DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 293–302

NATURAL T–FUNCTIONS ON THE COTANGENT BUNDLES OF SOME WEIL BUNDLES

JIŘÍ TOMÁŠ

ABSTRACT. For some Weil algebras A, we present a simple procedure giving all natural T-functions on the cotangent bundle of a Weil bundle T^*T^A. For some other cases, we describe the additional natural functions. We outline the relations of this problem to the problem of lifting of vector fields to the cotangent bundle of a Weil bundle.

1. This paper is a contribution to the theory of Weil bundles. We follow some results by Kolář concerning the classification of all natural operators $T \to TT^A$ transforming vector fields on manifolds to vector fields on a Weil bundle T^A, [4], [5] and some construction of his, transforming natural operators of this kind to natural T-functions on T^*T^A, the cotangent bundle of T^A, [3].

For some cases of Weil algebras A, all natural T-functions in question can be obtained by this construction. We show, that in some other cases it is not so.

We start from the concept of a natural T-function. For a natural bundle F, a natural T-function f is a natural operator f_M transforming vector fields on a manifold M to functions on FM. The naturality condition reads as follows. For a local diffeomorphism $\varphi : M \to N$ between manifolds M, N and for vector fields X on M and Y on N satisfying $T\varphi \circ X = Y \circ \varphi$ it holds $f_N \circ F\varphi = f_M$. An absolute natural operator of this kind, i.e. independent of the vector field, is called a natural function on F.

Now we remind some basic results of Kolář. The first one, [4], [5], giving the classification of all natural operators $T \to TT^A$ reads as follows

All natural operators $T \to TT^A$ are of the form

$$L(c)\mathcal{T}^A + \lambda_D$$

1991 *Mathematics Subject Classification.* 58A20, 53A55.
Key words and phrases. Natural bundle, natural operator, vector field, Weil bundle.
The author was supported by the grant No. 201/96/0079 of the GA ČR.
This paper is in final form and no version of it will be submitted for publication elsewhere.

where $\mathcal{T}^A$ denotes the flow prolongation on a natural bundle T^A, $L(c)$ denotes the natural affinor on T^A induced by $c \in A$ in the sense of Koszul, [4], [5] and λ_D denotes the absolute operator induced by a derivation $D \in \operatorname{Der} A$.

We essentially use the following construction of Kolář, [3]. For a natural bundle F and a vector field Y on FM, Kolář defined an operation $\widetilde{\ }$ transforming Y into a function $\widetilde{Y} : T^*FM \to \mathbb{R}$ as follows

$$\widetilde{Y}(\omega) = \langle Y(p(\omega)), \omega\rangle$$

where p is the cotangent bundle projection and $\omega \in T^*FM$. One can immediately verify that this operation transforms a natural operator $A : T \to TF$ into a natural T-function $\widetilde{A}$ on T^*F.

2. Let $F = T^A$, where A is considered as a factor algebra of the algebra of jets $\mathbb{D}^r_k = J^r_0(\mathbb{R}^k, \mathbb{R})$ or as the factor of the algebra of polynomials $\mathbb{R}[\tau_1, \dots, \tau_k]$.

Let $\mathcal{D}$ be a basis of $\operatorname{Der} A$ and C be a basis of A. Put $I_D = \widetilde{\lambda}_D$ and $J_c = \widetilde{L(c)\mathcal{T}^A}$. Let M be a manifold satisfying $\dim M \geq w(A) + 1$, where $w(A)$ denotes the width of A. We are going to prove, that for some cases of A, all natural T-functions $f_M : T^*T^AM \to \mathbb{R}$ are of the form

$$h(J_{c,M}, I_{D,M}), \qquad D \in \mathcal{D},\ c \in C \tag{1}$$

for an arbitrary smooth function $h : \mathbb{R}^{\dim A + \dim \operatorname{Der} A} \to \mathbb{R}$.

In what follows, we need the coordinate expressions of some natural operators. Let x^i_α denote the canonical coordinates on T^r_kM and p^α_i define the additional coordinates on $T^*T^r_kM$. A map $\tau_i \mapsto \delta^i_j \tau^\beta$ determines a derivation $D^\beta_j \in \operatorname{Der} \mathbb{D}^r_k$, [2], [8]. Denote by λ^β_j the associated natural operator $\lambda_{D^\beta_j}$. Its coordinate expression is of the form $\frac{\alpha_j}{\alpha_j+\beta_j}\frac{(\alpha+\beta)!}{\alpha!}x^i_\alpha\frac{\partial}{\partial x^i_{\alpha+\beta-\{j\}}}$. The following assertion yields all natural T-functions on $T^*T^r_k$, which are constructed by means of natural operators $T \to TT^r_k$.

Proposition 1. *For a manifold M satisfying $\dim M \geq k+1$, all natural T-functions on $T^*T^r_kM$ are of the form*

$$h(J_{c,M}, I_{D,M}), \qquad D \in \mathcal{D},\ c \in C \tag{2}$$

where C is a basis of $\mathbb{D}^r_k$, $\mathcal{D}$ is a basis of $\operatorname{Der}\mathbb{D}^r_k$ and h is any smooth function $\mathbb{R}^{\dim \mathbb{D}^r_k + \dim \operatorname{Der} \mathbb{D}^r_k} \to \mathbb{R}$.

Proof. By the immersion theorem, there is an immersion element $i \in T_0^{\mathbb{D}^r_k}\mathbb{R}^m$ satisfying $x^i_\alpha = 0$ for $|\alpha| \geq 2$ and $x^i_j = \delta^i_j$. By the general theory, [5], natural operators in question are of order $r+1$ and they are determined by their values over $j^{r+1}_0(\frac{\partial}{\partial x^{k+1}})$ on $(T^*T^r_k)_0\mathbb{R}^m$. By the action of the jet group G^{r+1}_m we can achieve on a dense subset of $(J^{r+1}T)_0\mathbb{R}^m \times (T^*T^r_k)_0\mathbb{R}^m$ the element $j^{r+1}_0(\frac{\partial}{\partial x^{k+1}})$ and an element over i in $(T^*T^r_k)_0\mathbb{R}^m$. Applying a general lemma from [5], Chapter

VI, Proposition 26.1 we obtain that every natural operator in question evaluated over i is of the form $h(p^\alpha_j)$, $0 \le |\alpha| \le r$, $j \in \{1, \dots, k+1\}$ for any smooth function h. It follows from the coordinate expressions of J_{τ^α} and $\widetilde{\lambda}^\beta_j$ that p^α_{k+1} coincide with J_{τ^α}, $0 \le |\alpha| \le r$ over $j^{r+1}_0(\frac{\partial}{\partial x^{k+1}})$ and i, while p^β_j, $1 \le |\beta| \le r$, $j \in \{1, \dots, k\}$ coincide with natural functions $I^\beta_j = \widetilde{\lambda}^\beta_j$ over i. The transformation laws of G^{r+1}_m on $(T^*T^r_k)_0\mathbb{R}^m$ are of the form

$$\bar{y}^i_\alpha = a^i_{l_1\dots l_p} y^{l_1}_{\alpha_1} \cdots y^{l_p}_{\alpha_p}, \qquad \bar{p}^\alpha_i = \tilde{a}^j_{l_1\dots l_p} \bar{y}^{l_1}_{\gamma_1} \cdots \bar{y}^{l_s}_{\gamma_s} p^{\alpha\gamma}_j \tag{3}$$

where $a^i_{l_1\dots l_p}$ denote the canonical coordinates of G^{r+1}_m and $\tilde{a}^i_{l_1\dots l_p}$ denote its inverse.

It follows the existence of a subgroup $H \subseteq G^{r+1}_{k+1}$ such that $H \times \mathrm{id}_{\mathbb{R}^{m-k}}$ stabilizes i and $j^{r+1}_0(\frac{\partial}{\partial x^{k+1}})$. Put $B^{s+1}_m = \{j^{s+1}_0\varphi \in B^{r+1}_m ; j^s_0\varphi = j^s_0 \,\mathrm{id}_{\mathbb{R}^m}\}$.

All p^0_j, $j \in \{1, \dots, k\}$ are annihilated by $H \times \mathrm{id}_{\mathbb{R}^{m-k}} \cap B^{r+1}_m$ namely by an element satisfying $a^i_{j_1\dots j_{r+1}} = 0$ except $a^j_{j\dots j}$, which completes the proof. $\square$

3. In the following propositions, we investigate in more details the case $A = \mathbb{D} \otimes \mathbb{D}^r_k$, where $\mathbb{D} = \mathbb{D}^1_1$ denotes the algebra of dual numbers. Such a Weil algebra corresponds to the Weil bundle TT^r_k. The following lemma yields the description of all natural operators $T \to TTT^A$.

Let $E \to M$ be a vector bundle and $f : E \to E$ be a vector bundle morphism over the identity on M. The vertical tangent bundle VE is canonicaly identified with $E \times_M E$. Let us denote by Wf the map $\mathrm{id}_E \times f$ in this identification. Hence Wf is a vertical vector field on E. The following result was deduced in [8].

Lemma 2. *All natural operators $T \to TTT^A$ are of the form*

$$\mathcal{T} \circ (L(a)\mathcal{T}^A + \lambda_D) + \mathcal{V} \circ (L(b)\mathcal{T}^A + \lambda_E) + W(L(c))$$

for a, b, $c \in A$, D, $E \in \mathrm{Der}\, A$, $\mathcal{T}$, $\mathcal{T}^A$ the flow operators associated to natural bundles T, T^A and the vertical lift $\mathcal{V} : T \to TT$.

Let x^i_α denote the canonical coordinates on $T^r_k M$ and $y^i_\alpha = dx^i_\alpha$ the additional ones on $TT^r_k M$. The additional coordinates on $T^*T^r_k M$ are defined by $p^\alpha_i dx^i_\alpha + q^\alpha_i dy^i_\alpha$.

We are going to prove that all natural T-functions on $T^*TT^r_k$ are obtained by the construction (1) only if $r = 1$ or $k = 1$ or $r = 2$ and $k = 2$.

Proposition 3. *Let $r = 1$ or $k = 1$ or $r = 2$ and $k = 2$. Then for a manifold M satisfying $\dim M \ge k+2$, all natural T-functions $T^*TT^r_k M \to \mathbb{R}$ are of the form*

$$h(\widetilde{\mathcal{T} \circ (L(a)\mathcal{T}^A + \lambda_D)}, \widetilde{\mathcal{V} \circ (L(b)\mathcal{T}^A + \lambda_E)}, \widetilde{W(L(c))})$$

where a, b, c are elements of the basis C of $\mathbb{D}^r_k$, D as well as E are the elements of the basis $\mathcal{D}$ of $\mathbb{D}^r_k$ and h is any smooth function.

Proof. Similarly as in the proof of Proposition 1, consider the immersion element $i \in (TT^r_k)_0\mathbb{R}^m$ which is of the form $x^i_\alpha = 0$ for $|\alpha| \ge 2$, $x^i_j = \delta^i_j$ for $j \in \{1, \dots, k\}$

and $y_0^i = \delta_{k+1}^i$, $y_\alpha^i = 0$ for $|\alpha| \geq 1$. On the dense subset of $(J^{r+2}T)_0\mathbb{R}^m \times (T^*TT_k^r)_0\mathbb{R}^m$ we can achieve by the action of the jet group G_m^{r+2} an element in $(T^*TT_k^r)_0\mathbb{R}^m$
over i and $j_0^{r+2}(\frac{\partial}{\partial x^{k+2}})$. In this case p_{k+2}^α coincide with $\widetilde{\mathcal{T}\circ(L(\tau^\alpha)\mathcal{T}_k^r)}$ and q_{k+2}^α with $\widetilde{\mathcal{V}\circ(L(\tau^\alpha)\mathcal{T}_k^r)}$ for $0 \leq |\alpha| \leq r$. Moreover, p_j^α coincide with $\widetilde{\mathcal{T}\circ\lambda_j^\beta}$ and q_j^α with $\widetilde{\mathcal{V}\circ\lambda_j^\beta}$ for $1 \leq |\alpha| \leq r$ and $j \in \{1,\dots,k\}$. Finally, q_{k+1}^α coincide with natural operators $\widetilde{W(L(\tau^\alpha)}$ for $0 \leq |\alpha| \leq r$.

Consider a subgroup $H \subseteq G_{k+1}^{r+2} \times \mathrm{id}_{\mathbb{R}^{m-k-1}}$ stabilizing i and $j_0^{r+2}(\frac{\partial}{\partial x^{k+2}})$. Its existence follows from the following transformation laws of the action of G_m^{r+2} on $(J^{r+1}T)_0\mathbb{R}^m$ and on $(TT_k^r)_0\mathbb{R}^m$ given by the formulas

$$\begin{aligned}
(4) \qquad & \bar{y}_\alpha^i = a_{l_1\dots l_p}^i y_{\alpha_1}^{l_1}\dots y_{\alpha_p}^{l_p}, && |\alpha_1|,\dots|\alpha_p| \geq 1\\
& \bar{z}_\alpha^i = a_{l_1\dots l_s}^i y_{\alpha_1}^{l_1}\dots y_{\alpha_{s-1}}^{l_{s-1}} z_{\alpha_s}^{l_s} && |\alpha_1|,\dots|\alpha_{s-1}| \geq 1, |\alpha_s| \geq 0
\end{aligned}$$

It follows that $a_j^i = \delta_j^i$ and for $2 \leq |\alpha| \leq r$, $a_\alpha^i = 0$ and $a_{k+1\alpha}^i = 0$ whenever α does not contain $k+1$.

The following formulas yield the transformation laws of $H \times \mathrm{id}_{\mathbb{R}^{m-k-1}}$ over i and $j_0^{r+2}(\frac{\partial}{\partial x^{k+2}})$ for p_{k+1}^α and q_i^0

$$\begin{aligned}
(5) \qquad & \bar{p}_{k+1}^\alpha = \tilde{a}_{k+1k+1l_1\dots l_s}^j q_j^{\alpha l_1\dots l_s} && 0 \leq s \leq r-|\alpha|\\
& \bar{q}_i^0 = q_i^0 + \tilde{a}_{il_1\dots l_r}^j q_j^{l_1\dots l_r}
\end{aligned}$$

It follows that q_i^0 can be annihilated by $B_m^{r+1} \cap (H \times \mathrm{id}_{\mathbb{R}^{m-k-1}})$. All p_{k+1}^α can be annihilated by means of $(H \times \mathrm{id}_{\mathbb{R}^{m-k-1}}) \cap B_m^{r-|\alpha|+2}$ even if $|\alpha| = r$ whenever $k+1 \geq \mathrm{C}(r+k-1,k-1)$ which is the number of multiindices of order r. It holds only for $r=1$ or $k=1$ or for $r=2$ and $k=2$. This completes the proof. □

4. We are searching all natural functions $T^*TT_k^rM \to \mathbb{R}$. In order to be simple, suppose $M = \mathbb{R}^{k+1}$, since for $m > k+1$ all our investigations can be easily modified. Consider a subgroup H stabilizing the immersion element i, which is of the form $x_j^i = \delta_j^i$, $j \in \{1,\dots,k\}$, $y_0^i = \delta_{k+1}^i$ and $x_\alpha^i = 0$ for $|\alpha| \geq 2$ and $y_\alpha^i = 0$ for $|\alpha| \geq 1$. From the transformation laws for x_α^i and y_α^i we obtain $a_\alpha^i = 0 = a_{\beta k+1}^i$ for $|\alpha| \geq 2$ and $|\beta| \geq 1$. Moreover, $a_j^i = \delta_j^i$ for $j \in \{1,\dots,k\}$ and $a_{k+1}^i = \delta_{k+1}^i$.

Consider the group projection $\pi_H : G_{k+1}^{r+1} \to H$. Then $G_{k+1}^{r+1} = H \oplus \operatorname{Ker}\pi_H$ and every $j_0^{r+1}\varphi$ can be decomposed as $\pi_H(j_0^{r+1}\varphi) \circ (\pi_H(j_0^{r+1}\varphi^{-1}) \circ j_0^{r+1}\varphi)$. In coordinates, every element of $\operatorname{Ker}\pi_H$ must satisfy $a_{k+1k+1\alpha}^i = 0$ for any multiindex α including the empty one.

Weil algebra $\mathbb{D} \otimes \mathbb{D}_k^r$ can be considered as the factor of the algebra of $\mathbb{D}_{k+1}^{r+1}$ by the ideal $< \tau_{k+1}^2 >$. Consider an inclusion $\iota : TT_k^r \to T_{k+1}^{r+1}$ defined by $\bar{x}_\alpha^i = x_\alpha^i$ and $\bar{x}_{k+1\alpha}^i = y_\alpha^i$ if α is any multiindex not containing $k+1$, $0 \leq |\alpha| \leq r$ and other $\bar{x}^i_\alpha$ put zeros. Further, let $p : T^{r+1}_{k+1} \to TT^r_k$ be the canonical projection induced by the homomorphism $\tilde{p} : \mathbb{D}^{r+1}_{k+1} \to \mathbb{D} \otimes \mathbb{D}^r_k$ of Weil algebras. Denote by $\bar{\Lambda}^\beta_{k+1}$ the absolute natural operator on T^{r+1}_{k+1} determined by the derivation $D^\beta_{k+1} \in \mathrm{Der}(\mathbb{D}^{r+1}_{k+1})$, $D^\beta_{k+1}(\tau^i) = \delta^i_{k+1}\tau^\beta$. Put $A^\beta_{k+1} = Tp \circ \bar{\Lambda}^\beta_{k+1} \circ \iota$.

The operators A^β_{k+1} are not natural but the following lemma presents some properties of theirs, used later in the construction of the additional natural T-functions on $T^*TT^r_k$.

Lemma 4. *The functions* $\widetilde{A^\beta_{k+1}}$ *are natural with respect to the action of* $\mathrm{Ker}\,\pi_H$.

Proof. The coordinate expression of $\widetilde{A^\beta_{k+1}}$ is of the form $y^i_\alpha p^{\alpha+\beta}_i$ if we distinguish the order in multiindices. Let $\bar{p}^\beta_{k+1}$ denote the value of p^β_{k+1} over the immersion element i, which can be achieved by the action of $\mathrm{Ker}\,\pi_H$.

It follows from the transformation laws that $\bar{p}^\beta_{k+1} = \tilde{b}^j_{k+1} p^{\alpha\beta}_j$, where (b^i_α) denote the coordinates of the inverse element to (a^i_α) transforming i onto (x^i_α, y^i_α). Thus we have $\bar{p}^\beta_{k+1} = y^i_\alpha p^{\alpha\beta}_i$, which completes the proof. □

In what follows, we shall need one notation for absolute natural operators $T \to TTT^r_k$ of all three kinds and for operators A^β_{k+1} defined before Lemma 4.

Let β not contain $k+1$ and $j \in \{1, \dots, k\}$. We have four sets of operators $T \to TTT^r_k$, the first three of them being natural.

(i) $A^\beta_j = Tp \circ \bar{\Lambda}^\beta_j \circ \iota = \mathcal{T} \circ \lambda^\beta_j$ for $1 \le |\beta| \le r$, where $\bar{\Lambda}^\beta_j$ are absolute natural operators $T \to TT^{r+1}_{k+1}$ corresponding to derivations $D \in \mathrm{Der}(\mathbb{D}^{r+1}_{k+1})$, which are of the form $D : \tau_i \to \delta^i_j \tau^\beta$.

(ii)$A^{k+1\beta}_j = Tp \circ \bar{\Lambda}^{k+1\beta}_j \circ \iota = \mathcal{V} \circ \lambda^\beta_j$ for $1 \le |\beta| \le r$, where $\bar{\Lambda}^{k+1\beta}_j$ are absolute operators $T \to TT^{r+1}_{k+1}$ corresponding to derivations $D \in \mathrm{Der}(\mathbb{D}^{r+1}_{k+1})$ of the form $D : \tau_i \to \delta^i_j \tau^{k+1}\tau^\beta$.

(iii)$A^{k+1\beta}_{k+1} = Tp \circ \bar{\Lambda}^{k+1\beta}_{k+1} \circ \iota = W(L(\tau\beta))$, where $\bar{\Lambda}^{k+1\beta}_{k+1}$ are natural operators $T \to TT^{r+1}_{k+1}$ corresponding to derivations on $\mathrm{Der}(\mathbb{D}^{r+1}_{k+1})$, which are of the form $D : \tau_i \to \delta^i_{k+1}\tau^{k+1}\tau^\beta$.

(iv)$A^\beta_{k+1} = Tp \circ \bar{\Lambda}^\beta_{k+1} \circ \iota$ considered in Lemma 4 and corresponding to derivations on $\mathrm{Der}\,\mathbb{D}^{r+1}_{k+1}$ which are of the form $\tau^i = \delta^i_{k+1}\tau^\beta$.

Now, the value of every natural function in question over i is of the form $h(p^\alpha_i, q^\alpha_i)$. Clearly, p^α_j and q^α_j for $j \in \{1, \dots, k\}$ and $1 \le |\alpha| \le r$ as well as q^α_j for $0 \le |\alpha| \le r$ correspond to natural functions constructed from natural operators $T \to TTT^r_k$ by the operations $\tilde{}$. Further, q^0_j for $j \in \{1, \dots, k\}$ and p^0_i can be easily annihilated by H. It follows from Lemma 4, that p^α_{k+1}, $1 \le |\alpha| \le r$ correspond to A^α_{k+1} invariant with respect to the action of $\mathrm{Ker}\,\pi_H$.

Thus we fix i and consider the action of H on p^α_{k+1}. It follows from the transformation laws for the action of H on p^α_i over i that

$$\bar{p}^\alpha_{k+1} = p^\alpha_{k+1} - a^j_{k+1k+1\gamma} q^{\alpha\gamma}_j \tag{5}$$

Let $0 \leq s \leq r$. Then there are $N_s = \mathrm{C}(s+k-1, k-1)$ multiindices α satisfying $|\alpha| = s$. The number of $a^j_{k+1k+1\gamma}$ is $(k+1)N_{r-s}$ and all p^α_{k+1} can be annihilated by $B^{r-s+2}_{k+1} \cap H$ whenever $(k+1)N_{r-s} \geq N_s$.

Let $(k+1)N_{r-s} < N_s$ and M be a minimal number of this property. Then (5) can be considered in the matrix form $\bar{p}_{k+1} = p_{k+1} - Qa$, which will be in more details disscussed later. This way we obtain that almost every G^{r+1}_{k+1} orbit in the sense of density forms a $(k+1)\,\Sigma^r_{s=M} N_{r-s}$-dimensional affine subspace in the $\Sigma^r_{s=M} N_s$-dimensional affine space parametrized by p^α_i and q^α_i corresponding to natural functions constructed from natural operators $T \to TTT^r_k$. Thus we obtain $\Sigma^r_{s=M} N_s - (k+1)\,\Sigma^r_{s=M} N_{r-s}$ base H-invariants and consequentely we obtain $\Sigma^r_{s=M} N_s - (k+1)\,\Sigma^r_{s=M} N_{r-s}$ base additional natural functions, while every natural function in question is obtained as a smooth combination of those additional ones and those constructed from natural operators $T \to TTT^r_k$.

Now we describe the matrix Q. It can be considered as a block-matrix Q^s_t of $r - M + 1$ block-lines and block-columns, where s denote indices of block-lines and t indices of block-columns for $s, t \in \{0, 1, \dots, r - M + 1\}$. A block $Q^s_t = 0$ if $t > s$. Let $s \geq t$. Then Q^s_t is a block-column matrix of N_t block-columns. Every block $Q^s_{t,l}$ for $l \in \{1, \dots, N_t\}$ contains of N_s lines and $k+1$ columns.

We are going to construct the additional natural functions. The equation (5) in the matrix form $\bar{p}_{k+1} = p_{k+1} - Qa$ can be considered as systems of linear equations in $a^j_{k+1k+1\gamma} q^{\alpha\gamma}_j$ cosidered by block-lines if we put for some multindices the left-hand sides zeros. Then $a^j_{k+1k+1\gamma}$ are uniquely determined and the values of other $\bar{p}^\alpha_{k+1}$ in the block-line determine the H-invariants. In the next block-line Q_s we use the already expressed $a^j_{k+1k+1\delta}$ for $|\delta| \leq s$ and by them we express further H-invariants. We use the identifications

$$p^\alpha_{k+1} \cong \widetilde{A^\alpha_{k+1}} \qquad \text{and} \qquad q^\alpha_j \cong \widetilde{A^{\alpha k+1}_j} \tag{6}$$

and this way obtain the additional natural functions on $T^*TT^r_k$, which follows from Lemma 4.

Now we construct the base of the additional natural functions. Let Mult_s denote the set of all multiindices of order s.

In the first step we costruct natural functions from p^α_{k+1}, where α are multiindices of order r. Let I_r denote a selection of multiindices $\alpha_1, \dots, \alpha_{k+1}$ for which the vectors $q^{\alpha_1}, \dots, q^{\alpha_{k+1}}$ are linearly independent, which is possible on a dense subset of $\mathbb{R}^{(k+1)N_r}$. In the first block-line of (5) considered in the matrix form put $\bar{p}^{\alpha_1}_{k+1} = \dots = \bar{p}^{\alpha_{k+1}}_{k+1} = 0$, which uniquely determines all a^j_{k+1k+1}. Their coordinate expressions are of the form

$$a^j_{k+1k+1} = \frac{\widetilde{A^{\alpha_s}_{k+1}} \cdot \mathrm{Ort}^j(\widetilde{A^{k+1\alpha_s}_i})}{\widetilde{A^{k+1\alpha_s}_j} \cdot \mathrm{Ort}^j(\widetilde{A^{k+1\alpha_s}_i})} \qquad \alpha_s \in I_r,\, i, j \in \{1, \dots, k+1\} \tag{7}$$

where $\mathrm{Ort}^j(\widetilde{A_i^{k+1\alpha_s}})$ denotes the orthogonal complement of column vectors forming the matrix $(\widetilde{A_i^{k+1\alpha_s}})$ of order $k+1$ and $\cdot$ denotes the scalar product. By means of a^j_{k+1k+1} we obtain natural functions $F^l_{I_r}$ corresponding to the selection of multiindices I_r

$$(8) \qquad F^l_{I_r} = \widetilde{A^{\alpha_l}_{k+1}} - \frac{\widetilde{A^{\alpha_s}_{k+1}} \cdot \mathrm{Ort}^j(\widetilde{A_i^{k+1\alpha_s}})}{\widetilde{A_j^{k+1\alpha_s}} \cdot \mathrm{Ort}^j(\widetilde{A_i^{k+1\alpha_s}})} \widetilde{A_j^{k+1\alpha_s}} \qquad \alpha_s \in I_r, l \in \mathrm{Mult}_r \backslash I_r$$

In the points, where the denominator is equal zero, $F^l_{I_r}$ can be multiplied in the neighbourhood and $F^l_{I_r}$ can be replaced by $G^l_{I_r}$, which are of the form

$$(9) \qquad G^l_{I_r} = (\widetilde{A_j^{k+1\alpha_s}} \cdot \mathrm{Ort}^j(\widetilde{A_i^{k+1\alpha_s}}))\widetilde{A^{\alpha_l}_{k+1}} - (\widetilde{A^{\alpha_s}_{k+1}} \cdot \mathrm{Ort}^j(\widetilde{A_i^{k+1\alpha_s}}))\widetilde{A_j^{k+1\alpha_s}}$$

Clearly, $G^l_{I_r}$ takes the zero value in the point, where the denominator is zero and it is a natural $T^*TT^r_k$-function.

In the next steps, we are searching the additional natural $T^*TT^r_k$-functions corresponding to multiindices of lower orders by a reccurent procedure. Fix an element $\omega = (p_i^\alpha, p_i^\alpha)$ over i. Let $r-s \geq M$ and suppose the selections $I_r, \dots, I_{r-s+1}$ of multiindices for which the corresponding $G^l_{I_r}, \dots, G^l_{I_r,\dots,I_{r-s+1}}$ take a non-zero value. It is satisfied on the dense subset.

Select a set of multiindices of order $r-s$ denoted by I_{r-s} so that the vectors $q^{\alpha_1}, \dots, q^{\alpha_{(k+1)N_{r-s}}}$ be linearly independent. Similarly as in the first step we consider the formula (5) in the matrix form and put $\bar{p}^{\alpha_1}_{k+1} = \dots = \bar{p}^{\alpha_{(k+1)N_{r-s}}}_{k+1} = 0$ on the left hand side. Using the identifications (6) we have

$$(10) \qquad F^l_{I_r,\dots,I_{r-s}} = \widetilde{A^{\alpha_l}_{k+1}} - a^j_{k+1k+1\gamma}\widetilde{A_j^{\alpha_l\gamma k+1}}, \qquad 0 \leq |\gamma| \leq s$$

where all $a^j_{k+1k+1\gamma}$ for $|\gamma| \leq s-1$ are determined in the previous steps. By the following formula we can reccurently compute $a^j_{k+1k+1\gamma}$ for $|\gamma| = s$

$$(11) \qquad a^j_{k+1k+1\gamma} = \frac{(\widetilde{A^{\alpha_t}_{k+1}} - a^h_{k+1k+1\delta}\widetilde{A_h^{k+1\alpha_t\delta}}) \cdot \mathrm{Ort}^j_\gamma(\widetilde{A_i^{\alpha_t\varepsilon k+1}})}{\widetilde{A_j^{\alpha_t\gamma k+1}} \cdot \mathrm{Ort}^j_\gamma(\widetilde{A_i^{\alpha_t\varepsilon k+1}})}$$

where δ, ε are multiindices satisfying $|\varepsilon| = |\gamma|$ and $0 \leq |\delta| < |\gamma|$. If the denominator in (11) is zero, we can consider the natural $T^*TT^r_k$- functions $G^l_{I_r,\dots,I_{r-s}}$ instead $F^l_{I_r,\dots,I_{r-s}}$, where

$$(12) \qquad G^l_{I_r,\dots,I_{r-s}} = (\widetilde{A_j^{\alpha_t\gamma k+1}} \cdot \mathrm{Ort}^j_\gamma(\widetilde{A_i^{\alpha_t\varepsilon k+1}}))F^l_{I_r,\dots,I_{r-s}}$$

Then $G^l_{I_r,\dots,I_{r-s}}$ take zero value, in the points where the denominator in (10) is zero.

Thus we proved the following theorem, giving all natural T-functions on $T^*TT^r_k$, since the step including the non-absolute natural T-functions is the same as in Proposition 1 or in Proposition 3.

Theorem 5. *Let M be a manifold satisfying* $\dim M \geq k+2$. *Then all natural T-functions $T^*TT^r_kM \to \mathbb{R}$ are of the form*

$$h(\widetilde{\mathcal{T}\circ(\,L(a)\,\mathcal{T}^r_k\,)}, \widetilde{\mathcal{V}\circ(\,L(b)\,\mathcal{T}^A\,)}, \widetilde{W(L(c))}), G^l_{I_r,\dots,I_{r-s}}) \tag{13}$$

for any smooth function h, $0 \leq s \leq M-r$ and $l \in \{(k+1)N_{r-s}+1,\dots,N_s\}$.

Moreover, in the s-th step we can choose only $N_{r-s}-(k+1)N_s$ base additional natural functions $G^l_{I_r,\dots,I_{r-s}}$.

5. We present a corollary of Theorem 5, giving the classification of all natural operators $T \to TT^*T^r_k$, which is the result published in [8] . This result is obtained in the different form and we present it little modified.

Let x^i_α denote the canonical coordinates on T^r_kM and $q^\alpha_i dx^i_\alpha$ define the additional coordinates on $T^*T^r_kM$. Let

$$A_MX(x^i_\alpha, q^\alpha_i) = Y^i_\alpha(j^{r+1}_0X, x^i_\alpha, q^\alpha_i)\frac{\partial}{\partial x^i_\alpha} + Q^\alpha_i(j^{r+1}_0X, x^i_\alpha, q^\alpha_i)\frac{\partial}{\partial q^\alpha_i} \tag{14}$$

be the expression of a natural operator A_M in coordinates, X being a vector field on M. Let us define by $\pi^\alpha_i dx^i_\alpha + \rho^i_\alpha dq^\alpha_i$ the additional coordinates on $T^*T^*T^r_kM$.

Since natural operators $TM \to TT^*T^r_kM$ are in the canonical bijection with natural T-functions $g_M : T^*T^*T^r_kM \to \mathbb{R}$ linear on fibers of $T^*T^*T^r_kM \to T^*T^r_kM$, (14) can be identified with

$$g_M = Y^i_\alpha(j^{r+1}_0X, x^i_\alpha, q^\alpha_i)\pi^\alpha_i + Q^\alpha_i(j^{r+1}_0X, x^i_\alpha, q^\alpha_i)\rho^i_\alpha. \tag{15}$$

Natural T-functions g_M are in the canonical bijection with natural T-functions $f_M : T^*TT^r_kM \to \mathbb{R}$, given by $f_M = g_M \circ t_{T^r_kM} \circ s^{-1}_{T^r_kM}$, where $s_M : TT^*M \to T^*TM$ is the natural equivalence by Modugno-Stefani, [7] and $t_M : TT^*M \to T^*T^*M$ is the natural equivalence by Kolář-Radziszewski, [6].

Let $y^i_\alpha = dx^i_\alpha$ define the additional coordinates on TT^r_kM and $p^\alpha_i dx^i_\alpha + s^\alpha_i dy^i_\alpha$ on $T^*TT^r_kM$. The coordinate expression of $t_{T^r_kM} \circ s^{-1}_{T^r_kM}$ yields

$$y^i_\alpha = -\rho^i_\alpha \qquad p^\alpha_i = \pi^\alpha_i \qquad s^\alpha_i = q^\alpha_i \tag{16}$$

Thus natural operators $A_M : TM \to TT^*T^r_kM$ are identified with natural T-functions $f_M : T^*TT^r_kM \to \mathbb{R}$ which are of the form $Y^i_\alpha(j^{r+1}_0X, x^i_\alpha, q^\alpha_i)p^\alpha_i - Q^\alpha_i(j^{r+1}_0X, x^i_\alpha, q^\alpha_i)y^i_\alpha$. Having a natural operator $A_M : TM \to TT^r_kM$ such that the corresponding natural T-function $\widetilde{A_M}$ on $T^*TT^r_kM$ is linear in (y^i_α, p^α_i), we obtain a natural operator $\widetilde{\widetilde{A_M}} : TM \to TT^*T^r_kM$. The base of natural opeators $A_M : TM \to TT^r_kM$ satisfying these properties of linearity is formed by $\mathcal{T}\circ L(c)\mathcal{T}^r_k$, $\mathcal{T}\circ\lambda^\beta_j$ and $W(L(c))$.

Proposition 6. *For a manifold M satisfying* $\dim M \geq k+2$ *all natural operators* $TM \to TT^*T^r_k M$ *are of the form*

$$h(\widetilde{\mathcal{V} \circ (\, L(\,d)\, \mathcal{T}^r_k\,)}, \widetilde{\mathcal{V} \circ \lambda^\mu_t})\, \widetilde{\widetilde{\mathcal{T} \circ (\, L(c)\, \mathcal{T}^r_k\,)}} + h^j_\beta(\widetilde{\mathcal{V} \circ (\, L(\,d)\, \mathcal{T}^r_k\,)}, \widetilde{\mathcal{V} \circ \lambda^\mu_t})\, \widetilde{\widetilde{\mathcal{T} \circ \lambda^\beta_j}} +$$

$$h_c(\widetilde{\mathcal{V} \circ (\, L(\,d)\, \mathcal{T}^r_k\,)}, \widetilde{\mathcal{V} \circ \lambda^\mu_t}\,)\, \widetilde{\widetilde{W(L(c))}}$$

Proof. Among all natural T-functions $f_M : T^*TT^r_k M \to \mathbb{R}$ we are searching for those, which are linear in (y^i_α, p^α_i). All natural T-functions f_M are of the form $h(\widetilde{\mathcal{T} \circ (\, L(\,a)\, \mathcal{T}^r_k\,)}, \widetilde{\mathcal{V} \circ (\, L(\,b)\, \mathcal{T}^A\,)}, \widetilde{W(L(c))}), G^l_{I_r,\dots,I_{r-s}})$. The coordinate expression of $G^l_{I_r,\dots,I_{r-s}}$ is a function of products $y^i_\alpha p^\beta_i$ and natural T-functions constructed from natural operators $T \to TTT^r_k$. It holds $Y^i_\alpha = \frac{\partial f}{\partial p^\alpha_i} = \frac{\partial f}{\partial p^\alpha_i}{}_{|y^i_\beta = 0 = p^\gamma_i} = \frac{\partial f}{\partial G^l_{I_r,\dots,I_{r-s}}} \cdot 0 + f_1(j^{r+1}_0 X, x^i_\alpha, q^\alpha_i)$. Thus Y^i_α does not depend on $G^l_{I_r,\dots,I_{r-s}}$ and analogously we prove that neither does Q^α_i. This completes the proof. $\square$

We are going to give a more geometrical form of this proposition. We need the following lemma.

Lemma 7. *Let $Y : TM \to TTM$ be a linear vector field, ([5]). Then there is a unique vector field $\widetilde{\widetilde{Y}} : T^*M \to TT^*M$ satisfying $\widetilde{\widetilde{\widetilde{Y}}} \circ t_M \circ s^{-1}_M = \widetilde{Y}$. Moreover, $\widetilde{\widetilde{Y}}$ is dual to Y.*

Proof. Let $Y = \xi^i(x) \frac{\partial}{\partial x^i} + \eta^i_j(x) y^j \frac{\partial}{\partial y^i}$. Further, let $p_i dx^i + q_i dx^i$ define the additional coordinates on T^*TM. Then $\widetilde{Y} = \xi^i(x) p_i + \eta^i_j(x) y^j q_i$. Define the additional coordinates on T^*M by $\alpha_i dx^i$ and on T^*T^*M by $\rho_i dx^i + \sigma^i d\alpha_i$. If $\widetilde{\widetilde{Y}} = U^i \frac{\partial}{\partial x^i} + V_i \frac{\partial}{\partial \alpha_i}$, then $\widetilde{\widetilde{\widetilde{Y}}} = U^i \rho_i + V_i \sigma^i$. It follows from the definitions of the natural equivalences by Modugno-Stefani, ([7]), Kolář-Radziszewski, ([6]) and their coordinate descriptions that $\rho_i = p_i$ and $\sigma^i = -y^i$ and consquentely we have $\widetilde{\widetilde{\widetilde{Y}}} = p_i U^i - y^i V_i$. Thus we have $V_i = -\eta^j_i(x) q_j$, which completes the proof. $\square$

Lemma 8. *Let $Y = \mathcal{T} X$ be the flow prolongation of a vector field X. Then $\widetilde{\widetilde{Y}} = \mathcal{T}^* X$.*

Proof. Using the coordinates from Lemma 7, we have

$$\mathcal{T} X = X^i(x) \frac{\partial}{\partial x^i} + X^i_j(x) y^j \frac{\partial}{\partial y^i}\,.$$

One can easily verify that

$$\mathcal{T}^* X = X^i(x) \frac{\partial}{\partial x^i} - X^j_i(x) \frac{\partial}{\partial q_i}\,.$$

Thus we have $\widetilde{\widetilde{\mathcal{T}\circ(L(c)\mathcal{T}^r_k)}} = \mathcal{T}^*\circ(L(c)\mathcal{T}^r_k)$ and $\widetilde{\widetilde{\mathcal{T}\circ\lambda^\beta_j}} = \mathcal{T}^*\circ\lambda^\beta_j$. Moreover, we have $\widetilde{\widetilde{W(L(c))}} = W^*(L(c)) = -W(L^*(c))$, where $L^*(c)$ denotes the dual map to $L(c)$. Now we can present Proposition 6 in the following form.

Proposition 9. *All natural operators $T\to TT^*T^r_k$ are of the form*

$$h_c(\widetilde{L(d)\mathcal{T}^r_k},\widetilde{\lambda^\mu_t})\mathcal{T}^*\circ(L(c)\mathcal{T}^r_k) + h^j_\beta(\widetilde{L(d)\mathcal{T}^r_k},\widetilde{\lambda^\mu_t})\mathcal{T}^*\circ\lambda^\beta_j + h_c(\widetilde{L(d)\mathcal{T}^r_k},\widetilde{\lambda^\mu_t})W(L^*(c))$$

for any basis of $\mathbb{D}^r_k$ formed by c, d, $1\le|\beta|,|\mu|\le r$ and for any smooth functions h_c, h^j_β.

Proof. It remains to investigate the arguments of functions, h_c, h^j_β. This is quite similar to [8].

I thank prof. I. Kolář for his useful help and suggestions.

References

[1] Kobak, P., *Natural liftings of vector fields to tangent bundles of 1-forms*, Mathematica Bohemica **116** (1991), 319–326.

[2] Kolář, I., *Covariant Approach to Natural Transformations of Weil Bundles*, Comment. Math. Univ. Carolinae (1986).

[3] Kolář, I., *On Cotagent Bundles of Some Natural Bundles*, Rendiconti del Circolo Matematico di Palermo Serie II-Numero **37** (1994), 115–120.

[4] Kolář, I., *On the Natural Operators on Vector Fields*, Ann. Global Anal. Geometry **6** (1988), 109–117.

[5] Kolář, I., Michor, P.W., Slovák, J., *Natural Operations in Differential Geometry*, Springer – Verlag, 1993.

[6] Kolář, I., Radziszewski, Z., *Natural Transformations of Second Tangent and Cotangent Bundles*, Czechoslovak Math. (1988), 274–279.

[7] Modugno, M., Stefani, G., *Some results on second tangent and cotangent spaces*, Quaderni dell'Università di Lecce (1978).

[8] Tomáš, J., *Natural operators on vector fields on the cotangent bundles of the bundles of (k,r)-velocities*, Rendiconti del Circolo Matematico di Palermo Serie II **Numero 54** (1998), 113–124.

[9] Tomáš, J., *Some natural operators on vector fields*, Archivum Mathematicum **31** (1995), 239–249.

Department of Mathematics, Faculty of Civil Engineering
Technical University Brno
Žižkova 17, 602 00 Brno, CZECH REPUBLIC
E-mail: mdtom@fce.vutbr.cz