DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 215–222

ON THE SPECIAL CLASS OF CURVES ON SOME FOUR-DIMENSIONAL SEMIPARALLEL SUBMANIFOLDS

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Abstract. A submanifold M^m in the Euclidean space E^n is said to be *parallel* if it satisfies the property $\bar{\nabla} h = 0$, where h is the second fundamental form and $\bar{\nabla}$ denotes the van der Waerden-Bortolotti connection. A submanifold is said to be *semiparallel* if it satisfies the property $\bar{R} \circ h = 0$, where $\bar{R}$ denotes the curvature operator of the connection. It has been proved by Lumiste [11] that the semiparallel submanifolds are the second order envelopes of the parallel ones. Using the Cartan moving frame method and the exterior differential calculus the paper describes some special classes of curves on irreducible envelopes of the reducible symmetric submanifolds $V^2(r_1) \times S^1(r_2) \times S^1(r_3)$ with a Veronese component, which is a Veronese surface in E^5.

0. Introduction

Let us examine a submanifold M^m in Euclidean space E^n by the Cartan moving frame method with the help of exterior differential calculus (see [6], Ch I §1). Then in the formulae of infinitesimal displacement

$$d\vec{x} = \omega^I \vec{e}_I, \quad d\vec{e}_I = \omega_I^K \vec{e}_K \tag{0.1}$$

of an orthonormal frame bundle $O(E^n) = \{x, \vec{e}_I (I = 1, \ldots, n)\}$, $x \in M^m$, we have $\omega_I^K + \omega_K^I = 0$, where $I, K = 1, \ldots, n$ and in addition the structure equations

$$d\omega^I = \omega^K \wedge \omega_K^I, \quad d\omega_I^K = \omega_I^L \wedge \omega_L^K\ . \tag{0.2}$$

If the bundle $O(E^n)$ is reduced to

$$O(M^m, E^n) = \{x, \vec{e}_i\, (i = 1, \ldots, m),\ \vec{e}_\alpha\, (\alpha = m+1, \ldots, n)\}$$

at every point $x \in M^m$, then in (0.1)

$$\omega^\alpha = 0$$

1991 *Mathematics Subject Classification.* 53B25, 53C40.

Key words and phrases. Curves on semiparallel submanifolds, parallel submanifolds, semiparallel submanifolds, Veronese surfaces.

This paper is in final form and no version of it will be submitted for publication elsewhere.

and the exterior differentiation of this system leads by means of (0.2) and the Cartan lemma to

$$\omega_i^\alpha = h_{ij}^\alpha \omega^j , \quad h_{ij}^\alpha = h_{ji}^\alpha .$$

These determine the *second fundamental form* with values in the normal vector bundle $T^\perp M^m$ of the manifold

$$h : (\vec{e}_i X^i , \vec{e}_j Y^j) \mapsto \vec{e}_\alpha h_{ij}^\alpha X^i Y^j .$$

The next step of exterior differentiation leads to the *third fundamental form*

$$\bar{\nabla} h : (\vec{e}_i X^i , \vec{e}_j Y^j , \vec{e}_k Z^k) \mapsto \vec{e}_\alpha h_{ijk}^\alpha X^i Y^j Z^k ,$$

whose components are given by

$$\bar{\nabla} h_{ij}^\alpha = \bar{\nabla}_k h_{ij}^\alpha \omega^k = h_{ijk}^\alpha \omega^k .$$

Here the coefficients h_{ijk}^α are symmetric with respect to the tangent indices due to the formulae of Peterson-Mainardi-Codazzi $\bar{\nabla}_k h_{ij}^\alpha = \bar{\nabla}_j h_{ik}^\alpha$. Continuing by exterior differentiation we get the *fourth fundamental form*

$$\bar{\nabla}(\bar{\nabla} h) \equiv \bar{\nabla}^2 h : (\vec{e}_i X^i , \vec{e}_j Y^j , \vec{e}_k Z^k , \vec{e}_l W^l) \mapsto \vec{e}_\alpha h_{ijkl}^\alpha X^i Y^j Z^k W^l ,$$

whose components are given by

$$\bar{\nabla} h_{ijk}^\alpha = \bar{\nabla}(\bar{\nabla}_k h_{ij}^\alpha) = \bar{\nabla}_l (\bar{\nabla}_k h_{ij}^\alpha) \omega^l = h_{ijkl}^\alpha \omega^l .$$

Here the components h_{ijkl}^α need not to be symmetric with respect to the lower indices. Continuing analogously with exterior differentiation we obtain the set of fundamental forms $\bar{\nabla}^s h$ $(s \geq 0)$. The submanifold M^m in Euclidean space (or even more generally, in space form $N^n(c)$) is said to be s-*parallel* if $\bar{\nabla}^s h = 0$, $\bar{\nabla}^{s-1} h \neq 0$ [13]. In other words, an s-parallel submanifold has a parallel fundamental form of order $s+1$ and a nonparallel fundamental form of order s. Thus 0-*parallel* submanifolds are totally geodesic, 1-*parallel* submanifolds are the well-known parallel ones, which have been completely described by Ferus [5] in terms of Lie algebras. The parallel submanifolds are analytically determined by the system $\bar{\nabla} h = 0$ of differential equations whose integrability condition is $\bar{R} \circ h = 0$. Here the components of $\bar{R} \circ h$ are given by

$$\bar{\nabla}(\bar{\nabla}_k h_{ij}^\alpha) \wedge \omega^k = -h_{kj}^\alpha \Omega_i^k - h_{ik}^\alpha \Omega_j^k + h_{ij}^\beta \Omega_\beta^\alpha ,$$

where $\Omega_i^k = \omega_i^\alpha \wedge \omega_\alpha^k$ and $\Omega_\beta^\alpha = \omega_\beta^k \wedge \omega_k^\alpha$ are the curvature forms of Levi-Civita and the normal connections, respectively. If $\bar{R} \circ h = 0$, the submanifold M^m is said to be *semiparallel* [1], [2]. This condition leads to the symmetry of coefficients h_{ijkl}^α with respect to the lower indices. It was proved by Lumiste [11] that the semiparallel submanifolds are the second order envelopes of the parallel ones. It is proved in [7] that the system $\bar{\nabla}^2 h = 0$ of differential equations determining 2-parallel submanifolds implies $\bar{R} \circ h = 0$. Thus a submanifold with a parallel third fundamental form is always a semiparallel one. This means that the classification of semiparallel submanifolds describes the connections between different classes of submanifolds. The classification has not yet been completed even for M^m in

Euclidean spaces E^n. Surveys of results have been given by Deprez [3] up to 1989 and by Lumiste [13] up to 1991. The classification problem has been solved for $m = 2$ [1], 3 [10,15] (partly together with the present author), $n-2$ [9], $n-1$ [2]. All normally flat semiparallel M^m have been described by Lumiste [12] in E^n and by Dillen and Nölker [4] in the space forms $N^n(C)$. Normally non-flat reducible cases for $m = 4$ have been described in [14,15] except the case in which M^4 is a second order envelope of the reducible symmetric submanifold $V^2(r_1) \times S^1(r_2) \times S^1(r_3)$ with a Veronese component, which is a Veronese surface in E^5, and its special case $V^2(r_1) \times S^1(r_2) \times E^1$ $(r_3 = \infty)$. For the most general case $r_2 < \infty$, $r_3 < \infty$ of the submanifolds M^4 mentioned above, the following results have been proved [16].

Proposition 1. *The tangent distributions $TV^2(r_1)$ and $TS^1(r_2) \times TS^1(r_3)$ on M^4 determine foliations.*

The integral submanifolds of the foliations are Veronese leaves M^2_{Ver} and Clifford leaves M^2_{Cliff}, respectively. The Veronese leaves M^2_{Ver} are the most general semiparallel surfaces with non-flat Levi-Civita and non-flat normal connections. The Clifford leaves M^2_{Cliff} are surfaces with flat van der Waerden-Bortolotti connection. In this paper the investigations will be continued by a description of special classes of curves on the submanifolds M^4 under consideration.

1. Preliminaries and apparatus

Let a semiparallel submanifold M^4 in Euclidean space E^n be a second order envelope of the normally non-flat reducible symmetric submanifold $V^2(r_1) \times S^1(r_2) \times S^1(r_3)$ with a Veronese component $V^2(r_1)$, which is a Veronese surface in E^5. Let the reduced orthonormal frame bundle $O(M^4, E^n)$ at every point $x \in M^4$ be specialized so that $\vec{e}_1, \vec{e}_2$ belong to $T_xV^2(r_1)$, $\vec{e}_3$ to $T_xS^1(r_2)$ and $\vec{e}_4$ to $T_xS^1(r_3)$. Further, let $\vec{e}_5$, $\vec{e}_6$, $\vec{e}_7$ determine the first normal space $N^1_xV^2(r_1)$ and $\vec{e}_8$, $\vec{e}_9$ the normal spaces $N^1_xS^1(r_2)$ and $N^1_xS^1(r_3)$, respectively. Let the rest frame vectors be denoted by $\vec{e}_\xi$, $\xi = 10, \dots, n$. In the formulae of the infinitesimal displacement of such a subbundle $O(M^4, E^n)$,

$$d\vec{x} = \omega^I \vec{e}_I\,, \quad d\vec{e}_I = \omega_I^K \vec{e}_K\,, \tag{1}$$

with $\omega_I^K + \omega_K^I = 0$, where $I, K, \dots = 1, \dots, n$, and also in the structure equations,

$$d\omega^I = \omega^K \wedge \omega_K^I\,, \quad d\omega_K^I = \omega_K^L \wedge \omega_L^I\,, \tag{2}$$

we have

$$\omega^5 = \omega^6 = \omega^7 = \omega^8 = \omega^9 = \omega^\xi = 0\,, \tag{3}$$

$$\omega_1^\xi = \omega_2^\xi = \omega_3^\xi = \omega_4^\xi = 0, \xi = 10, \dots, n\,. \tag{4}$$

Due to the product structure of the symmetric submanifold $V^2(r_1) \times S^1(r_2) \times S^1(r_3)$, which by assumption has second order tangency with M^4 at every point $x \in M^4$, the following equations are satisfied:

$$\omega_1^8 = \omega_2^8 = \omega_1^9 = \omega_2^9 = 0\,, \quad \omega_3^5 = \omega_3^6 = \omega_3^7 = \omega_3^9 = 0\,,$$

$$\omega_4^5 = \omega_4^6 = \omega_4^7 = \omega_4^8 = 0\,. \tag{5}$$

A further adaption of the frame part $\{\vec{e}_1, \vec{e}_2, \vec{e}_5, \vec{e}_6, \vec{e}_7\}$ to $V^2(r_1)$, as has been done in [15], leads to

$$\begin{gathered}
\omega_1^5 = \rho_1\sqrt{3}\omega^1\,, \quad \omega_1^6 = \rho_1\omega^1\,, \quad \omega_1^7 = \rho_1\omega^2\,, \\
\omega_2^5 = \rho_1\sqrt{3}\omega^2\,, \quad \omega_2^6 = -\rho_1\omega^2\,, \quad \omega_2^7 = \rho_1\omega^1\,, \\
\omega_3^8 = \rho_2\omega^3\,, \quad \omega_4^9 = \rho_3\omega^4\,,
\end{gathered} \tag{6}$$

where $\rho_1 = r_1^{-1}$, $\rho_2 = r_2^{-1}$, $\rho_3 = r_3^{-1}$. Let us suppose that ρ_2 and ρ_3 do not vanish, i.e. the circles $S^1(r_2)$, $S^1(r_3)$ do not become straight lines. The semiparallel submanifold M^4 under consideration is determined by the system (3)-(6) of Pfaff equations.

Due to the structure equations (2) and the Cartan lemma, exterior differentiation of the equations of the system (3)-(6) leads to the system

$$\begin{gathered}
\omega_1^3 = \rho_1 C\omega^1\,, \quad \omega_2^3 = \rho_1 C\omega^2\,, \quad \omega_5^8 = -(\sqrt{3})^{-1}\rho_1 C\omega^3\,, \quad \omega_6^8 = \omega_7^8 = 0\,, \\
\omega_1^4 = \rho_2 D\omega^1\,, \quad \omega_2^4 = \rho_2 D\omega^2\,, \quad \omega_5^9 = -(\sqrt{3})^{-1}\rho_2 D\omega^3\,, \quad \omega_6^9 = \omega_7^9 = 0\,,
\end{gathered} \tag{7}$$

and the following consequences:

$$\begin{gathered}
\omega_5^\xi = A_1^\xi\omega^1 + A_2^\xi\omega^2\,, \quad \omega_6^\xi = A_3^\xi\omega^1 + A_4^\xi\omega^2\,, \\
\omega_7^\xi = (\sqrt{3}A_2^\xi + A_4^\xi)\omega^1 + (\sqrt{3}A_1^\xi - A_3^\xi)\omega^2\,, \\
\omega_8^\xi = B^\xi\omega^3\,, \quad \omega_9^\xi = C^\xi\omega^4\,, \\
\omega_3^4 = \rho_2 E_1\omega^3 - \rho_3 E_2\omega^4\,, \quad \omega_8^9 = \rho_2 E_2\omega^3 - \rho_3 E_1\omega^4\,, \\
\omega_5^6 = \sqrt{3}\lambda_1\omega^1 + \sqrt{3}\lambda_2\omega^2\,, \quad \omega_5^7 = -\sqrt{3}\lambda_2\omega^1 + \sqrt{3}\lambda_1\omega^2\,, \\
\omega_6^7 = 2\omega_1^2 - 5\lambda_2\omega^1 - 5\lambda_1\omega^2\,, \\
dln\rho_1 = -2\lambda_1\omega^1 + 2\lambda_2\omega^2 + \rho_1 C\omega^3 + \rho_1 D\omega^4\,, \\
d\rho_2 = \gamma_3\omega^3 + \rho_2^2 E_1\omega^4\,, \quad d\rho_3 = \rho_3^2 E_2\omega^3 + \gamma_4\omega^4\,.
\end{gathered} \tag{8}$$

After the exterior differentiation of (7) there arise, in particular, the expressions

$$\begin{gathered}
dC = 2\lambda_1 C\omega^1 - 2\lambda_2 C\omega^2 + \rho_2 DE_1\omega^3 - \rho_3 DE_2\omega^4\,, \\
dD = 2\lambda_1 D\omega^1 - 2\lambda_2 D\omega^2 - \rho_2 CE_1\omega^3 + \rho_3 CE_2\omega^4\,.
\end{gathered} \tag{9}$$

If we denote

$$\hat\omega^3 = C\omega^3 + D\omega^4\,, \quad \hat\omega^{12} = \lambda_1\omega^1 - \lambda_2\omega^2\,, \quad \hat\omega^{34} = \rho_2 E_1\omega^3 - \rho_3 E_2\omega^4\,, \tag{10}$$

then it follows from (9) that

$$dC = 2C\hat\omega^{12} + D\hat\omega^{34}\,, \quad dD = 2D\hat\omega^{12} - C\hat\omega^{34}\,, \tag{11}$$

and the exterior differentiation of (10), in conjunction with the structure equations (2), leads to

$$d\hat{\omega}^3 = 2\hat{\omega}^{12} \wedge \hat{\omega}^3 \,, \quad d\hat{\omega}^{12} = 0 \,, \quad d\hat{\omega}^{34} = 0 \,.$$

2. Results

As by the Proposition 1 the tangent distributions $TV^2(r_1)$ and $TS^1(r_2) \times TS^1(r_3)$ on M^4 determine foliations, then let us denote the corresponding integral submanifolds by $M^2_{Ver} = M^4(mod\,\omega^3, \omega^4)$ and $M^2_{\mathrm{Cliff}} = M^4(mod\,\omega^1, \omega^2)$. Here M^2_{Ver} is the 2nd order envelope of Veronese surfaces *(Veronese leaf)* and M^2_{Cliff} is the 2nd order envelope of the Clifford tori *(Clifford leaf).*

Let us introduce the function $\gamma^2 = C^2 + D^2$, which shall be called the *function of reducibility* as its values describe the reducibility properties of the submanifolds M^4 under consideration [16,17]. From the decomposition theorems [7,8] and (7) there follows

Proposition 2. *The second order envelope M^4 of the reducible symmetric submanifold $V^2(r_1) \times S^1(r_2) \times S^1(r_3)$ is irreducible iff neither of the parameters C and D vanish.*

The orthogonal vectors $\vec{\varepsilon}_3 = C\vec{e}_3 + D\vec{e}_4$ and $\vec{\varepsilon}_4 = -D\vec{e}_3 + C\vec{e}_4$ with $|\vec{\varepsilon}_3| = |\vec{\varepsilon}_4| = \gamma^2 \neq 0$ determine an orthogonal net of curves on the Clifford leaf M^2_{Cliff}. Let us denote the curves with tangent vectors $\vec{\varepsilon}_3$ by $s(I)$ and the curves with tangent vectors $\vec{\varepsilon}_4$ by $s(II)$, respectively. By (1) and (11) we have

$$d\vec{\varepsilon}_3 = 2\hat{\omega}^{12}\vec{\varepsilon}_3 - \rho_1\gamma^2(\omega^1\vec{e}_1 + \omega^2\vec{e}_2) + C\rho_2\omega^3\vec{e}_8 + D\rho_3\omega^4\vec{e}_9,$$

$$d\vec{\varepsilon}_4 = 2\hat{\omega}^{12}\vec{\varepsilon}_4 - \rho_2 D\omega^3\vec{e}_8 + \rho_3 C\omega^4\vec{e}_9, \tag{12}$$

Proposition 3. *The first curvatures of the curves $s(I)$ and $s(II)$ are $k(I) = \gamma^{-1}\sqrt{\rho_2^2C^4 + \rho_3^2D^4}$ and $k(II) = \gamma^{-1}\sqrt{\rho_2^2D^4 + \rho_3^2C^4}$, respectively. Their vectors of normal curvature have a nonzero component in the direction of one of the normal vectors of M^2_{Ver} and the components in the direction of $\vec{e}_\xi$ can be non-zero if at least one of the vectors $\vec{b} = B^\xi\vec{e}_\xi$, $\vec{c} = C^\xi\vec{e}_\xi$ does not vanish.*

Proof. Let us transform the part $\vec{e}_3$, $\vec{e}_4$ of the reduced orthonormal frame bundle $O(M^4, E^n)$ by the rotation

$$\vec{\varepsilon}_{30} = \vec{e}_3\cos\alpha + \vec{e}_4\sin\alpha\,, \quad \vec{\varepsilon}_{40} = -\vec{e}_3\sin\alpha + \vec{e}_4\cos\alpha\,,$$

or

$$\vec{e}_3 = \vec{\varepsilon}_{30}\cos\alpha - \vec{\varepsilon}_{40}\sin\alpha\,, \quad \vec{e}_4 = \vec{\varepsilon}_{30}\sin\alpha + \vec{\varepsilon}_{40}\cos\alpha\,,$$

where $\cos\alpha = \gamma^{-1}C$, $\sin\alpha = \gamma^{-1}D$, $\alpha \neq 0$, $\alpha \neq \frac{\pi}{2}$ on the Clifford leaves M^2_{Cliff} under consideration. Then the first formula of an infinitesimal displacement of the described subbundle $O(M^4, E^n)$ along the curve $s(I)$ is as follows:

$$d\vec{x} = \omega^{30}\vec{\varepsilon}_{30} = \omega^{30}\gamma^{-1}C\vec{e}_3 + \omega^{30}\gamma^{-1}D\vec{e}_4\,.$$

From here it is obvious that

$$\omega^3 = \gamma^{-1} C\omega^{30} = \omega^{30}\cos\alpha\,, \quad \omega^4 = \gamma^{-1} D\omega^{30} = \omega^{30}\sin\alpha\,.$$

Then from (12) we get

$$d\vec{\varepsilon}_{30} = \omega^{30}(\rho_2\cos^2\alpha\vec{e}_8 + \rho_3\sin^2\alpha\vec{e}_9)$$

and denoting

$$\vec{\varepsilon}_{31} = \rho_2\cos^2\alpha\vec{e}_8 + \rho_3\sin^2\alpha\vec{e}_9$$

we have

$$|\vec{\varepsilon}_{31}|^2 = \rho_2^2\cos^4\alpha + \rho_3^2\sin^4\alpha\,.$$

Now due to

$$d\vec{\varepsilon}_{30} = \omega^{30}k(I)\vec{\varepsilon}_{310} = \omega^{30}|\vec{\varepsilon}_{31}|\vec{\varepsilon}_{310}$$

it follows that $k(I) = |\vec{\varepsilon}_{31}| > 0$. In particular,

$$k(I) = \sqrt{\rho_2^2\cos^4\alpha + \rho_3^2\sin^4\alpha} = \gamma^{-1}\sqrt{\rho_2^2 C^4 + \rho_3^2 D^4}\,.$$

Analogously the formulae of infinitesimal displacement of the bundle $O(M^4, E^n)$ along the curve $s(II)$ on the Clifford leaf are, due to (12), as follows:

$$d\vec{x} = \hat{\omega}^4\vec{\varepsilon}_{40} = -\hat{\omega}^4\gamma^{-1}D\vec{e}_3 + \hat{\omega}^4\gamma^{-1}C\vec{e}_4 = \hat{\omega}^4(-\sin\alpha\vec{e}_3 + \cos\alpha\vec{e}_4)\,,$$

$$d\vec{\varepsilon}_{40} = \hat{\omega}^4(\rho_2\sin^2\alpha\vec{e}_8 + \rho_3\cos^2\alpha\vec{e}_9)\,.$$

Recall that here $\vec{\varepsilon}_4 = -D\vec{e}_3 + C\vec{e}_4 = \gamma\vec{\varepsilon}_{40}$. Denoting

$$\vec{\varepsilon}_{41} = \rho_2\sin^2\alpha\vec{e}_8 + \rho_3\cos^2\alpha\vec{e}_9 = |\vec{\varepsilon}_{41}|\vec{\varepsilon}_{410}$$

we have

$$|\vec{\varepsilon}_{41}|^2 = \rho_2^2\sin^4\alpha + \rho_3^2\cos^4\alpha\,.$$

Now from

$$d\vec{\varepsilon}_{40} = \hat{\omega}^4 k(II)\vec{\varepsilon}_{410} = \hat{\omega}^4|\varepsilon_{41}|\vec{\varepsilon}_{410}$$

it follows that $k(II) = |\varepsilon_{41}| > 0$, and in particular,

$$k(II) = \sqrt{\rho_2^2\sin^4\alpha + \rho_3^2\cos^4\alpha} =$$

$$= \gamma^{-1}\sqrt{\rho_2^2 D^2 + \rho_3^2 C^2}\,.$$

Description of the second normal spaces of considered curves are given by the differentials $d\vec{\varepsilon}_{310}$ and $d\vec{\varepsilon}_{410}$, respectively:

$$d\vec{\varepsilon}_{310} = -k(I)\omega^{30}\vec{\varepsilon}_{30} + \omega^{30}(\kappa_I^{40}\vec{\varepsilon}_{40} + \kappa_I^5\vec{e}_5 + \kappa_I^8\vec{e}_8 + \kappa_I^9\vec{e}_9 + \kappa_I^\xi\vec{e}_\xi)\,,$$

$$d\vec{\varepsilon}_{410} = -k(II)\hat{\omega}^4\vec{\varepsilon}_{40} + \hat{\omega}^4(\kappa_{II}^{30}\vec{\varepsilon}_{30} + \kappa_{II}^5\vec{e}_5 + \kappa_{II}^8\vec{e}_8 + \kappa_{II}^9\vec{e}_9 + \kappa_{II}^\xi\vec{e}_\xi)\,.$$

Calculations, which have been done with the help of (1), (8)-(11), lead to

$$\kappa_I^{40} = \frac{\cos\alpha\cdot\sin\alpha(\rho_2^2\cos^2\alpha - \rho_3^2\sin^2\alpha)}{k(I)}\,,$$

$$\kappa_I^5 = \frac{\gamma}{\sqrt{3}} k(I) \neq 0\,,$$

$$\kappa_I^\xi \vec{e}_\xi = \frac{1}{k(I)}(\rho_2 \cos^3 \alpha \vec{b} + \rho_3 \sin^3 \alpha \vec{c}) = \frac{1}{k(I)}(B^\xi \rho_2 \cos^3 \alpha + C^\xi \rho_3 \cos^3 \alpha)\vec{e}_\xi\,,$$

$$\kappa_{II}^{30} = \frac{\cos\alpha \cdot \sin\alpha(\rho_2^2 \sin^2\alpha - \rho_3^2 \cos^2\alpha)}{k(II)}\,,$$

$$\kappa_{II}^5 = -\frac{\gamma}{\sqrt{3}k(II)} \cos\alpha \cdot \sin\alpha(\rho_2^2 \sin^2\alpha - \rho_3^2 \cos^2\alpha) = -\frac{\gamma}{\sqrt{3}} k_{II}^{30}\,,$$

$$\kappa_{II}^\xi \vec{e}_\xi = \frac{1}{k(II)}(-\rho_2 \sin^3 \alpha \vec{b} + \rho_3 \cos^3 \alpha \vec{c}) = -\frac{1}{k(II)}(-B^\xi \rho_2 \sin^3 \alpha + C^\xi \rho_3 \cos^3 \alpha)\vec{e}_\xi\,.$$

These equalities prove the second assertion of the proposition. The reason being that the normal curvature vector $s(I)$ has a nonzero component k_I^5 in the direction of the normal $\vec{e}_5$ of the Veronese leaf. The non-vanishing or otherwise of the corresponding quantity for $s(II)$ is then of no consequence for this proof. The components of these normal curvature vectors in the direction of $\vec{e}_\xi$ are given by coefficients k_I^ξ and k_{II}^ξ, which depend linearly on the parameters B^ξ and C^ξ and can vanish only if the vectors $\vec{b}$ and $\vec{c}$ vanish simultaneously.

Remark. The irreducibilty of the semiparallel submanifold M^4 under consideration is described by the coefficients k_I^5 and k_{II}^5 (see the Proposition 2).

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