DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 207–213

ON A VARIETY OF WEIERSTRASS DATA FOR BRANCHED MINIMAL SURFACES IN EUCLIDEAN SPACE

KATSUHIRO MORIYA

Abstract. A moduli space of Weierstrass data for a certain class of algebraic minimal surfaces in $\mathbb{R}^3$ or translation spaces becomes a real analytic variety. In this paper, the defining equations of the variety is obtained in the case the surface is of genus 0 with two puncture points which coincide with two ramification points of the Gauss map and whose order are equal each other. The moduli space becomes a real algebraic variety.

1. Introduction

The purpose of this paper is to give a moduli space of a certain class of non-planar algebraic minimal surfaces in $\mathbb{R}^3$ and in a translation space $\mathbb{R}^3/T(v)$. A translation space $\mathbb{R}^3/T(v)$ is defined by the quotient space of $\mathbb{R}^3$ by the discrete group of isometries $T(v)$ generated by a translation by $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. A possibly branched complete conformal minimal immersion $X : M \to \mathbb{R}^3/T(v)$ from a Riemann surface M is called an *algebraic minimal surface* if it is of finite total curvature.

The notable property of an algebraic minimal surface is that the Riemann surface M is compactified conformally (cf. [3] and [4]). Hence, this surface is denoted by $(X, \bar{M}, \mathbb{R}^3/T(v))$ in this paper, where $\bar{M}$ is the Riemann surface compactified from M.

There are two kinds of methods to study the moduli space of algebraic minimal surfaces. One is to investigate certain operators characterizing minimal surfaces. The other is to investigate the moduli space of the data corresponding to minimal surfaces. For example, J. Peréz and A. Ros [5], [6] proved the moduli space of nondegenerate, unbranched, properly immersed algebraic minimal surfaces in $\mathbb{R}^3$ with embedded horizontal ends becomes a real analytic manifold by applying the

---

1991 *Mathematics Subject Classification.* Primary 53A10; Secondary 53C42.
Key words and phrases. Minimal surface, moduli space.
This paper is in final form and no version of it will be submitted for publication elsewhere.

implicit function theorem to a certain real analytic map. In [7], the Mo's construction of the moduli space of algebraic minimal surfaces in $\mathbb{R}^3$ with fixed topology is appeared. This is done by investigating the moduli space of their Weierstrass data. R. Kusner and N. Schmitt [2] proved the moduli space of algebraic minimal surfaces of genus zero with embedded planar ends becomes a certain complex space by investigating the moduli space of their spinor data of the spinor representation.

The interesting problems about the study of moduli space are to find geometric structures defined on the moduli space and to obtain results about existence or characterization of the surfaces by the geometric properties. The above studies obtain such results.

To investigate the geometric property of the moduli space in detail, it is necessary to clarify the precise parametrization of the moduli space. This is possible under the formulation by X. Mo when the compactified Riemann surface is $\mathbb{C}P^1$.

In this paper, the moduli space $\mathcal{W} = \mathcal{W}(\tilde{B}, P, dz/z, v)$ is studied by the X. Mo's method where $\mathcal{W} = \mathcal{W}(\tilde{B}, P, dz/z, v)$ is consisted of the Weierstrass data of algebraic minimal surfaces $(X, \mathbb{C}P^1, \mathbb{R}^3/T(v))$ which have 0 and ∞ as puncture points of order P, and whose Gauss map ramifies at the puncture points with the same order and the total order of branch points is $\tilde{B}$. The terminologies used above are described the following section. Let U be an open subset of $\mathbb{R}^{4J+2}$ defined by

$$U := \{(e_{1,1}, e_{1,2}, \ldots, e_{2J,1}, e_{2J,2}, c_1, c_2) \in \mathbb{R}^{4J+2} \mid e_{2J,1}^2 + e_{2J,2}^2 \neq 0, c_1^2 + c_2^2 \neq 0\}. \tag{1.1}$$

Then, the set $\mathcal{W}$ is considered as a subset of U. Moreover, the following is proved:

Theorem 1.1. *If the moduli space $\mathcal{W} = \mathcal{W}(\tilde{B}, P, dz/z, v)$ is nonempty, it is a nonsingular real algebraic variety of U whose dimension is $4J-1$ if $k \leq J-1$, $4J+1$ if $k \geq J+1$, and $4J$ if $k = J \neq 0$.*

The author would like to express his gratitude to Y. Ohnita for his encouragement.

2. The Weierstrass representation

In this section, a brief explanation of the Weierstrass representation for algebraic minimal surfaces used in this paper is given (cf. K. Yang [7], D. Hoffman and H. Karcher [1], and K. Moriya [3], [4]).

Let $(X = \mathbb{C}P^1, \mathbb{R}^3/T(v))$ be an algebraic minimal surface and M a Riemann surface which is the domain of X. In the following, when $v \neq (0,0,0)$, assume that there exists the minimal immersion $\tilde{X}\colon \tilde{M} \to \mathbb{R}^3$ from a holomorphic covering $\tilde{M}$ of M which make the following diagram commutative:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\exists \tilde{X}} & \mathbb{R}^3 \\ {\scriptstyle \exists\pi}\downarrow & & \downarrow{\scriptstyle \Pi} \\ M & \xrightarrow[X]{} & \mathbb{R}^3/T(v) \end{array}$$

where Π is the natural projection. In the case where $v = (0,0,0)$, assume that $\tilde{X} = X$.

Let $\tilde{z}$ be a local holomorphic coordinate of $\tilde{M}$. Then, local holomorphic differentials $\tilde{\Psi}_i := (\partial \tilde{X}_i / \partial \tilde{z})\, d\tilde{z}$, $i = 1, 2, 3$ are extended to a meromorphic 1-forms on $\mathbb{C}P^1$. Hence, $\tilde{\Psi}_i := (\partial \tilde{X}_i / \partial \tilde{z}) d\tilde{z}$ is denoted by $\Psi_i := (\partial X_i / \partial z) dz$ in the following, where z is a local holomorphic coordinate of M. The original surface is obtained by integration:

$$X_i(z) := \operatorname{Re} \int_{z_0}^{z} \Psi_i, \; i = 1, 2, 3, \tag{2.1}$$

where z_0 is a point on $\mathbb{C}P^1$ such that $X(z_0) = (0,0,0)$. From the conformality of X, the relation $(\Psi_1 \otimes \Psi_1) + (\Psi_2 \otimes \Psi_2) + (\Psi_3 \otimes \Psi_3) = 0$ holds. For $\Psi = (\Psi_1, \Psi_2, \Psi_3)$, the divisor (Ψ) of $d\Psi$ is defined by

$$(\Psi) := \sum_{p \in \mathbb{C}P^1} \left(\min_{i=1,2,3} \operatorname{ord}_p \Psi_i \right) \cdot p \,. \tag{2.2}$$

Assume that $(\Psi) = \sum_{j=1}^{l} B_j \cdot b_j - \sum_{i=1}^{r} P_i \cdot p_i$, $B_j > 0, P_i > 0$.

Definition 2.1. A point b_j is called a *branch point* of order B_j and a point p_i a *puncture point* of order P_i. The sum $\tilde{B} := \sum_{j=1}^{l} B_j$ is called the *total order of branch points* and $\tilde{P} := \sum_{i=1}^{r} P_i$ the *total order of puncture points.*

The total curvature of X is equal to $2\pi(2 + \tilde{B} - \tilde{P})$ (cf. K. Moriya [3, Corollary 2.2]).

In the following, only the case where $r = 2$ is considered. The 1-dimensional complex projective space $\mathbb{C}P^1$ is identified with $\mathbb{C} \cup \{\infty\}$.

The two puncture points may be assumed to be 0 and ∞ since any two points in $\mathbb{C}P^1$ is transformed to 0 and ∞ simultaneously by a Möbius transformation. Hence, it may be assumed that $(\Psi) = \sum_{j=1}^{l} B_j \cdot b_j - P_0 \cdot 0 - P_\infty \cdot \infty$. Let γ be a simple closed curve around 0 whose orientation is chosen as the origin lies in the left side. The number $s_i := \int_\gamma \Psi_i = 2\pi\sqrt{-1}\operatorname{Res}(0; \Psi_i)$ is called the period of Ψ_i around 0, $i = 1, 2, 3$, where $\operatorname{Res}(0; \Psi_i)$ is the residue of Ψ_i at 0. Then, $(\operatorname{Re} s_1, \operatorname{Re} s_2, \operatorname{Re} s_3)$ is equal to $v = (v_1, v_2, v_3)$.

Definition 2.2. The condition

$$(\Psi_1 \otimes \Psi_1) + (\Psi_2 \otimes \Psi_2) + (\Psi_3 \otimes \Psi_3) = 0 \,, \tag{2.3}$$

is called the *conformality condition.* The condition

$$(\Psi) = \sum_{j=1}^{l} B_j \cdot b_j - P_0 \cdot 0 - P_\infty \cdot \infty \,, \tag{2.4}$$

is called the *divisor condition.* The condition

$$(\operatorname{Re} s_1, \operatorname{Re} s_2, \operatorname{Re} s_3) = (v_1, v_2, v_3) \,, \tag{2.5}$$

is called the *period condition.*

Conversely, if three meromorphic 1-forms Ψ_i, $i = 1, 2, 3$, satisfying the conformality, the divisor, and the period condition, that is (2.3), (2.4), and (2.5), are given, then an algebraic minimal surface $(X, \mathbb{C}P^1, \mathbb{R}^3/T(v))$ is obtained by integration (2.1). If the other base point z_0 in the integral is chosen, the image of the immersion shifts by a translation.

Let Ω be a meromorphic 1-form dz/z, such that $(\Omega) = -1\cdot 0 - 1\cdot\infty$. Then, three meromorphic 1-forms Ψ_i, $i = 1, 2, 3$, satisfying the conformality, the divisor, and the period condition is equivalent to a pair of rational functions (g, h) via the following relations:

$$(g,h) = \left(\frac{\Psi_3}{\Psi_1 - \sqrt{-1}\Psi_2}, \frac{\Psi_3}{\Omega}\right), \tag{2.6}$$

$$(\Psi_1, \Psi_2, \Psi_3) = \left(\frac{1}{g} - g, \sqrt{-1}\left(\frac{1}{g} + g\right), 2\right)\frac{h}{2}\Omega. \tag{2.7}$$

There is the following relation among (g), (h), (Ω), and (Ψ):

$$(\Psi) = -(g)_0 - (g)_\infty + (h) + (\Omega). \tag{2.8}$$

Since the relation (2.4) holds, the relation (2.8) becomes as follows:

$$\begin{aligned} -(g)_0 - (g)_\infty + (h) \\ = \sum_{j=1}^{k} B_j \cdot b_j - (P_0 - 1)\cdot 0 - (P_\infty - 1)\cdot\infty. \end{aligned} \tag{2.9}$$

Definition 2.3. The pair (g, h) of meromorphic functions on $\mathbb{C}P^1$ associated to an algebraic minimal surface $(X, \mathbb{C}P^1, \mathbb{R}^3/T(v))$ is called the *Weierstrass data* of X.

Fix a holomorphic coordinate z on $\mathbb{C}$ centered at the origin. If g ramifies at each puncture point with ramification index $k-1$, then it may be assumed that $g = az^k$ for some $a \in \mathbb{C}^\times := \mathbb{C} - \{0\}$ and $k > 0, k \in \mathbb{Z}$. Then, $(g) = k\cdot 0 - k\cdot\infty$.

Assume that $P_0 = P_\infty$. Tthen from the divisor condition (2.4), the divisor of h become as follows:

$$(h) = \sum_{j=1}^{k} B_j \cdot b_j - (P - k - 1)\cdot 0 - (P - k - 1)\cdot\infty, \tag{2.10}$$

where $P := P_0$. Hence, $h = c\prod_{i=1}^{2J}(z - b_j)/z^{P-k-1}$ for some $c \in \mathbb{C}^\times$.

Since h is a meromorphic function on $\mathbb{C}P^1$, the degree of (h) is equal to 0. Hence, $\tilde{B} - 2P + 2k + 2 = 0$. Thus, $\tilde{B}$ is even. Let $\tilde{B} = 2J$. Then $k = P - J - 1 > 0$ and $P - k - 1 = J$.

The Weierstrass data

$$(g,h) = \left((az)^{P-J-1}, \frac{ca^J\prod_{i=1}^{2J}(z - b_j/a)}{z^J}\right), \tag{2.11}$$

produces the same algebraic minimal surface as

$$(g,h)=\left(z^{P-J-1},\frac{c\prod_{i=1}^{2J}(z-b_j)}{z^J}\right), \tag{2.12}$$

since the difference of these data is caused by only the difference of the choice of a holomorhic coordinate on $\mathbb{C}^\times$ fixing 0 and ∞.

Let $\mathcal{A}=\mathcal{A}(\tilde{B},P,v)$ be the set $\{(X,\mathbb{C}P^1,\mathbb{R}^3/T(v))\}$ of algebraic minimal surfaces whose puncture points are 0 and ∞ of order P, whose total order of branch points is $\tilde{B}$, and whose Gauss map ramifies at the puncture points with the same ramification index. Let $X\cong Y$ for X and Y in $\mathcal{A}$ mean that $X+x=Y$ for some $x\in\mathbb{R}^3$. Let $\mathcal{W}=\mathcal{W}(\tilde{B},P,dz/z,v)$ be the set of Weierstrass data corresponding to algebraic minimal surfaces belonged with $\mathcal{A}$ by (2.6) and (2.7). Summarizing the above discussion, the following lemma has been proved.

Lemma 2.4. *There is a bijective corresponding between $\mathcal{A}(\tilde{B},P,v)/\cong$ and $\mathcal{W}(\tilde{B},P,dz/z,v)$. The set $\mathcal{W}(\tilde{B},P,dz/z,v)$ is considered as the set $\{(g,h)\}$ of pairs of rational functions such that*

$$g=z^{P-J-1},\, h=\frac{c\prod_{i=1}^{2J}(z-b_j)}{z^J}, \tag{2.13}$$

$$\left(\mathrm{Re}\int_\gamma\Psi_1,\mathrm{Re}\int_\gamma\Psi_2,\mathrm{Re}\int_\gamma\Psi_3\right)=(v_1,v_2,v_3). \tag{2.14}$$

where $c,b_1,\dots,b_B\in\mathbb{C}^\times$.

Remark 2.5. The rational function g of a Weierstrass data (g,h) of f is the stereographic projection of the extended normal Gauss map of f. Hence, Weierstrass data $(\exp[\sqrt{-1}\theta]g,h),\theta\in\mathbb{R}$ produce the same algebraic minimal surface as (g,h) rotated by an angle θ around the x_3-axis. The Weierstrass data $(g,mh), m\in\mathbb{R}^\times:=\mathbb{R}-\{0\}$ produce the similar algebraic minimal surface as (g,h) transformed similarly by the center of similitude the origin of $\mathbb{R}^3$ and the ratio of similitude m.

3. The moduli space of Weierstrass data

In this section, the proof of Theorem 1.1 is given.

Proof of Theorem 1.1 . Let the set $\mathcal{R}=\mathcal{R}(\tilde{B},P)$ be the set of pairs of rational functions (g,h) where

$$g=z^{P-J-1},\, h=\frac{c\prod_{i=1}^{2J}(z-b_j)}{z^J}, \tag{3.1}$$

and $\tilde{B}=2J$. Then, $\mathcal{R}$ is identified with the set $\mathrm{Div}_+^{2J}(\mathbb{C}^\times)\times\mathbb{C}^\times$ by the identification

$$(g,h)\cong\left(\sum_{j=1}^{2J}b_j,c\right), \tag{3.2}$$

where the space of positive divisors of degree $2J$ on $\mathbb{C}^\times$ is denoted by $\mathrm{Div}_+^{2J}(\mathbb{C}^\times)$. The space $\mathrm{Div}_+^{2J}(\mathbb{C}^\times)$ is considered as $\mathbb{C}^{2J-1}\times\mathbb{C}^\times$ by the identification

$$\sum_{j=1}^{2J}(1\cdot b_j)\cong(\sigma_1(b_j),\ldots,\sigma_{2J}(b_j)),\tag{3.3}$$

where $\sigma_i(b_j)$ is the elementary symmetric polynomial with degree i, $i=1,\ldots,2J$. After this, $\sigma_i(b_j)$ is denoted by e_i. By the above identifications, $\mathcal{R}$ is considered as $\mathbb{C}^{2J-1}\times(\mathbb{C}^\times)^2$.

On the other hand, the set $\mathcal{R}$ is considered as an open subset of $\mathbb{R}^{4J+2}$ defined by

$$\{(e_{1,1},e_{1,2},\ldots,e_{2,1},e_{2,2},c_1,c_2)\in\mathbb{R}^{4J+2}\,|\,e_{2J,1}^2+e_{2J,2}^2\neq 0, c_1^2+c_2^2\neq 0\}.\tag{3.4}$$

Thus, $\mathcal{R}$ is identified with U.

Let $\tau\colon U\to\mathbb{C}^3$ be the period map defined by

$$\tau(g,h)=\left(\int_\gamma\Psi_1,\int_\gamma\Psi_2,\int_\gamma\Psi_3\right).\tag{3.5}$$

Then

$$\begin{aligned}\operatorname{Re}\tau_1(g,h)&=-\,\pi(c_1(e_{J-k,2}-e_{J+k,2})+c_2(e_{J-k,1}-e_{J+k,1}))\,,\\ \operatorname{Re}\tau_2(g,h)&=-\,\pi(c_1(e_{J-k,2}+e_{J+k,2})+c_2(e_{J-k,1}+e_{J+k,1}))\,,\\ \operatorname{Re}\tau_3(g,h)&=-\,2\pi(c_1e_{J,2}+c_2e_{J,1})\,,\end{aligned}\tag{3.6}$$

where $k=P-J-1>0$ and $e_{i,1}=e_{i,2}=0$ if $i>2J$ or $i<0$ and $e_{i,1}=1, e_{i,2}=0$ if $i=0$. The set $\mathcal{W}(\tilde{B},P,dz/z,v)$ consists of the element (g,h) of U satisfying $\operatorname{Re}\tau(g,h)=(v_1,v_2,v_3)$. Simplifying these equations the following equations are obtained:

$$\begin{aligned}\phi_1&:=2\pi(c_2e_{J-k,1}+c_1e_{J-k,2})+v_1+v_2=0\,,\\ \phi_2&:=2\pi(c_2e_{J+k,1}+c_1e_{J+k,2})+v_2-v_1=0\,,\\ \phi_3&:=2\pi(c_2e_{J,1}+c_1e_{J,2})+v_3=0\,,\end{aligned}\tag{3.7}$$

if $k\le J-1$,

$$\begin{aligned}&\phi_1:=v_1+v_2=0,\phi_2:=v_2-v_1=0\,,\\ &\phi_3:=2\pi(c_2e_{J,1}+c_1e_{J,2})+v_3=0\,,\end{aligned}\tag{3.8}$$

if $k\ge J+1$ and if $J\neq 0$,

$$\begin{aligned}&\phi_1:=v_1+v_2=0,\phi_2:=v_2-v_1=0\,,\\ &\phi_3:=2\pi c_2+v_3=0\,,\end{aligned}\tag{3.9}$$

if $k\ge J+1$ and if $J=0$, and

$$\begin{aligned}&\phi_1:=2\pi c_2+v_1+v_2=0,\phi_2:=v_2-v_1=0\,,\\ &\phi_3:=2\pi(c_2e_{J,1}+c_1e_{J,2})+v_3=0\,,\end{aligned}\tag{3.10}$$

if $k=J\neq 0$. Hence, the following is easily obtained:

Lemma 3.1. *The set $\mathcal{W}(\tilde{B},P,dz/z,v)$ is nonempty if and only if $k\ge J+1$ and $v_1=v_2=0$, $k=J$ and $v_1=v_2$, or $k\le J-1$.*

From the above equations, the following are obtained:

$$
\begin{aligned}
d\phi_1 &= 2\pi(e_{J-k,1}dc_2 + c_2 de_{J-k,1} + e_{J-k,2}dc_1 + c_1 de_{J-k,2}) \\
d\phi_2 &= 2\pi(e_{J+k,1}dc_2 + c_2 de_{J+k,1} + e_{J+k,2}dc_1 + c_1 de_{J+k,2}) \\
d\phi_3 &= 2\pi(e_{J,1}dc_2 + c_2 de_{J,1} + e_{J,2}dc_1 + c_1 de_{J,2})\,,
\end{aligned} \tag{3.11}
$$

if $k \leq J-1$,

$$d\phi_3 = 2\pi(e_{J,1}dc_2 + c_2 de_{J,1} + e_{J,2}dc_1 + c_1 de_{J,2})\,, \tag{3.12}$$

if $k \geq J+1$ and if $J \neq 0$,

$$d\phi_3 = 2\pi dc_2\,, \tag{3.13}$$

if $k \geq J+1$ and if $J = 0$, and

$$
\begin{aligned}
d\phi_1 &= 2\pi dc_2\,, \\
d\phi_3 &= 2\pi(e_{J,1}dc_2 + c_2 de_{J,1} + e_{J,2}dc_1 + c_1 de_{J,2})\,,
\end{aligned} \tag{3.14}
$$

if $k = J \neq 0$. Hence, if $\mathcal{W}(\tilde{B}, P, dz/z, v)$ is nonempty, then it is a nonsingular real algebraic variety of dimension $4J-1$ if $k \leq J-1$, $4J+1$ if $k \geq J+1$, and $4J$ if $k = J \neq 0$. □

References

[1] Hoffman, D. and Karcher, H., *Complete embedded minimal surfaces of finite total curvature*, Geometry, V, Springer, Berlin, 1997, pp. 5–93, 267–272.

[2] Kusner, R. and Schmitt, N., *The spinor representation of surfaces in space*, (dg-ga/9610005).

[3] Moriya, K., *On a variety of algebraic minimal surfaces in Euclidean* 4*-space*, Tokyo J. Math. **21** (1998), no. 1, 121–134.

[4] ———, *On a moduli space of minimal annuli*, preprint.

[5] Pérez, J. and Ros, A., *The space of properly embedded minimal surfaces with finite total curvature*, Indiana Univ. Math. J. **45** (1996), no. 1, 177–204.

[6] ———, *The space of complete minimal surfaces with finite total curvature as Lagrangian submanifold*, Trans. Amer. Math. Soc., to appear.

[7] Yang, K., *Complete Minimal Surfaces of Finite Total Curvature*, Math. Appl., Kluwer Academic Publishers, Dordrecht, Boston, London, 1994.

Department of Mathematics, Graduate School of Science
Tokyo Metropolitan University, Minami-Ohsawa1-1, Hachioji-shi
Tokyo, 192-0397, JAPAN
E-mail: katsum@comp.metro-u.ac.jp