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INFINITESIMAL GEODESIC DEFORMATIONS OF THE TOTALLY GEODESIC MANIFOLDS

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Abstract. We study infinitesimal geodesic deformations of the totally geodesic manifolds. We find conditions for which totally geodesic manifolds admit nontrivial geodesic deformation. The conditions give fundamental equations of these deformations. These equations are the main goal in this paper.

1. Let V_m be the Riemannian space with local coordinates $(y^1, y^2, \dots, y^m)$, then the formulas

$$y^\alpha = f^\alpha(x^1, x^2, \dots, x^n), \qquad \operatorname{Rank}\left\|\frac{\partial f^\alpha}{\partial x^i}\right\| = n < m \tag{1}$$

define a subspace $V_n \subset V_m$.

If $a_{\alpha\beta}$ and g_{ij} are metric tensors of V_m and V_n respectively, then [1]

$$g_{ij} = a_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\,.$$

Here and further $\alpha, \beta, \gamma, \dots = 1, 2, \dots, m$; $i, j, k, \dots = 1, 2, \dots, n$, if not stipulated otherwise.

Let $\xi^\alpha(x^1, x^2, \dots, x^n)$ be some contravariant vector field of the space V_m given in points V_n. Formulas

$$\widetilde{y}^\alpha = y^\alpha(x) + \varepsilon\xi^\alpha(x) \tag{2}$$

where ε is a small real parameter, define some $\widetilde{V}_n$, which we shall name an *infinitesimal deformations* of V_n. The vector ξ^α is called *the field of displacements* or *vector of displacements*. We will neglect the ε in k-th degree, $k = 2, 3, \dots$ because it is very small.

Thus, we consider infinitesimal deformations of the first order.

Definition. Infinitesimal deformations of the first order are called *geodesic* if the geodesic lines of V_n are preserved under these deformations [2,3,4].

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The following Theorem holds [2,4]:

Theorem 1. *The vector field $\xi^\alpha(x)$ defines a geodesic deformation of V_n if and only if it is a solution of the following system of equations*

$$\begin{gathered} a_{\alpha\beta}\, y^\alpha_{,\,i}\xi^\beta_{,\,jk} + a_{\alpha\beta}\xi^\alpha_{,\,i}\, y^\beta_{,\,jk} + \\ \Gamma_{\gamma\beta\alpha}(y^\alpha_{,\,i}\, y^\beta_{,\,j}\, \xi^\gamma_{,\,k} + y^\alpha_{,\,i}\, \xi^\beta_{,\,j}\, y^\gamma_{,\,k} + \xi^\alpha_{,\,i}\, y^\beta_{,\,j}\, y^\gamma_{,\,k}) + \\ \xi^\gamma \left(\frac{\partial a_{\alpha\beta}}{\partial y^\gamma} y^\alpha_{,\,i} y^\beta_{,\,jk} + \frac{\partial \Gamma_{\gamma\beta\alpha}}{\partial y^\gamma} y^\alpha_{,\,i} y^\beta_{,\,j} y^\gamma_{,\,k} \right) = \psi_i g_{jk} + \psi_j g_{ik}\,, \end{gathered} \tag{3}$$

where $\Gamma_{\gamma\beta\alpha}$ are the Christoffel symbols constructed for $a_{\alpha\beta}$, $\psi_i \equiv \psi_{,i}$ is a gradient vector, the comma means covariant derivative of V_n.

The rather complicated system (3) gains a more convenient aspect in case $m = n+1$, i.e. V_n is hypersurface of V_m.

The condition (1) allows vectors $y^\alpha_{,\,i}$ to be chosen as basis of V_n. We consider, that $\det \|g_{ij}\| \neq 0$, hence, normal $\eta^\alpha(x)$ to hypersurface is nonisotropic [1], i.e.

$$a_{\alpha\beta}\, y^\alpha_{,\,i}\eta^\beta = 0; \qquad a_{\alpha\beta}\,\eta^\alpha\eta^\beta = e, \qquad e = \pm 1.$$

The system of the vectors $\{y^\alpha_{,\,i}\,;\, \eta^\alpha\}$ forms a basis of V_{n+1}.

Let

$$\xi^\alpha(x) = \lambda^i\, y^\alpha_{,\,i} + \mu\,\eta^\alpha \tag{4}$$

where

$$\lambda_j = g_{ij}\lambda^i = a_{\alpha\beta}\xi^\alpha\, y^\beta_{,\,j}\,; \qquad \mu = e\, a_{\alpha\beta}\xi^\alpha\, y^\beta.$$

If we use (4), Gauss and Weingarten derivation formulas for hypersurfaces, then the system (3) can be simplified to [2,3,4]

$$\lambda_{i,jk} = \lambda_s R^s_{kji} + \psi_j g_{ik} + \psi_k g_{ij} + (\mu\Omega_{ij})_{,k} + (\mu\Omega_{ik})_{,j} - (\mu\Omega_{jk})_{,i}\,, \tag{5}$$

where R^h_{ijk} is the Riemannian tensor and Ω_{ij} is the second fundamental tensor $V_n \subset V_{n+1}$.

The system (5) also gives necessary and sufficient conditions for geodesic deformations of hypersurface V_n, we will name it *fundamental system*.

2. Let V_n be a totally geodesic hypersurface. It means, that geodesics of V_n are geodesics of V_{n+1} at the same time [1].

The condition

$$\Omega_{ij} = 0$$

is an analytical feature of the totally geodesic hypersurface and then (5) has the simple form

$$\lambda_{i,jk} = \lambda_s R^s_{kji} + \psi_j g_{ik} + \psi_k g_{ij}\ . \tag{6}$$

The system (6) does not put any conditions on the invariant μ and we have

Theorem 2. *The vector field of displacements*

$$\xi^\alpha(x) = \lambda^i\, y^\alpha_{\ ,i} + \mu\, \eta^\alpha$$

has been defined up to the normal component μ *at the infinitesimal geodesic deformation of the totally geodesic hypersurface* $V_n \subset V_{n+1}$.

Remark. If V_n coincides with V_m, (2) defines the infinitesimal geodesic transformations of V_n, which have been studied in [1]. In this case $y^\alpha = x^\alpha$, and $\mu = 0$, $\xi^\alpha = \lambda^\alpha$ in (4), hence (5) has the form (6).

Thus, the problems of the infinitesimal geodesic deformations of the totally geodesic hypersurfaces and the problems of the infinitesimal transformations of any V_n, which preserve the geodesics of these V_n, are equivalent [1, page 275].

3. The problem of the infinitesimal geodesic deformations of the totally geodesic manifolds is reduced to solving the fundamental system (6). Let's find the conditions of integrability.

Differentiating (6) covariantly with respect to x^l and alternating (without division) the result on k and l, we obtain

$$\psi_{,jl}\, g_{ki} - \psi_{,jk}\, g_{li} = \lambda_{s,j} R^s_{ikl} + \lambda_{s,i} R^s_{jkl} + \lambda_{s,i} R^s_{kij} - \lambda_{s,k} R^s_{lij} + \lambda_s R^s_{kij,l} - \lambda_s R^s_{lij,k}\,. \tag{7}$$

Contracting equation (7) by g^{il}, we obtain

$$\psi_{,jk} = \frac{1}{n-1}(\lambda_s R_{jk,}{}^{s} + \lambda_{s,k} R^s_j + \lambda_{s,j} R^s_k) \tag{8}$$

where $R_{ij} = g_{is} R^s_j$ is the Ricci tensor.

If we put

$$\lambda_{s,p} = b_{sp}, \tag{9}$$

then (6) and (8) can be rewritten as

$$b_{ij,k} = \lambda_s R^s_{kji} + \psi_j g_{ik} + \psi_k g_{ij}\,, \tag{10}$$

$$\psi_{,jk} = \frac{1}{n-1}(\lambda_s R_{jk,}{}^{s} + b_{sp}(R^s_j \delta^p_k + R^s_k \delta^p_j) \tag{11}$$

where δ^p_k is the Kronecker's symbol.

The set of the equations (9), (10), (11) is the system of the Cauchy type with respect to b_{ij}, λ_i, ψ_j.

We obtain

Theorem 3. *A totally geodesic manifold* V_n *admits nontrivial infinitesimal geodesic deformations if and only if the system of the equations* (9), (10), (11) *has a nontrivial solution.*

The common solution of this system contains no more than $n(n+2)$ arbitrary constants, generated by initial values of searched functions. The number of linearly independent solutions of this system does not exceed $n(n+2)$ and solutions cannot have a functional arbitrariness [4,5].

The system of the equations (9), (10), (11) is solvable only in case of realization of integrability conditions.

They hold identically for the equations (9).

The conditions of integrability for the equations (10) are the relations (7). From (8) and (9), formula (7) is given by

$$b_{pq}A^{pq}_{ijkl} + \lambda_s B^s_{ijkl} = 0, \tag{12}$$

where

$$A^{pq}_{ijkl} = -\delta^q_j W_{i\,.\,kl}^{\ p} - \delta^q_k W_{ij\,.\,l}^{\ \ p} + \delta^q_i W_{ijk\,.}^{\ \ \ p} + \delta^p_i W^{\ q}_{\ jkl}\,, \qquad B^s_{ijkl} = -g^{sh} W_{ijkl,\,h}$$

and W^h_{ijk} is the Weyl tensor [1 – 6].

If we use the concept of Lie derivative with respect to λ^s [6], (12) gets an especially convenient and compact form

$$L_\lambda W_{ijkl} = (\lambda_{i,s} + \lambda_{s,i}) W^s_{jkl}\,. \tag{13}$$

Let's find the conditions of integrability of (11). Having differentiated (11) with respect to x^l, and alternating result on k and l, applying the Ricci identity, we obtain

$$b_{sp} D^{sp}_{jkl} + \lambda_p R_{sjkl,\,.\,.}^{\quad\ \ sp} + (1-n)\psi_s W^s_{jkl} = 0 \tag{14}$$

where

$$D^{sp}_{jkl} = -g^{sh}\left(\delta^p_l R^q_{jhk,q} + \delta^p_k R^q_{jhl,q} + \delta^p_j R^q_{hlk,q}\right). \tag{15}$$

We rewrite the formulas (14) and (15)

$$L_\lambda R^q_{jlk,q} = (n-1)\psi_p W^p_{jlk} \tag{16}$$

or

$$L_\lambda W^q_{jkl,q} = (n-2)\psi_p W^p_{jlk}\,.$$

The conditions of integrability (13) and (16) of the system of the equations (9), (10), (11) are linear homogeneous algebraic equations with respect to b_{ij}, λ_i, ψ_i, with factors, which are well defined on the space V_n.

Their differential prolongations of any order will be the same. But the number of linearly independent among them cannot exceed $n(n+2)$ [4,5].

Theorem 4. *The totally geodesic manifold V_n admits nontrivial infinitesimal geodesic deformations if and only if the system of the equations (13), (16) and its linearly independent differential prolongations have nontrivial solutions with respect to b_{ij}, λ_i, ψ_i.*

The common solution of the systems of the equations (9), (10), (11) consists of no more than $n(n+2)$ essential parameters. We will name their amount as *the degree of mobility* V_n concerning tangent geodesic deformations, i.e. when the field of displacements is given by

$$\xi^\alpha = \lambda^i y^\alpha_{,i}\,.$$

The space V_n will have the maximum degree of mobility, for which (13) and (16) hold identically. It is obvious that it will hold if and only if $W_{ijkl} = 0$ or, in other words, for spaces of constant curvature.

Theorem 5. *Only the Riemannian spaces of constant curvature have the maximum degree of mobility concerning tangent geodesic deformations, which equals* $n(n+2)$.

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