ∞–spinor (γ_5) together with its representation in algebra of bounded linear operators on $\wedge W^{-k}(X) \oplus \wedge W^k(X)$ ([5]). By using these results, $(\infty - p)$–forms and Hodge operators on a mapping space $\mathrm{Map}(X, M)$ (and some of its extensions) are investigated and Clifford bundle is constructed. If D is positive definite, such construction is always possible, while D is Dirac type, vanishing of string class $\mathrm{Map}(X, M)$ is necessary to the construction of Clifford bundle. Difficulties with the construction of spinor bundle and definition of Dirac-Ramond operator are investigated.

1. Regularized dimension of $W^k(X)$

Let X be a d–dimensional compact spin manifold, E an Hermitian vector bundle over X. $L^2(X)$ the Hilbert space of section of E. We fix the norm and inner product of $L^2(X)$ and denote them by $||f||$ and (f, g), respectively. Let D be a non-degenerate 1-st order self-adjoint elliptic pseudo-differential operator acting on smooth sections of E. Then D allows spectral decomposition

$$Df = \sum \lambda(f, e_\lambda)e_\lambda, \ De_\lambda = \lambda e_\lambda \ \{e_\lambda\} \ \text{ is an O.N.} - \text{basis of } L^2(X)\,. \tag{1}$$

By (1), we can define operators $|D|, D_+$ and D_- by

$$\begin{aligned} &|D|f = \sum |\lambda|(f, e_\lambda)e_\lambda\,, \\ &D_+ = (|D| + D)/2, \ D_- = (|D| - D)/2\,. \end{aligned} \tag{2}$$

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D_+ and D_- are positive, hence their k–th powers ($k > 0$) are defined. Since D is non-degenerate, D has the Green operator $G, |G|, G_+$, and G_- are defined similarly. We also set

$$P_+ = G_+D_+,\ P_- = G_-D_-\,,\ J = P_+ - P_- = G|D|\,. \tag{3}$$

By using D, we fix the k–th Sobolev metric $||f||_k$ of a section of E (and $(-k)$–th Sobolev metric $||f||_{-k}$) by

$$||f||_k = ||D^{(k)}f||\,,\ ||f||_{-k} = ||G^{(k)}f||\,. \tag{4}$$

The k–th Sobolev space and $(-k)$–th Sobolev space of sections of E are denoted by $W^k(X)$ and $W^{-k}(X)$, respectively. There is the natural pairing $\langle u, f\rangle$ of $u \in W^{-k}(X)$ and $f \in W^k(X)$ given by

$$\langle u, f\rangle = (G^k u, D^k f)\,. \tag{5}$$

Let $\eta(D, s)$, $\zeta(|D|, s)$, etc., the spectral eta or zeta function of $D, |D|$, etc. They are meromorphic on whole complex plane with possible poles at $s = d, d-1, \ldots, 1$, are at most order 1, and holomorphic at $s = 0$ ([10]). We say $\nu = \zeta(|D|, 0)$ the regularized dimension of $W^k(X)$. We also set

$$\nu_+ = \zeta(D_+, 0),\ \nu_- = \zeta(D_-, 0)\,. \tag{6}$$

Since we have

(7)

$$\zeta(D_\pm + P_\pm, 0) \equiv \nu_\pm - m\mathrm{Res}_{s=1}\zeta(D_\pm, s) + \ldots + (-1)^d \frac{m^d}{d}\mathrm{Res}_{s=d}\zeta(D_\pm, s) \ \mathrm{mod}.2\,,$$

selecting constants m, n, and replacing D by $D + mI + nJ$, we may assume ν and ν_- both integers with arbitrary parity.

The Ray-Singer determinant $\det|D| = \exp(-\zeta'(|D|, 0))$ can be thought to be the volume of cube $\{\sum c_\lambda e_\lambda | 0 \le c_\lambda \le |\lambda|\} \subset W^k(X)$, $k < -d/2$. Since we have

$$\det|D^k| = (\det|D|)^k\,,\ \det(t|D|) = t^\nu \det|D|\,, \tag{8}$$

we may define the regularized volume of a cube $\{\sum c_\lambda e_\lambda | 0 \le c_\lambda \le |\lambda|^k\}$ by using $\det|D|$. $\det D$ is defined to be $(-1)^{\nu_-}\det|D|$.

2. Differential forms on an open set of $W^k(X)$

Since $\{e_\lambda\}$ is an O.N.-basis of $L^2(X)$, setting $e_{\lambda,k} = \mathrm{sgn}\lambda\ |\lambda|^{-k} e_\lambda$, $\{e_{\lambda,k}\}$ becomes an O.N.-basis of $W^k(X)$. Then taking $\{e_\lambda\}$ and $\{e_{\lambda,k}\}$ the coordinates of $L^2(X)$ and $W^k(X)$, we have

$$\frac{\partial f}{\partial e_{\lambda,k}} = \mathrm{sgn}\lambda \cdot |\lambda|^k \frac{\partial f}{\partial e_\lambda} \tag{9}$$

By (9) and $df = \sum \partial f/\partial e_\lambda de_\lambda = \sum \partial f/\partial e_{\lambda,k} de_{\lambda,k}$, we may identify $de_{\lambda,k}$ and $e_{\lambda,-k} = \mathrm{sgn}\lambda|\lambda|^k e_\lambda$. This shows that a 1-form on $\mathcal{U}$, an open set of $W^k(X)$, is a smooth map from $\mathcal{U}$ to $W^{-k}(X)$. Hence a p–form on $\mathcal{U}$ is a smooth map from $\mathcal{U}$

to $\wedge^p W^{-k}(X)$, the Sobolev $(-k)$–space of alternating sections of $\overbrace{E \times \cdots \times E}^{k}$ (on $\overbrace{X \times \cdots \times X}^{k}$).

An $(\infty - p)$–form on $\mathcal{U}$ is a smooth map from $\mathcal{U}$ to $\wedge^p W^k(X)$. Since the volume form on $\mathcal{U}$ is $(\det D)^{-k-d/2}$, an ∞–form on $\mathcal{U}$ is a scalar function $\mathcal{U}$ multiplied by $(\det D)^{-k-d/2}$ ([4], [7]).

By using Sobolev duality, we define the wedge product of a p–form u and an $(\infty - q)$–form f, $p \leq q$. The sign convension is

$$u \wedge f = (-1)^{p(\nu - q)} f \wedge u\,, \tag{10}$$

that is, ∞ is replaced by ν. Then to be consistent in this definition, we need to assume ν to be an integer.

If $f(x) = f(x, x_1, \ldots, x_p)$ is an $(\infty - p)$–form, its Fréchet differential is denoted by F-df. Regarding $x_1, \ldots, x_{p-1}$ to be parameters, F-$df(x, x_1, \ldots, x_p)$ is a linear operator on $W^k(X)$. Under the assumption this operator to be trase class, the exterior differential df of f is defined to be

$$df(x, x_1, \ldots, x_{p-1}) = (-1)^{p-1} \mathrm{trF}\!-\!df(x, x_1, \ldots, x_{p-1}, x),\ p \geq 1\,. \tag{11}$$

One of the striking property of $(\infty - p)$–form is the following Theorem ([3], [4])

Theorem 2.1. *If an $(\infty - p)$–form f is exterior differentiable, then f is globally exact.*

As a conseqence, d is not nilponent on the space of $(\infty - p)$–forms.

In the rest, we denote by $\wedge W^{-k}(X)$ and $\wedge W^k(X)$ the Grassmann algebras over $W^{-k}(X)$ and $W^k(X)$. We consider them to be Hilbert spaces with the norms

$$\begin{aligned} ||\sum u^p||^2 &= \sum ||u^p||^2_{-k}\,,\ u^p \in \wedge^p W^{-k}(X) \subset W^{-k}(X \times \cdots \times X)\,, \\ ||\sum f^p||^2 &= \sum ||f^p||^2_k,\ f^p \in \wedge^p W^k(X) \subset W^k(X \times \cdots \times X)\,. \end{aligned} \tag{12}$$

The Grassmann algebra over $W^{-k}(X)$ with $(\infty - p)$–form is the Hilbert space $\wedge W^{-k}(X) \oplus \wedge W^k(X)$.

3. Clifford algebras over Sobolev spaces

The Clifford algebra $C(W^{-k})$ over $W^{-k}(X)$ is the algebra over C generated by $\{e_\lambda\}$ with the relation

$$e_\lambda e_\mu + e_\mu e_\lambda = -2\mathrm{sgn}\lambda\ |\lambda|^{-2k} \delta_{\lambda,k}\,. \tag{13}$$

It has the representation $r : C(W^{-k}) \to B(\wedge W^{-k})$, the algebra of bounded linear operators on $\wedge W^{-k}$. Let 1 be the unit of $C = \wedge^0 W^{-k} \subset \wedge W^{-k}$. Then set $s(a) = r(a) \cdot 1$, we have the module isomorphism

$$s : C(W^{-k}) \cong \wedge W^{-k}\,. \tag{14}$$

We define $||a||$ by $||s(a)||$. Precisely saying, we consider $C(W^{-k})$ to be the completion of the algebra generated by $\{e_\lambda\}$.

If the regularized dimension ν of $W^k(X)$ is an integer, the infinite spinor $e^\infty(\gamma_5)$ thought to be the infinite product $\cdots e_{\lambda_1} \cdots e_{\lambda_2} \cdots, \cdots \leq \lambda_1 \leq \cdots \leq \lambda_2 \leq \cdots$, must satisfy

$$e_\lambda \cdot e^\infty = (-1)^{\nu-1} \cdot e_\lambda,\ e^\infty \cdot e^\infty = (-1)^{\nu(\nu+1)/2+\nu_-}(\det|D|)^{-2k}. \tag{15}$$

The algebra $C(W^{-k}[e^\infty])$ is said to be the Clifford algebra over $W^{-k}(X)$ with an infinite spinor. A representation $R : C(W^{-k})[e^\infty] \to B(\wedge W^{-k} \oplus \wedge W^k)$, the algebra of bounded linear operators on $\wedge W^{-k} \oplus \wedge W^k$, is given by

$$R(e_\lambda) = \begin{pmatrix} r(e_\lambda) & 0 \\ 0 & (-1)^{\nu-1} G^{2k} D^{2k} \end{pmatrix} \tag{16}$$

$$\begin{aligned} &R(e^\infty) \\ &= \begin{pmatrix} 0 & (-1)^{\frac{1}{2}(\nu_- + \frac{\nu(\nu+1)}{2})}(\det|D|)^{-2k} D^{2k} \\ (-1)^{\frac{1}{2}(\nu_- + \frac{\nu(\nu+1)}{2})} G^{2k} & 0 \end{pmatrix}. \end{aligned} \tag{17}$$

The isometry $s : C(W^{-k})[e^\infty] \to \wedge W^{-k} \oplus \wedge W^k$ is defined by

$$s(y) = R(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{18}$$

Then we have

$$s(C(W^{-k})) = \wedge W^{-k},\ s(C(W^{-k})e^\infty) = \wedge W^k. \tag{19}$$

4. Sobolev structures of mapping spaces

Let M be an N–dimensional smooth manifold. We denote $\mathrm{Map}(X, M)$ the space of Sobolev k–class maps from X to M, $k > d/2$. In the rest, we only consider the subspace of $\mathrm{Map}(X, M)$ consisting of 0-homotopic maps. For sake of simplicity, this space is also denoted by $\mathrm{Map}(X, M)$.

If $f \in \mathrm{M}ap(X, M)$ is homotopic to a constant map, then its graph

$$\mathrm{G}r(f) = \{(x, f(x)) | x \in X\} \subset X \times M \tag{20}$$

is homotopic to $X \times *$, $* \in M$. Since the normal bundle of $X \times *$ in $X \times M$ is trivial, the normal bundle of $\mathrm{Gr}(f)$ in $X \times M$ is trivial. Hence there is a diffeomorphism h_u such that

(21)

$$h_u : \mathcal{U}(\mathrm{G}r(f)) \cong \mathrm{Gr}(f) \times R^N,\ \mathcal{U}(\mathrm{Gr}(f)) \text{ is a neighborhood of } \mathrm{Gr}(f) in\ X \times M.$$

By (21), $\mathrm{Map}(X, M)$ is a Sobolev manifold modeled by $W^k(X) \otimes R^N$, where $W^k(X)$ is the k–Sobolev space of scalar fields on X.

If M is real algebraic manifold defined by the equations

$$F_1(X_1, \ldots, X_m) = \cdots = F_i(X_1, \ldots, X_m) = 0, \tag{22}$$

in R^m, $\mathrm{Map}(X, M)$ is a connected component of the algebraic set in $W^k(X) \otimes R^m$ determined by

$$F_1(f_1, \ldots, f_m) = \cdots = F_i(f_1, \ldots, f_m) = 0,\ f_i \in W^k(X),\ 1 \leq i \leq m. \tag{23}$$

Hence $\mathrm{Map}(X, M)$ is an algebraic set in $W^k(X) \otimes R^m$.

Since $F_1, \ldots, F_i$ are polynomials, replacing $W^k(X)$ by the k–Sobolev space of spinor fields $W^k(X, E)$ on X, the equations (23) have meanings. Then $\mathrm{Map}(X, M)$ is contained in a connected compact C of the algebraic set in $W^k(X, E) \otimes R^m$ defined by the equations (23), and, there is a neighborhood $\mathcal{U}$ of $\mathrm{Map}(X, M)$ in C such that $\mathcal{U}$ is a non-singular manifold modeled by $W^k(X, E) \otimes R^N$. Taking $\mathcal{U}$ sufficiently small, we may assume $\mathrm{Map}(X, M)$ is the deformation retract of $\mathcal{U}$. For simply, this $\mathcal{U}$ is also denoted by $\mathrm{Map}(X, M)$ in the rest.

5. Connection with respect to D

Let $F : \mathcal{U} \to S$, $\mathcal{U}$ an open set of M, be a function, then

$$F^{\#}(f)(x) = F(f(x)), \ x \in X, \ f \in \mathrm{Map}(X, \mathcal{U}) , \tag{24}$$

is a map $F^{\#} : \mathrm{Map}(X, \mathcal{U}) \to \mathrm{Map}(X, S)$.

Let $E = \{g_{uv}\}$ be a G–bundle over M, G a Lie group, then $\{g^{\#}_{uv}\}$ defines a $\mathrm{Map}(X, G)$–bundle over $\mathrm{Map}(X, M)$. This bundle is denoted by $E^{\#}$. When G is a unitary group (or an orthogonal group), we may consider $B^{\#}$ to be a loop group bundle ([2], cf. [8]). Especially, if T is the tangent bundle of M, $T^{\#}$ becomes the tangent bundle of $\mathrm{Map}(X, M)$, and we may consider $T^{\#}$ to be a loop group bundle.

Let $B = \{b_{uv}\}$ be a $\mathrm{Map}(X, G)$–bundle. We assume G acts on E, the Hermitian vector bundle over X and D acts on its sections. Let $D(0)$ be the space of 0-th order pseudo-differential operators acting on the sections of E. Then a collection $\{A_u\}$; $A_u : \mathcal{U} \to D(0)$, such that

$$(D + A_u) b_{uv} = b_{uv} (D + A_v) \tag{25}$$

is said to be a connection of B with respect to D. $\{A_u\}$ always exist and if G is a unitary group, we may assume A_u is Hermitian for any $x \in \mathcal{U}$.

Taking B to be the (co)tangent bundle of $\mathrm{Map}(X, M)$ and D the operator by which the Sobolev metric of $W^k(X)$ is determined, we have the following facts (cf. [2], [6]).

(i) If D positive definite, then D has a connection $\{A_u\}$ such that $D + A_u(x)$ is non-degenerate for any $\mathcal{U}$ and $x \in \mathcal{U}$.

(ii) If D is of the Dirac type, then $D + A_u(x)$ degenerates for some $\mathcal{U}$ and $x \in \mathcal{U}$, whenever the tangent bundle of $\mathrm{Map}(X, M)$ is non-trivial as a loop group bundle.

By (25), the spectral type of $D + A_u(x)$ does not depend on $\mathcal{U}$. Hence the regularized dimension $\nu(x) = \zeta(|D + A_u(x)|, 0)$ is a globally defined function on $\mathrm{Map}(X, M)$. Therefore selecting mass-term $m = m(x)$, we may assume ν is a constant integer on $\mathrm{Map}(X, M)$. While if D is a Dirac type, we can take both ν and ν_- to be constant integers if and only if structure group of the tangent bundle of $\mathrm{Map}(X, M)$ is reduced to the connected component of the identity (of the loop group, cf. [6]).

6. $(\infty - p)$–forms and Hodge operators on mapping space

If $|m|$ is small, $\det(D_\pm + mP_\pm)$ is holomorphic in m. Its analytic continuation is an entire function on whole m–plane. In this n°, we denote this function by $\det(D_\pm + mP_\pm)$ and set

$$\det(D + mP) = \det(D_+ + mP_+)\det(D_- + mP_-)\,. \tag{26}$$

If $(D + A_u(x)$ has 0-modes for some $x \in \mathcal{U}$, we choose a scalar $m = m_u$ such that $D + A_u(x) + mI$ does not have 0-modes for any $x \in \mathcal{U}$, and set

$$\det_m(D + A_u) = \det((D + A_u(x) + mI) - mI)\,. \tag{27}$$

Then $\{\gamma_{uv}\} = \{\det_m(D+A_u)/\det_m(D+A_v)\}$ defines a line bundle over $\mathrm{Map}(X, M)$. It is shown this bundle is determined by the tangent bundle of $\mathrm{Map}(X, M)$ and does not depend on the choice of $\{A_u\}$ and $\{m_u\}$ ([3]).

By the local definition of ∞–forms f^∞ on $\mathrm{Map}(X, M)$ takes the form

$$f^\infty = \{f_u \det_m(D + A_u)^{-k-d/2}\}\,. \tag{28}$$

Hence $\{f_u\}$ should be a cross-section of $\{\gamma_{uv}\}^{k+d/2}$. Therefore, to define ∞–form on $\mathrm{Map}(X, M)$ we need to assume the bundle $\{\gamma_{uv}\}^{k+d/2}$ can be defined. That is, denoting C the Chern class of $\{\gamma_{uv}\}$, $(k+d/2)C$ should be an integral class. In the rest, we denote $\{\gamma_{uv}\}^{k+d/2}$ by τ and call it the determinant bundle of $\mathrm{Map}(X, M)$.

Let T be the tangent bundle of $\mathrm{Map}(X, M)$, then we say a cross-section of the associated $\wedge^p W^k(X)$–bundle of $T \otimes \tau$ to be an $(\infty - p)$–form of $\mathrm{Map}(X, M)$.

We assume ν and ν_- are both integers. Let u^p be a p–form of $\mathrm{Map}(X, M)$ and $f^{\infty-p} = \{f_u\}$, $f_u = f\det_m(D+A_u)^{k+d/2}$ be an $(\infty-p)$–form of $\mathrm{Map}(X, M)$. Then we define the Hodge $*$–operator by

$$*u^p = (-1)^p \det_m(D + A_u)^{k+d/2} G^{2k}_{A_u}\,,$$

$G^{2k}_{A_u}$ is the Green operator of $D + A_u$,

$$*f^{\infty-p} = (-1)^{\nu p+\nu_-}\det_m(D + A_u)^{2k}\det_m\,(D + A_u)^{-(k+d/2)} f_u\,. \tag{29}$$

If $D + A_u(x)$ has 0-modes, then $G_{A_u(x)}$ can not be defined. Hence $* = *(x)$ can not be defined. But if u^p takes the values in $\wedge^p L^2(X)$ $(\subset \wedge^p W^{-k}(X))$, near x then the pairing $\langle u^p, f^{\infty-p}\rangle$ is defined at x.

We note that if $\dim\ker(D + A_u(x)) = 1$, then $u^1 \wedge *u^1$ becomes integrable near x if $k < d/2 + 1$. Since k is larger than $d/2$, the appropriate selection of k is

$$k = (d + 1)/2\,. \tag{30}$$

By (30), we have $k + d/2 = d + 1/2$. Therefore τ is defined on $\mathrm{Map}(X, M)$ if and only if the Chern class of $\{\gamma_{uv}\}$ is divisible by 2.

By (29), if $*$ is defined, then we have

$$*^2 u^p = (-1)^{p(\nu-p)+\nu_-} u^p\,, \quad *^2 f^{\infty-p} = (-1)^{(\nu-p)p+\nu_-} f^{\infty-p}\,. \tag{31}$$

7. Construction of Clifford bundle

To define the Clifford bundle, we modify $R(e^\infty)$ to the following operator $E^\infty = E^\infty_u$

$$E^\infty = \begin{pmatrix} 0 & (-1)^{\frac{\nu(\nu+1)}{2} - \frac{\nu_-}{2} + p} D_u^{2k} \det D_u^{-k-d/2} \\ (-1)^{\frac{\nu(\nu+1)}{2} + \nu_- + \nu p} \det D_u^{k+d/2} G^{2k}_{A_u} & 0 \end{pmatrix} \tag{32}$$

acting on the space $\wedge^p(W^{-k}(X)) \oplus \wedge^p(W^k(X)) \otimes \tau$, where we put $D_u = D + A_u$. Here we assume ν and ν_- both integers.

By definitions, we have

$$(E^\infty)^2 u^p = (-1)^{p(\nu-p)+\nu_-} u^p, \ (E^\infty)^2 f^{\infty-p} = (-1)^{(p-\nu)p+\nu_-} f^{\infty-p}\,. \tag{33}$$

That is, we may consider the action of E^∞ to be the Hodge operation.

Since $(E^\infty)^{-1}$ is equal to $(-1)^{(p-\nu)p+\nu_-} E^\infty$ on the space $\wedge^p W^{-k}(X) \oplus \wedge^p W^k(X) \otimes \tau$, we have

$$E^\infty_u (E^\infty_v)^{-1} = \begin{pmatrix} (D + A_u)^{2k} \det D_u^{-k-d/2} \det D_u^{k+d/2} G^{2k}_{A_v} & 0 \\ 0 & \det D_u^{k+d/2} G^{2k}_{A_u} D_v^{2k} \det D_v^{-k-d/2} \end{pmatrix}. \tag{34}$$

The Clifford bundle Cliff on $\mathrm{Map}(X, M)$ is the bundle with the transition $\{E^\infty_u (E^\infty_v)^{-1}\}$ and the fibre $\wedge(W^{-k}(X)) \oplus \wedge(W^k(X)) \otimes \tau = C(W^k)[e^\infty]$. We set

$$\begin{aligned} Y &= \{x \in \mathrm{Map}(X, M) | \ker(D + A_u(x)) \neq \{0\}\}\,, \\ Y_r &= \{x \in \mathrm{Map}(X, M) | \dim \ker(D + A_u(x)) = r\}\,, \end{aligned} \tag{35}$$

and assume $\{Y_r\}$ is a non singular submanifold of $\mathrm{Map}(X, M)$. Originally, E_u is defined only on $E_u - Y$.

Let x be an element of Y_r, $\{e_\lambda\}$, the proper function of $D - A_u(x)$ such that

$$(D - A_u(x)) e_{\lambda_1} = \cdots = (D - A_u(x)) e_{\lambda_r} = 0\,. \tag{36}$$

Then Y_r is defined by $e_{\lambda_1} = \cdots = e_{\lambda_r} = 0$ near x. Hence we may identify $\ker(D + A_u(x))$ and the normal space of Y_r at x. In other word, the normal bundle $N(Y_r)$ of Y_r is given by

$$N(Y_r) = \cup_{x \in Y_r} \ker(D + A_u(x))\,. \tag{37}$$

Therefore, if $N(Y_r)$ is trivial, $E^\infty_u (E^\infty_v)^{-1}$ is defined on Y_r.

Then applying results in [1] and [6], we have

Theorem 7.1. *The Clifford bundle associated to the tangent bundle of $\Omega M = Map(S^1, M)$ is defined on $\overline{Y}_r$ if the string classes $S^1(T), \cdots, S^{[(r+1)/2]}(T)$ of T vanish.*

8. Parity exchange and Dirac-Kähler type operator

$C(W^{-k})$ is the direct sum of modules of even spinors and odd spinors. Corresponnding subbundles of Cliff are denoted by S^0 and S^1. We also define the subbundle $S^0(+)$, $S^0(-)$, $S^1(+)$ and $S^1(-)$ of Cliff by

$$S^0(\pm) = \{u \pm *u | u \in S^0\}\,,$$
$$S^1(\pm) = \{u \pm *u | u \in S^1\}\,. \tag{38}$$

Here $*u$ means $E^\infty \cdot u$. By definition, we have

$$\mathrm{Cliff} = S^0(+) \oplus S^0(-) \oplus S^1(+) \oplus S^1(-)\,. \tag{39}$$

We also set

$$S(+) = S^0(+) \oplus S^1(+)\,,\ S(-) = S^0(-) \oplus S^1(-)\,. \tag{40}$$

The formal adjoint δ of d is defined by

$$\delta u^p = (-1)^p *^{-1} d * u^p\,,\ \delta f^{\infty - p} = (-1)^p *^{\nu - p} d * df^{\infty - p}\,. \tag{41}$$

We note δ is not nilpotent on the space of finite degree forms, that is $\delta^2 \neq 0$ on the space of sections of $C(W^{-k})$, because d is not nilpotent on the space of $(\infty - p)$–forms.

We define the operators $D(+)$ and $D(-)$ by

$$D(+)u^p = (d+\delta)u^p,\ D(+) * u^p = (-1)^{p-1}(d-\delta) * u^p\,,$$
$$D(-)u^p = (-1)^{p-1}(d-\delta)u^p,\ D(-) * u^p = (d+\delta) * u^p\,, \tag{42}$$

then $D(+)$ and $D(-)$ both map the space of sections of $S(\pm)$ to the space of sections of $S(\mp)$ (cf. [13]).

Since d and δ are not nilpotent, $(d+\delta)^2$ is not equal to $\Delta = d\delta + \delta d$. But we have

$$\Delta = \frac{1}{2}(D(+)^2 + D(-)^2)\,. \tag{43}$$

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