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SOME EXAMPLES OF THE HERMITE-LIOUVILLE STRUCTURE ON THE CLASSICAL HOPF SURFACE

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Abstract. It is rewarding to investigate compact riemannian manifolds whose geodesic flows are completely integrable. A Hermite-Liouville structure on a complex surface is defined as a couplet (g, L) of a hermitian metric g and a 2-dimensional real vector space L of first integrals on the cotangent bundle of its geodesic flow such that (1) L contains the energy function (2) all the first integrals contained in L are fiberwise homogenous polynomials of degree 2 and are hermitian. The author constructs some examples of the Hermite-Liouville structure on the classical Hopf surface. The geodesic flows of these examples are completely integrable.

Introduction

It is rewarding to investigate compact riemannian manifolds whose geodesic flows are completely integrable. In 19th century, Jacobi investigated the 2-dimensional ellipsoid, and then Liouville generalized it as the theory of the Liouville line elements. The Liouville line element on a real surface is defined by

$$(ds)^2 = (f_1(x_1) + f_2(x_2))((dx_1)^2 + (dx_2)^2)$$

by using a certain local coordinate (x_1, x_2) in the surface. A real surface with a Liouville line element has the following remarkable property:

We set

$$\begin{aligned} E &= \frac{1}{2}\frac{1}{f_1(x_1) + f_2(x_2)}((\xi_1)^2 + (\xi_2)^2) \\ F &= \frac{1}{f_1(x_1) + f_2(x_2)}(f_2(x_2)(\xi_1)^2 - f_1(x_1)(\xi_2)^2)\,, \end{aligned}$$

where (x_1, x_2, ξ_1, ξ_2) is the canonical coordinate in the cotangent bundle. We notice that E is the energy function on the cotangent bundle.

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Then, we have

$$\{F, E\} = 0\,.$$

This implies that F is a first integral of the geodesic flow, and hence this geodesic flow is completely integrable.

In relation to the present viewpoint, this argument can be recognized as a local theory in the differential geometry. However, in 1991 K. Kiyohara began to develop a global theory in this area [1]. In [1] he defined the compact Liouville surface, i.e., the compact real surface with the Liouville structure, and then classified the family called "proper." The Liouville structure can be defined as the couplet (g, L) of the riemannian metric g and the 2-dimensional real vector space L of the first integrals of its geodesic flow which are fiberwise homogenous functions of degree 2 on the cotangent bundle.

After that the author studied compact Kähler-Liouville surfaces, i.e., compact Kähler surfaces with the Liouville structures [2].

Subsequently, K. Kiyohara generalized these concepts to the higher dimensional Liouville manifolds and Kähler-Liouville manifolds [3].

The Hermite-Liouville structure is a couplet (g, L) of the hermitian metric g and a 2-dimensional real vector space L of first integrals of its geodesic flow such that (1) L contains the energy function; (2) each first integral belonging to L is a fiberwise homogenous polynomial function of degree 2 and is hermitian.

The purpose of this paper is to study the Hermite-Liouville structures on compact complex surfaces and to construct some examples of the Hermite-Liouville structure on a compact non-Kähler hermitian surface.

In §1 we have the definition of the Hermite-Liouville structure. In §2 we discuss the case that g is a Kähler metric; we present only the statement of the results in [2]. In §3 and §4 we argue about the case that g is a non-Kähler hermitian metric; actually, we construct some examples of the Hermite-Liouville structure on the classical Hopf surface.

Throughout this paper, we assume the differentiability of class C^∞.

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§1. The definition of the Hermite-Liouville structure

In this section we define the Hermite-Liouville structure on a compact complex surface. Let (M, J) be a compact complex surface. A Hermite-Liouville structure on (M, J) is defined as a couplet (g, L) of a hermitian metric g on (M, J) and a 2-dimensional real vector space L of real valued functions on the cotangent bundle T^*M such that

(1) L contains the energy function on T^*M corresponding to g;

(2) For any $F \in L$ and for any point $p \in M$,

$$F_p \equiv F|_{T_p^*M} : T_p^*M \longrightarrow \boldsymbol{R}$$

is a homogenous polynomial of degree 2;

(3) For any $F \in L$, F is hermitian, i.e., $F \circ J = F$;

(4) For any $F_1, F_2 \in L$,

$$\{F_1, F_2\} = 0\,,$$

where $\{*, *\}$ is the canonical Poisson bracket on T^*M.

We call the quadruplet $(M, g, J; L)$ of such M, g, J, L a Hermite-Liouville surface.

Let $(M, g, J; L)$ be a Hermite-Liouville surface. We set $L_p \equiv \{F_p = F|_{T_p^*M} \mid F \in L\ \}$. We call the point p such that $\dim L_p = 1$ an L-singular point of (M, J) and denote the set of them by M_{sing}. On the other hand, we call the point p such that $\dim L_p = 2$ an L-regular point of (M, J) and denote the set of them by M_{reg}. Then, it follows that $M = M_{sing} \cup M_{reg}$ (disjoint union).

§2. Kähler case

In this section, we discuss the case that g is a Kähler metric on (M, J). The argument in this section is that of [2]. Only the statement of the results is described here. See [2] for its proof.

Let $(M, g, J; L)$ be a compact Hermite-Liouville surface. We call the couplet (g, L) a Kähler-Liouville structure on (M, J) and call the quadruplet $(M, g, J; L)$ a compact Kähler-Liouville surface if (M, g, J) is a compact Kähler surface.

A compact Kähler-Liouville surface $(M, g, J; L)$ is said to be proper if it satisfies the following conditions:

(1) $M_{sing} \neq \emptyset$;

(2) For any $F \in L\backslash\{0\}$ and for $p \in M$ such that $F_p = 0$, there exists a covector $w \in T_p^*M$ which satisfies $(dF)_w \neq 0$.

Now, we have the statement of the results of the Kähler case below.

Let $(M, g, J; L)$ be a compact proper Kähler-Liouville surface. Then,

(1) *The geodesic flow of (M, g) is completely integrable.*

Actually, taking a certain basis E, F of L, where E denotes the energy function on T^*M, we can obtain certain Killing vector fields Y_1, Y_2 such that

(i) $\{E, F\} = 0,\quad \{E, Y_1\} = \{E, Y_2\} = 0,$
$\{F, Y_1\} = \{F, Y_2\} = 0,\quad [Y_1, Y_2] = 0;$

(ii) *E, F, Y_1, Y_2 are functionally independent almost everywhere on T^*M.*

(2) *(M, J) is bi-holomorphic with $\boldsymbol{C}P^2$.*

(3) *(M, g, J) has three points q_0, q_1, q_2 and three totally geodesic 1-dimensional complex submanifolds H_0, H_1, H_2 which are bi-holomorphic with $\boldsymbol{C}P^1$ such that*

(i) $H_1 \cap H_2 = \{q_0\}$, $H_0 \cap H_1 = \{q_2\}$, $H_0 \cap H_2 = \{q_1\}$;

(ii) *H_0 coincides with the subset of M on which $F \in L$ such that $F_p = 0$ for $p \in M_{sing}$ is degenerate, and hence includes M_{sing};*

(iii) *$H_0 \cup H_1 \cup H_2$ is the subset of M on which $F \in L$ described in (ii) can be said to be critical in a certain sense.*

(4) *M_{sing} forms a compact real submanifold of H_0, and is diffeomorphic with S^1.*

(5) *There exists an effective action Φ of $S^1 \times S^1$ on M (as automorphisms of $(M, g, J; L)$) such that*

(i) *Φ leaves the three points q_0, q_1, q_2 fixed;*

(ii) *Φ leaves the three submanifolds H_0, H_1, H_2 invariant.*

(6) *There exists a family Σ of compact totally geodesic 2-dimensional real submanifolds of (M, g) such that*

(i) *Each compact real submanifold $S \in \Sigma$ associated with the Liouville structure inherited from L forms a compact real Liouville surface which is diffeomorphic with $\boldsymbol{R}P^2$;*

(ii) *The compact real Liouville surfaces belonging to Σ are isomorphically transferred to each other by the action Φ on M.*

§3. An example of the Hermite-Liouville structure on the classical Hopf surface

Now we consider the case that its riemannian metric is a non-Kähler one. In this section we construct an example of the Hermite-Liouville structure on the classical Hopf surface with the standard metric g_0.

The classical Hopf surface is defined by the quotient manifold

$$H \equiv (\boldsymbol{C}^2 \setminus \{0\})/G$$

of $\boldsymbol{C}^2 \setminus \{0\}$ by the infinite cyclic group

$$G = \left\{ r^m \,\middle|\, r : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \to e \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, m \in \boldsymbol{Z} \right\},$$

where $e = \sum_{n=1}^{\infty} \frac{1}{n}$. It is well known that the Hopf surface (H, J) does not admit any Kähler metric.

We define a linear Lie group $CU(2)$ by

$$CU(2) = \{A \in GL(2; \boldsymbol{C}) | {}^t\overline{A}A = c^2 I \text{ for a real number } c\}.$$

Then, $CU(2)$ acts on H transitively (as automorphisms of (H, J)). Let $cu(2)$ denote the Lie algebra of $SU(2)$. The matrices

$$X_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad Y_{11} = \begin{bmatrix} 2\sqrt{-1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad Y_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 2\sqrt{-1} \end{bmatrix},$$

$$X_{21} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad Y_{21} = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}$$

form a basis of $cu(2)$.

For any $X \in cu(2)$, we define a vector field $\sigma(X)$ on H by

$$\sigma(X)_p = \frac{d}{ds} \exp(-sX) \cdot p \bigg|_{s=0}, \qquad p \in H.$$

We here construct fiberwise homogenous polynomial functions F_1, F_2 of degree 2 on T^*H as follows:

$$\begin{aligned} F_1 &= (\sigma(X_{12}))^2 + \left(\sigma\left(\frac{-Y_{11}+Y_{22}}{2}\right)\right)^2 + (\sigma(X_{21}))^2 + (\sigma(Y_{21}))^2 \,, \\ F_2 &= -\sigma(Y_{11})\sigma(Y_{22}) + (\sigma(X_{21}))^2 + (\sigma(Y_{21}))^2 \,, \end{aligned}$$

where $\sigma(X_{ij})$, $\sigma(Y_{ij})$ are recognized as fiberwise linear functions on T^*H. Then, we have the following

Proposition 3.1. (1) F_1 *and* F_2 *commute with respect to the canonical Poisson bracket on* T^*H, *i.e.,*

$$\{\ F_1,\ F_2\ \} = 0\,.$$

(2) F_1 *and* F_2 *are hermitian, i.e.,*

$$F_1 \circ J^* = F_1, \quad F_2 \circ J^* = F_2\,,$$

where J^* *denotes the complex structure in* T^*H *corresponding to* J.

Proof. (1) Let $(cu(2))^*$ denotes the dual vector space of the Lie algebra $cu(2)$. Using the dual basis of $X_{12}, Y_{11}, Y_{22}, X_{21}, Y_{21}$ in $(cu(2))^*$, we have the complex matrix expression

$$\Xi = \begin{bmatrix} \xi_{12} + \sqrt{-1}\eta_{11} & -\xi_{21} + \sqrt{-1}\eta_{21} \\ \xi_{21} + \sqrt{-1}\eta_{21} & \xi_{12} + \sqrt{-1}\eta_{22} \end{bmatrix}$$

for each $\xi \in (cu(2))^*$. By means of the complex-valued function $\bar{F}_t(\xi) = \det(tI + \Xi)$ on $(cu(2))^*$, where I denotes the identity matrix, we obtain four $\mathrm{Ad}^*(CU(2))$-invariant homogenous polynomial functions

$$\begin{aligned} f_{11}(\xi) &= 2\xi_{12}\,, \\ f_{12}(\xi) &= \eta_{11} + \eta_{22}\,, \\ f_{21}(\xi) &= -\eta_{11}\eta_{22} + (\xi_{12})^2 + (\xi_{21})^2 + (\eta_{21})^2\,, \\ f_{22}(\xi) &= (\eta_{11} + \eta_{22})\xi_{12}\,. \end{aligned}$$

We moreover define two homogenous polynomial functions $\tilde{F}_1$, $\tilde{F}_2$ of degree 2 on $(cu(2))^*$ by

$$\begin{aligned} \tilde{F}_1 &= f_{21} + \frac{1}{4}(f_{12})^2 = (X_{12})^2 + \left(\frac{-Y_{11}+Y_{22}}{2}\right)^2 + (X_{21})^2 + (Y_{21})^2\,, \\ \tilde{F}_2 &= f_{21} - \frac{1}{4}(f_{11})^2 = -Y_{11}Y_{22} + (X_{21})^2 + (Y_{21})^2\,, \end{aligned}$$

where X_{ij}, Y_{ij} are recognized as linear forms on $(cu(2))^*$. It follows that $\tilde{F}_1$ and $\tilde{F}_2$ are $\mathrm{Ad}^*(CU(2))$-invariant and hence that

$$\{\tilde{F}_1, \tilde{F}_2\}_{(cu(2))^*} = 0\,,$$

where $\{*,*\}_{(cu(2))^*}$ denotes the canonical Poisson bracket defined on the space of the functions on $(cu(2))^*$. Using the moment map $\mu : T^*H \to (cu(2))^*$, we obtain

$$F_1 = \mu^*(\tilde{F}_1)\,, \qquad F_2 = \mu^*(\tilde{F}_2)\,,$$

and

$$\{F_1, F_2\} = 0\,,$$

where $\{*,*\}$ denotes the canonical Poisson bracket on T^*H.

(2) There exist vector fields V_1, V_2 on H which satisfy $F_2 = (V_1)^2 + (V_2)^2$ and $JV_1 = V_2$. Thus, we have $F_2 \circ J^* = F_2$. On the other hand, since $J(\sigma(X_{12})) = \sigma\left(\frac{Y_{11}+Y_{22}}{2}\right)$ and since $F_1 = (\sigma(X_{12}))^2 + \left(\sigma\left(\frac{Y_{11}+Y_{22}}{2}\right)\right)^2 + F_2$, we have $F_1 \circ J^* = F_1$. $\square$

The standard metric g_0 on H is defined as the inherited metric from $(\boldsymbol{C}^2 \setminus \{0\}, (ds)^2)$. Then, we have the following

Proposition 3.2. (1) *(H, g_0) is a riemanniann homogenous space.*

(2) *$\frac{1}{2}F_1$ coincides with the energy function on T^*H with respect to the standard metric g_0.*

Proof. (1) It is easy to verify that the action of $CU(2)$ on H preserves the metric g_0.

(2) The assertion follows from the fact that $\sigma(X_{12})$, $\sigma(\frac{-Y_{11}+Y_{22}}{2})$, $\sigma(X_{21})$, $\sigma(Y_{21})$ form an orthonormal frame on H with respect to g_0. $\square$

Let L denotes the vector space which is spanned by F_1 and F_2. Then, we have

Theorem 3.3. *The couplet (g_0, L) is an Hermite-Liouville structure on the Hopf surface (H, J).*

Proof. This is an immediate consequence of Proposition 3.1, Proposition 3.2 (2) and the definition of F_1, F_2. $\square$

We moreover have the following

Theorem 3.4. *The geodesic flow of (H, g_0) is completely integrable.*

Proof. Recall the argument of the proof of Proposition 3.1 (1) and keep the notation of it. Let $\tilde{F}_3, \tilde{F}_4$ be the linear forms on $(cu(2))^*$ which are defined by $Y_{11}, Y_{22} \in cu(2)$, respectively. Since $[Y_{11}, Y_{22}] = 0$, we have

$$\{\tilde{F}_3, \tilde{F}_4\}_{(cu(2))^*} = 0\,.$$

Since $\tilde{F}_1$ and $\tilde{F}_2$ are $\mathrm{Ad}^*(CU(2))$-invariant, we can conclude that

$$\{\tilde{F}_i, \tilde{F}_j\}_{(cu(2))^*} = 0\,, \qquad i,j = 1,2,3,4\,.$$

Setting $F_3 = \mu^*(\tilde{F}_3) = \sigma(Y_{11})$, $F_4 = \mu^*(\tilde{F}_4) = \sigma(Y_{22})$, we have

$$\{ F_i,\ F_j \} = 0, \qquad i, j = 1, 2, 3, 4\,.$$

From Proposition 3.2 (2), we notice that $\frac{1}{2}F_1$ is the energy function on T^*H. It is easy to see that F_1, F_2, F_3, F_4 are functionally independent almost everywhere on the cotangent bundle T^*H. □

§4. A family of the Hermite-Liouville structure on the classical Hopf surface

In this section we construct a family of the Hermite-Liouville structures on the Hopf surface (H, J).

$\pi : H \to \boldsymbol{C}P^1$ be the principal fiber bundle obtained by an obvious manner. Let K be the set of the fiberwise constant functions κ on H such that

$$(\sigma(Y_{11}))\kappa = (\sigma(Y_{22}))\kappa = 0.$$

Then, for each $\kappa \in K$, we set

$$\begin{aligned} F_1^{(\kappa)} &= (\sigma(X_{12}))^2 + \left(\sigma\left(\frac{Y_{11}+Y_{22}}{2}\right)\right)^2 \\ &\qquad + e^{2\kappa}\Big(-\sigma(Y_{11})\sigma(Y_{22}) + (\sigma(X_{21}))^2 + (\sigma(Y_{21}))^2 \Big)\,, \\ F_2^{(\kappa)} &= e^{2\kappa}\Big(-\sigma(Y_{11})\sigma(Y_{22}) + (\sigma(X_{21}))^2 + (\sigma(Y_{21}))^2 \Big)\,. \end{aligned}$$

The function $\frac{1}{2}F_1^{(\kappa)}$ on T^*H can be an energy function which corresponds to a certain riemannian metric. Then, we denote by g_κ such a riemannian metric, i.e., the riemannian metric corresponding to $\frac{1}{2}F_1^{(\kappa)}$. Let $L^{(\kappa)}$ denotes the vector space spanned by $F_1^{(\kappa)}$ and $F_2^{(\kappa)}$. We then obtain the following

Theorem 4.1. (1) *For each $\kappa \in K$, the couplet $(g_\kappa, L^{(\kappa)})$ is an Hermite-Liouville structure on the Hopf surface (H, J).*

(2) *The set $\{(g_\kappa, L^{(\kappa)})\}_{\kappa\in K}$ is a family of the Hermite-Liouville structures on the Hopf surface (H, J).*

Proof. It is sufficient to prove (1). Recalling the arguments of the proof of Proposition 3.1 (1), (2), we have

$$\left\{ (\sigma(X_{12}))^2 + \left(\sigma\left(\frac{Y_{11}+Y_{22}}{2}\right)\right)^2, \quad F_2 \right\} = 0\,.$$

Since the vector fields $\sigma(X_{12}), \sigma\left(\frac{Y_{11}+Y_{22}}{2}\right)$ are tangent to the fibers of the principal fiber bundle $\pi : H \to \boldsymbol{C}P^1$ and since κ is a fiberwise constant function, we have

$$\sigma(X_{12})\kappa = \sigma\left(\frac{Y_{11}+Y_{22}}{2}\right)\kappa = 0\,.$$

From the above equalities, we have

$$\{F_1^{(\kappa)}, F_2^{(\kappa)}\} = 0\,.$$

Recalling the argument of the proof of Proposition3.1 (2), we immediately conclude that $F_1^{(\kappa)} \circ J^* = F_1^{(\kappa)},\ F_2^{(\kappa)} \circ J^* = F_2^{(\kappa)}$. □

We moreover have the following

Theorem 4.2. *The geodesic flow of* (H, g_κ) *is completely integrable.*

Proof. From the proof of Theorem 3.4, we see immediately that

$$\left\{ (\sigma(X_{12}))^2 + \left(\sigma\left(\frac{Y_{11}+Y_{22}}{2}\right)\right)^2, \quad F_i \quad \right\} = \{\ F_2, F_i\ \} = 0\,, \quad i = 3, 4\,.$$

By the condition that $\sigma(Y_{11})\kappa = \sigma(Y_{22})\kappa = 0$, we have

$$\{\ F_1^{(\kappa)},\ F_i\ \} = \{\ F_2^{(\kappa)},\ F_i\ \} = 0\,, \quad i = 3, 4\,.$$

We have already verified that $\{F_3, F_4\} = 0$ in the proof of Theorem 3.4. We can easily see that $F_1^{(\kappa)}$, $F_2^{(\kappa)}$, F_3, F_4 are functionally independent almost everywhere on the cotangent bundle T^*H. □

Remark. We can construct this example so that (M, g_κ) is a non-homogenous space.

Besides, the following properties can be seen easily:

(1) $\dim(L^{(\kappa)}) = 2$ for every point $p \in H$, and hence $H_{sing} = \emptyset,\ H_{reg} = H$.

(2) For any $p \in H$, $(F_2^{(\kappa)})_p$ is positive semi-definite.

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