DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 171–184

SURFACES IN PROJECTIVE DIFFERENTIAL GEOMETRY AND INTEGRABLE SYSTEMS

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Abstract. Our aim here is to give a brief review of surfaces in projective 3-space with the emphasize on the nonlinear equations underlying them. As examples we have choosen isothermally asymptotic surfaces, projectively applicable surfaces, surfaces of Jonas and projectively minimal surfaces. It is demonstrated, that the corresponding projective 'Gauss-Codazzi' equations reduce to integrable systems which are familiar from the modern soliton theory and coincide with the stationary flows in the Davey-Stewartson and Kadomtsev-Petviashvili hierarchies, equations of the Toda lattice, etc. The corresponding Lax pairs can be obtained by inserting a spectral parameter in the equations of the Wilczynski moving frame.

Projective differential geometry of surfaces in P^3 has been extensively developed in the first half of the century in the works of Wilczynski, Fubini, Čech, Cartan, Tzitzeica, Demoulin, Rozet, Godeaux, Lane, Eisenhart, Finikov, Bol and many others. These investigations culminated in a number of beautifull geometric constructions and a list of particularly interesting classes of surfaces which now seem to be quite deeply forgotten.

1. Surfaces in projective differential geometry

Based on [24] (see also [6]), let us briefly recall the standard way of defining surfaces M^2 in projective space P^3 in terms of solutions of a linear system

$$\begin{aligned} r_{xx} &= \beta\ r_y + \tfrac{1}{2}(V - \beta_y)\ r \\ r_{yy} &= \gamma\ r_x + \tfrac{1}{2}(W - \gamma_x)\ r \end{aligned} \tag{1}$$

where β, γ, V, W are functions of x and y. If we cross-differentiate (1) and assume r, r_x, r_y, r_{xy} to be independent, we arrive at the compatibility conditions [15, p.

1991 *Mathematics Subject Classification.* 53A40, 53A20, 35Q58, 35L60.
Key words and phrases. Projective differential geometry, integrable systems.
This paper is in final form and no version of it will be submitted for publication elsewhere.

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$$\beta_{yyy} - 2\beta_y W - \beta W_y = \gamma_{xxx} - 2\gamma_x V - \gamma V_x$$

$$W_x = 2\gamma\beta_y + \beta\gamma_y \tag{2}$$

$$V_y = 2\beta\gamma_x + \gamma\beta_x \; .$$

For any fixed β, γ, V, W satisfying (2) the linear system (1) is compatible and possesses a solution $r = (r^0, r^1, r^2, r^3)$ where $r^i(x, y)$ can be regarded as homogeneous coordinates of a surface in projective space P^3. For our purposes, one may think of M^2 as a surface in a three-dimensional space with position vector $R = (r^1/r^0, r^2/r^0, r^3/r^0)$. If we choose any other solution $\tilde{r} = (\tilde{r}^0, \tilde{r}^1, \tilde{r}^2, \tilde{r}^3)$ of the same system (1) then the corresponding surface $\tilde{M}^2$ with position vector $\tilde{R} = (\tilde{r}^1/\tilde{r}^0, \tilde{r}^2/\tilde{r}^0, \tilde{r}^3/\tilde{r}^0)$ constitutes a projective transform of M^2 so that any fixed β, γ, V, W satisfying (2) define a surface M^2 uniquely up to projective equivalence. Moreover, a simple calculation yields

$$\begin{aligned} R_{xx} &= \beta R_y + a R_x \\ R_{yy} &= \gamma R_x + b R_y \end{aligned} \tag{3}$$

($a = -2r^0_x/r^0$, $b = -2r^0_y/r^0$) which implies that x, y are asymptotic coordinates of the surface M^2. System (3) can be viewed as an 'affine gauge' of system (1). In what follows, we assume that our surfaces are hyperbolic and the corresponding asymptotic coordinates x, y are real.[1] Since equations (2) specify a surface uniquely up to projective equivalence, they can be viewed as the 'Gauss-Codazzi' equations in projective geometry.

Even though the coefficients β, γ, V, W define a surface M^2 uniquely up to projective equivalence via (1), it is not entirely correct to regard β, γ, V, W as projective invariants. Indeed, the asymptotic coordinates x, y are only defined up to an arbitrary reparametrization of the form

$$x^* = f(x), \;\; y^* = g(y) \tag{4}$$

which induces a scaling of the surface vector according to

$$r^* = \sqrt{f'(x)g'(y)} \; r \, . \tag{5}$$

Thus [2, p. 1], the form of equations (1) is preserved by the above transformation with the new coefficients $\beta^*, \gamma^*, V^*, W^*$ given by

$$\begin{aligned} \beta^* &= \beta g'/(f')^2, \;\; V^*(f')^2 = V + S(f) \\ \gamma^* &= \gamma f'/(g')^2, \;\; W^*(g')^2 = W + S(g) \, , \end{aligned} \tag{6}$$

[1] The elliptic case is dealt with in an analogous manner by regarding x, y as complex conjugates.

where $S(\,\cdot\,)$ is the Schwarzian derivative, that is

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

The transformation formulae (6) imply that the symmetric 2-form

$$2\beta\gamma\, dx dy$$

and the conformal class of the cubic form

$$\beta\, dx^3 + \gamma\, dy^3$$

are absolute projective invariants. They are known as the projective metric and the Darboux cubic form, respectively, and play an important role in projective differential geometry since, in particular, they define a 'generic' surface uniquely up to projective equivalence.

Using (4)-(6), one can verify that the four points

$$\begin{gathered} r,\ r_1 = r_x - \tfrac{1}{2}\tfrac{\gamma_x}{\gamma} r,\ r_2 = r_y - \tfrac{1}{2}\tfrac{\beta_y}{\beta} r\,, \\ \boldsymbol{\eta} = r_{xy} - \tfrac{1}{2}\tfrac{\gamma_x}{\gamma} r_y - \tfrac{1}{2}\tfrac{\beta_y}{\beta} r_x + \left(\tfrac{1}{4}\tfrac{\beta_y\gamma_x}{\beta\gamma} - \tfrac{1}{2}\beta\gamma\right) r \end{gathered} \tag{7}$$

are defined in an invariant way, that is under the transformation formulae (4)-(6) they acquire a nonzero multiple which does not change them as points in projective space P^3. These points form the vertices of the so-called Wilczynski moving tetrahedral [2], [8], [24]. Since the lines passing through r, r_1 and r, r_2 are tangential to the x- and y-asymptotic curves, respectively, the three points r, r_1, r_2 span the tangent plane of the surface M^2. The line through r_1, r_2 lying in the tangent plane is known as the directrix of Wilczynski of the second kind. The line through $r, \boldsymbol{\eta}$ is transversal to M^2 and is known as the directrix of Wilczynski of the first kind. It plays the role of a projective 'normal'. The Wilczynski tetrahedral proves to be the most convenient tool in projective differential geometry.

Using (1) and (7), we easily derive for $r, r_1, r_2, \boldsymbol{\eta}$ the linear equations [8, p. 42]

$$\begin{gathered} \begin{pmatrix} r \\ r_1 \\ r_2 \\ \boldsymbol{\eta} \end{pmatrix}_x = \begin{pmatrix} \frac{1}{2}\frac{\gamma_x}{\gamma} & 1 & 0 & 0 \\ \frac{1}{2}b & -\frac{1}{2}\frac{\gamma_x}{\gamma} & \beta & 0 \\ \frac{1}{2}k & 0 & \frac{1}{2}\frac{\gamma_x}{\gamma} & 1 \\ \frac{1}{2}\beta a & \frac{1}{2}k & \frac{1}{2}b & -\frac{1}{2}\frac{\gamma_x}{\gamma} \end{pmatrix} \begin{pmatrix} r \\ r_1 \\ r_2 \\ \boldsymbol{\eta} \end{pmatrix} \\ \begin{pmatrix} r \\ r_1 \\ r_2 \\ \boldsymbol{\eta} \end{pmatrix}_y = \begin{pmatrix} \frac{1}{2}\frac{\beta_y}{\beta} & 0 & 1 & 0 \\ \frac{1}{2}l & \frac{1}{2}\frac{\beta_y}{\beta} & 0 & 1 \\ \frac{1}{2}a & \gamma & -\frac{1}{2}\frac{\beta_y}{\beta} & 0 \\ \frac{1}{2}\gamma b & \frac{1}{2}a & \frac{1}{2}l & -\frac{1}{2}\frac{\beta_y}{\beta} \end{pmatrix} \begin{pmatrix} r \\ r_1 \\ r_2 \\ \boldsymbol{\eta} \end{pmatrix}, \end{gathered} \tag{8}$$

where we introduced the notation

$$
\begin{gathered}
k=\beta\gamma-(\ln\beta)_{xy}\ ,\ l=\beta\gamma-(\ln\gamma)_{xy}\ ,\\
a=W-(\ln\beta)_{yy}-\tfrac{1}{2}(\ln\beta)_y^2\ ,\ b=V-(\ln\gamma)_{xx}-\tfrac{1}{2}(\ln\gamma)_x^2\ .
\end{gathered}
\tag{9}
$$

The compatibility conditions of equations (8) imply

$$
\begin{gathered}
(\ln\beta)_{xy}=\beta\gamma-k,\ (\ln\gamma)_{xy}=\beta\gamma-l\ ,\\
a_x=k_y+\tfrac{\beta_y}{\beta}k,\ b_y=l_x+\tfrac{\gamma_x}{\gamma}l,\\
\beta a_y+2a\beta_y=\gamma b_x+2b\gamma_x\ ,
\end{gathered}
\tag{10}
$$

which is just the equivalent form of the projective 'Gauss-Codazzi' equations (2).

Equations (8) can be rewritten in the Plücker coordinates. Let us consider a line l in P^3 passing through the points a and b with the homogeneous coordinates $a=(a^0:a^1:a^2:a^3)$ and $b=(b^0:b^1:b^2:b^3)$. With the line l we associate a point $a\wedge b$ in projective space P^5 with the homogeneous coordinates

$$
a\wedge b=(p_{01}:p_{02}:p_{03}:p_{23}:p_{31}:p_{12})\ ,
$$

where $p_{ij}=a_ib_j-a_jb_i$. The coordinates p_{ij} satisfy the well-known quadratic Plücker relation

$$
p_{01}p_{23}+p_{02}p_{31}+p_{03}p_{12}=0\ . \tag{11}
$$

Instead of a and b we may consider an arbitrary linear combinations thereof without changing $a\wedge b$ as a point in P^5. Hence, we arrive at the well-defined Plücker coorrespondence $l(a,b)\to a\wedge b$ between lines in P^3 and points on the Plücker quadric in P^5.

The Plücker correspondence plays an important role in the projective differential geometry of surfaces and often sheds some new light on those properties of surfaces which are not 'visible' in P^3 but acquire a precise geometric meaning only in P^5. Thus, let us consider a surface $M^2\in P^3$ with the Wilczynski tetrahedral $r,r_1,r_2,\boldsymbol{\eta}$ satisfying equations (8). Since the two pairs of points r,r_1 and r,r_2 generate two lines in P^3 which are tangential to the x- and y-asymptotic curves, respectively, the formulae

$$
\mathcal{U}=r\wedge r_1,\quad \mathcal{V}=r\wedge r_2
$$

define the images of these lines under the Plücker embedding. Hence, with any surface $M^2\in P^3$ there are canonically associated two surfaces $\mathcal{U}(x,y)$ and $\mathcal{V}(x,y)$ in P^5 lying on the Plücker quadric (11). In view of the formulae

$$
\mathcal{U}_x=\beta\,\mathcal{V},\quad \mathcal{V}_y=\gamma\,\mathcal{U},
$$

we conclude that the line in P^5 passing through a pair of points $(\mathcal{U},\mathcal{V})$ can also be generated by the pair of points $(\mathcal{U},\mathcal{U}_x)$ (and hence is tangential to the x-coordinate line on the surface $\mathcal{U}$) or by a pair of points $(\mathcal{V},\mathcal{V}_y)$ (and hence is tangential to the y-coordinate line on the surface $\mathcal{V}$). Consequently, the surfaces $\mathcal{U}$ and $\mathcal{V}$ are two focal surfaces of the congruence of straight lines $(\mathcal{U},\mathcal{V})$ or, equivalently, $\mathcal{V}$ is the Laplace transform of $\mathcal{U}$ with respect to x and $\mathcal{U}$ is the Laplace transform of $\mathcal{V}$ with respect to y. We emphasize that the x- and y-coordinate lines on the surfaces $\mathcal{U}$ and $\mathcal{V}$ are not asymptotic but conjugate. Continuation of the Laplace sequence in both directions, that is taking the x-transform of $\mathcal{V}$, the y-transform of $\mathcal{U}$, etc., leads, in the generic case, to an infinite Laplace sequence in P^5 known as the Godeaux sequence of a surface M^2 [2, p. 344]. The surfaces of the Godeaux sequence carry important geometric information about the surface M^2 itself.

Introducing

$$\mathcal{A} = r_2 \wedge r_1 + r \wedge \boldsymbol{\eta},\ \mathcal{B} = r_1 \wedge r_2 + r \wedge \boldsymbol{\eta},$$

$$\mathcal{P} = 2\, r_2 \wedge \boldsymbol{\eta},\ \mathcal{Q} = 2\, r_1 \wedge \boldsymbol{\eta},$$

we arrive at the following equations for the Plücker coordinates:

$$\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 & \beta & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -\beta a & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma_x}{\gamma} & 1 & 0 \\ 0 & 0 & 0 & b & 0 & 1 \\ -\beta a & 0 & \beta & 0 & b & -\frac{\gamma_x}{\gamma} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}$$

$$(12)\qquad \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_y = \begin{pmatrix} \frac{\beta_y}{\beta} & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 \\ 0 & a & -\frac{\beta_y}{\beta} & -\gamma b & 0 & \gamma \\ \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ -\gamma b & 0 & 0 & 0 & l & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}$$

Different types of surfaces can be defined by imposing additional constraints on β, γ, V, W (respectively, $\beta, \gamma, k, l, a, b$), so that, in a sense, projective differential geometry is the theory of (integrable) reductions of the underdetermined system (2) (respectively, (10)). Although the three linear systems (1), (8) and (12) are in fact equivalent, some of them prove to be more suitable for studying particular classes of projective surfaces.

2. Isothermally asymptotic surfaces

This class is specified by the condition $\beta = \gamma$, in which case equations (2) assume the form of the stationary modified Veselov-Novikov (mVN) equation

$$(13)\qquad \begin{aligned} \beta_{yyy} - 2\beta_y W - \beta W_y &= \beta_{xxx} - 2\beta_x V - \beta V_x, \\ W_x &= \tfrac{3}{2}(\beta^2)_y \\ V_y &= \tfrac{3}{2}(\beta^2)_x. \end{aligned}$$

Rewriting (12) in terms of $\mathcal{U}, \mathcal{V}$ (that is, expressing $\mathcal{A}, \mathcal{B}, \mathcal{P}, \mathcal{Q}$ through $\mathcal{U}, \mathcal{U}_y, \mathcal{U}_{yy}$ and $\mathcal{V}, \mathcal{V}_x, \mathcal{V}_{xx}$), we arrive at

$$
\begin{aligned}
\mathcal{U}_x &= \beta\ \mathcal{V} \\
\mathcal{V}_y &= \beta\ \mathcal{U}
\end{aligned}
$$

(14)
$$
\mathcal{U}_{yyy} - \mathcal{U}_{xxx} = 2\ W\ \mathcal{U}_y - 3\ \beta_x\ \mathcal{V}_x + W_y\ \mathcal{U} - 2\ \beta\ V \mathcal{V}
$$

$$
\mathcal{V}_{xxx} - \mathcal{V}_{yyy} = 2\ V\ \mathcal{V}_x - 3\ \beta_y\ \mathcal{U}_y + V_x\ \mathcal{V} - 2\ \beta\ W \mathcal{U}
$$

which coincides with the stationary limit of the mVN linear problem (the compatibility conditions of (14) coincide with (13)).

We recall that the (2+1)-dimensional mVN equation

$$
\beta_t = \beta_{xxx} - \beta_{yyy} - 2\ \beta_x\ V + 2\ \beta_y\ W - \beta\ V_x + \beta\ W_y
$$

(15)
$$
W_x = \tfrac{3}{2}(\beta^2)_y
$$

$$
V_y = \tfrac{3}{2}(\beta^2)_x
$$

was introduced in [1] and is associated with the two-dimensional Dirac operator

$$
\begin{aligned}
\mathcal{U}_x &= \beta\ \mathcal{V} \\
\mathcal{V}_y &= \beta\ \mathcal{U}
\end{aligned}
$$

(16)
$$
\mathcal{U}_t = \mathcal{U}_{xxx} - \mathcal{U}_{yyy} + 2\ W\ \mathcal{U}_y - 3\ \beta_x\ \mathcal{V}_x + W_y\ \mathcal{U} - 2\ \beta\ V\ \mathcal{V}
$$

$$
\mathcal{V}_t = \mathcal{V}_{xxx} - \mathcal{V}_{yyy} - 2\ V\ \mathcal{V}_x + 3\ \beta_y\ \mathcal{U}_y - V_x\ \mathcal{V} + 2\ \beta\ W\ \mathcal{U}.
$$

Replacing in the stationary limit $\mathcal{U}_t$ and $\mathcal{V}_t$ by $\lambda\mathcal{U}$ and $\lambda\mathcal{V}$, we can introduce a spectral parameter λ in the linear system (14) without violating its compatibility.

Isothermally asymptotic surfaces are attributed to Fubini [10] and can be equivalently defined by any of the following geometric properties:

– The 3-web, formed by asymptotic curves and Darboux's curves, is hexagonal. (Darboux's curves are the zero curves of the Darboux cubic form $\beta\, dx^3 + \gamma\, dy^3$).

– Isothermally asymptotic surfaces are the focal surfaces of special W-congruences, preserving Darboux's curves (in fact, this property implies a Bäcklund transformation for isothermally asymptotic surfaces).

Examples of isothermally asymptotic surfaces include arbitrary quadrics and cubics, quartics of Kummer, projective transforms of affine spheres and rotation surfaces. We refer to [15], [2], [8], [9], [7] for the further discussion of isothermally asymptotic surfaces.

3. Projectively applicable surfaces

Generically, the quadratic form

$$
\beta\gamma\ dxdy
$$

and the cubic form

$$\beta\, dx^3 + \gamma\, dy^3$$

define a surface $M^2 \subset P^3$ uniquely up to projective equivalence. There exists, however, a class of the so-called projectively applicable surfaces for which this is not the case. Analytically, projective applicability means that for a given β, γ equations (2) do not specify V and W uniquely (equivalently, equations $(10)_3$ – $(10)_5$ do not uniquely specify a and b). Projectively applicable surfaces naturally fall into two different classes: the so-called 'surfaces R_0' and 'surfaces R'.

Surfaces R_0 are specified by a condition $\beta = 1$ (or $\gamma = 1$), implying upon the substitution in (2) the system

$$\begin{aligned} &\gamma_{xxx} - 2\gamma_x V - \gamma V_x + W_y = 0 \\ &W_x = \gamma_y \\ &V_y = 2\gamma_x. \end{aligned} \tag{17}$$

Since W enters (17) only through it's derivatives, it is defined up to an additive constant $W \to W + \lambda$, providing thus a 1-parameter family of projectively nonequivalent surfaces having the same β and γ. The radius-vector of these surfaces satisfies the linear system

$$\begin{aligned} r_{xx} &= r_y + \tfrac{1}{2} V r \\ r_{yy} &= \gamma r_x + \tfrac{1}{2}(W - \gamma_x + \lambda) r \end{aligned}$$

containing a 'spectral parameter' λ. Rewriting it in the form

$$\begin{aligned} r_y &= r_{xx} - \tfrac{1}{2} V r \\ r_{xxxx} &= V r_{xx} + (V_x + \gamma) r_x + \tfrac{1}{2}(V_{xx} + \gamma_x - \tfrac{1}{2} V^2 + W + \lambda) r \end{aligned} \tag{18}$$

we recognise in $(18)_1$ the nonstationary Schrödinger equation which plays a role of the Lax operator of the (2+1)-dimensional integrable Kadomtsev-Petviashvili (KP) hierarchy. Meanwhile, $(18)_2$ coincides with the stationary limit of the linear problem generating the second flow in the KP hierarchy.

We recall that the second flow in the KP hierarchy is associated with the linear system

$$\begin{aligned} r_y &= r_{xx} - \tfrac{1}{2} V r \\ r_t &= r_{xxxx} - V r_{xx} - (V_x + \gamma) r_x - \tfrac{1}{2}(V_{xx} + \gamma_x - \tfrac{1}{2} V^2 + W) r \end{aligned} \tag{19}$$

which is related to (18) via a formal transformation $\frac{1}{2}\lambda r \to \partial_t r$. The compatibility conditions of (19) result in the 'second' KP flow:

$$V_t = \gamma_{xxx} - 2\gamma_x V - \gamma V_x + W_y$$

$$W_x = \gamma_y$$

$$V_y = 2\gamma_x.$$

Thus, equations (17) governing surfaces R_0 coincide with the stationary limit of the second flow in the KP hierarchy.

Surfaces R are specified by the condition $\beta_y = \gamma_x$. In this case V and W are defined by (2) up to an additive constant $W \to W + \lambda$, $V \to V + \lambda$, so that we again arrive at a 1-parameter family of projectively nonequivalent surfaces having the same β, γ. Their radius-vector satisfies the linear system

$$r_{xx} = \beta r_y + \tfrac{1}{2}(V - \beta_y + \lambda) r$$

$$r_{yy} = \gamma r_x + \tfrac{1}{2}(W - \gamma_x + \lambda) r$$

containing a 'spectral parameter' λ. Rewriting this linear system in the form

$$r_{xx} - r_{yy} = \beta r_y - \gamma r_x + \tfrac{1}{2}(V - W) r$$

$$r_{xx} + r_{yy} - \beta r_y - \gamma r_x - \tfrac{1}{2}(V + W - \beta_y - \gamma_x) r = \lambda r,$$

we recognize the stationary Davey-Stewartson (DS) linear problem (the relationship of surfaces R to the stationary DS hierarchy has been pointed out in [14]). Replacing λr by $\partial_t r$ we arrive at the (2+1)-dimensional DS linear problem

$$r_{xx} - r_{yy} = \beta r_y - \gamma r_x + \tfrac{1}{2}(V - W) r$$

$$r_t = r_{xx} + r_{yy} - \beta r_y - \gamma r_x - \tfrac{1}{2}(V + W - \beta_y - \gamma_x) r,$$

with the compatibility conditions

$$V_t + \beta_{yyy} - 2\beta_y W - \beta W_y = W_t + \gamma_{xxx} - 2\gamma_x V - \gamma V_x$$

$$\gamma_t = W_x - 2\gamma\beta_y - \beta\gamma_y$$

$$\beta_t = V_y - 2\beta\gamma_x - \gamma\beta_x$$

$$\beta_y = \gamma_x.$$

In the stationary limit they reduce to the equations governing surfaces R.

The coordinate net $\xi = x + y$, $\eta = x - y$ on the surface R is conjugate and has an important geometric property, namely, that both congruences of lines, tangent to the ξ- and η-coordinate curves, form a W-congruences. Conjugate nets with this property are called the nets R; they have been introduced by Demoulin and

Tzitzeica in a series of papers [3], [23]. Thus, surfaces R can be equivalently characterized by the existence of a net R. Bäcklund transformation for surfaces R was established by Jonas [12] and subsequently discussed in [8], [9], [5].

The most important examples of surfaces R include projective transforms of surfaces with constant Gaussian curvature $K = -1$ in the Euclidean 3-space, surfaces of constant curvature $K = 1$ in the Lorentzian 3-space and surfaces, carrying a net R which is periodic with period 4 under the Laplace transformations.

4. Surfaces of Jonas

are characterized by a condition

$$\beta_x = \gamma_y$$

which is equivalent to the requirement that the conjugate net $\xi = x + y,\ \eta = x - y$ on the surface M^2 has equal point and tangential Laplace invariants. This class of surfaces has been introduced by Jonas in [13] as a counterpart of surfaces R. With $\beta = \varphi_y,\ \gamma = \varphi_x$ equations (2) assume the form

$$\begin{gathered}\varphi_{yyyy} - 2\ \varphi_{yy}\ W - \varphi_y\ W_y = \varphi_{xxxx} - 2\ \varphi_{xx}\ V - \varphi_x\ V_x \\ (W + \tfrac{1}{2}\varphi_y^2)_x = 2\ (\varphi_x\varphi_y)_y \\ (V + \tfrac{1}{2}\varphi_x^2)_y = 2\ (\varphi_x\varphi_y)_x.\end{gathered} \tag{20}$$

Equations (20) are the compatibility conditions of the linear problem

$$\begin{aligned} f_x &= \varphi_y\ g \\ g_y &= \varphi_x\ f \end{aligned} \tag{21}$$

along with

$$\begin{gathered}\lambda\ f = f_{xxxx} + f_{yyyy} - (\varphi_y^2 + 2W)\ f_{yy} - 4\ \varphi_{xy}\ g_{xx} \\ +(\varphi_y\ \varphi_{yy} - 3\ W_y) f_y - (2\ \varphi_{xxy} + \varphi_y\ \varphi_x^2 + 2\ V\varphi_y) g_x + m\ f + n\ g, \\ \lambda\ g = g_{xxxx} + g_{yyyy} - (\varphi_x^2 + 2V)\ g_{xx} - 4\ \varphi_{xy}\ f_{yy} \\ +(\varphi_x\ \varphi_{xx} - 3\ V_x) g_x - (2\ \varphi_{xyy} + \varphi_x\ \varphi_y^2 + 2\ W\varphi_x) f_y + \tilde{m}\ g + \tilde{n}\ f,\end{gathered} \tag{22}$$

where $\lambda = const$ and $m,\ \tilde{m},\ n,\ \tilde{n}$ are given by the formulae

$$\begin{gathered} m = -W_{yy} - \varphi_y\ \varphi_{yyy} - 2\ \varphi_x\ \varphi_{xxx} + \varphi_{xx}^2 + 2\ V\ \varphi_x^2 + 2\ W\ \varphi_y^2 \\ \tilde{m} = -V_{xx} - \varphi_x\ \varphi_{xxx} - 2\ \varphi_y\ \varphi_{yyy} + \varphi_{yy}^2 + 2\ V\ \varphi_x^2 + 2\ W\ \varphi_y^2 \\ n = -2\ \varphi_{xxxy} + \varphi_x^2\ \varphi_{xy} + 3\ \varphi_x\ \varphi_y\ \varphi_{xx} + 2\ V\ \varphi_{xy} - \varphi_y\ V_x \\ \tilde{n} = -2\ \varphi_{xyyy} + \varphi_y^2\ \varphi_{xy} + 3\ \varphi_x\ \varphi_y\ \varphi_{yy} + 2\ W\ \varphi_{xy} - \varphi_x\ W_y \end{gathered}$$

Up to a gauge transformation this linear system coincides with that from [13], where it has been used for the construction of Bäcklund transformation for the surfaces of Jonas. The structure of linear systems (21), (22) clearly suggests that equations (20) should be certain reduction of the fourth order stationary flow in then DS hierarchy. Indeed, the fourth order flow in the DS hierarchy is generated by a 2-dimensional Dirac operator

$$\begin{aligned} f_x &= \beta\ g \\ g_y &= \gamma\ f \end{aligned} \tag{23}$$

where the time evolution of f and g is specified by

$$\begin{aligned} f_t &= f_{xxxx} + f_{yyyy} + \dots \\ g_t &= g_{xxxx} + g_{yyyy} + \dots. \end{aligned} \tag{24}$$

Linear problem corresponding to the stationary flow can be obtained by formally changing $f_t \to \lambda f,\ g_t \to \lambda g$ where λ plays a role of spectral parameter. The linear problem (21), (22) results after the additional reduction $\beta_x = \gamma_y$. We emphasize that the reduction $\beta_x = \gamma_y$ is compatible not with the (2+1)-dimensional flow generated by (23), (24) rather than only with its stationary limit.

Particular examples of the Jonas surfaces are provided by projective transforms of minimal surfaces in the Euclidean 3-space.

5. Projectively minimal surfaces

These surfaces are the extremals of the projective area functional

$$\int\int \beta\gamma\ dxdy. \tag{25}$$

The Euler-Lagrange equations for the functional (25) have been derived in [20], [19] and in our notation adopt the form

$$\beta_{yyy} - 2\beta_y W - \beta W_y = 0,\ \gamma_{xxx} - 2\gamma_x V - \gamma V_x = 0$$

$$W_x = 2\gamma\beta_y + \beta\gamma_y$$

$$V_y = 2\beta\gamma_x + \gamma\beta_x$$

which results after equating to zero both sides of $(2)_1$. Exploiting the obvious symmetry $\beta \to \lambda\beta,\ \gamma \to \frac{1}{\lambda}\gamma$, we can introduce the spectral parameter in the linear system (1):

$$r_{xx} = \lambda\ \beta\ r_y + \tfrac{1}{2}(V - \lambda\ \beta_y)\ r$$

$$r_{yy} = \tfrac{1}{\lambda}\ \gamma\ r_x + \tfrac{1}{2}(W - \tfrac{1}{\lambda}\ \gamma_x)\ r.$$

In terms of (10) the Euler-Lagrange equations reduce to

$$\beta a_y + 2a\beta_y = 0,\ \gamma b_x + 2b\gamma_x = 0,$$

and result after equating to zero both sides of equation $(10)_5$. In the Plücker coordinates the spectral problem assumes the form

$$
\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 & \lambda\beta & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & -\lambda\beta a & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma_x}{\gamma} & 1 & 0 \\ 0 & 0 & 0 & b & 0 & 1 \\ -\lambda\beta a & 0 & \lambda\beta & 0 & b & -\frac{\gamma_x}{\gamma} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}
$$

(26)

$$
\begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}_y = \begin{pmatrix} \frac{\beta_y}{\beta} & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 \\ 0 & a & -\frac{\beta_y}{\beta} & -\frac{\gamma b}{\lambda} & 0 & \frac{\gamma}{\lambda} \\ \frac{\gamma}{\lambda} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ -\frac{\gamma b}{\lambda} & 0 & 0 & 0 & l & 0 \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{A} \\ \mathcal{P} \\ \mathcal{V} \\ \mathcal{B} \\ \mathcal{Q} \end{pmatrix}
$$

with the spectral parameter λ. Setting

$$a = \frac{\varphi(x)}{\beta^2},\ b = \frac{\psi(y)}{\gamma^2},$$

we have three cases to distinguish:

General case. Both $\varphi(x)$ and $\psi(y)$ are nonzero. In this case, we can always normalize $\varphi(x), \psi(y)$ to ± 1 by means of the transformations (6). Let us assume, for instance, that $\varphi(x) = \psi(y) = 1$. With this normalization equations (10) take the form

(27)
$$(\ln\beta)_{xy} = \beta\gamma - k,\ (\ln\gamma)_{xy} = \beta\gamma - l,$$
$$(\beta k)_y + 2\tfrac{\beta_x}{\beta^2} = 0,\ (\gamma l)_x + 2\tfrac{\gamma_y}{\gamma^2} = 0.$$

Surfaces of Godeaux-Rozet [2, p. 318]. In this case, $\varphi = 0$, and hence $a = 0$, while ψ is nonzero and may be normalized to ± 1. Here, we assume that $\psi = 1$. Inserting this ansatz in (10) we obtain

$$k = \frac{s(x)}{\beta}\,.$$

Hence, if $s(x)$ is nonzero, it may be reduced to -1 by means of (6) so that the resulting equations take the form

(28)
$$(\ln\beta)_{xy} = \beta\gamma + \tfrac{1}{\beta},\ (\ln\gamma)_{xy} = \beta\gamma - l,$$
$$(\gamma l)_x + 2\tfrac{\gamma_y}{\gamma^2} = 0.$$

Surfaces of Demoulin. In this case, both φ and ψ are zero and hence $a = b = 0$, so that

$$k = \frac{s(x)}{\beta},\ l = \frac{t(y)}{\gamma}.$$

Once again, the analysis falls into three subcases depending on whether s, t are zero or not. In the generic situation $s \neq 0,\ t \neq 0$ both s and t may be normalized to -1 and the resulting equations reduce to the coupled Tzitzeica system

$$(\ln \beta)_{xy} = \beta\gamma + \tfrac{1}{\beta}, \qquad (\ln \gamma)_{xy} = \beta\gamma + \tfrac{1}{\gamma}. \tag{29}$$

In this form, the equations governing Demoulin surfaces have been set down in [8, p. 51]. The same system has been presented in [17] as a reduction of the two-dimensional Toda lattice. Inserting $a = b = 0,\ k = -\frac{1}{\beta},\ l = -\frac{1}{\gamma}$ in equations (26) and rewriting them in terms of $\mathcal{A}$ and $\mathcal{B}$, we arrive at a more convenient second-order linear problem [6]

$$\begin{gathered} \mathcal{B}_{xy} = -\tfrac{1}{\gamma}\ \mathcal{B}\quad \mathcal{A}_{xy} = -\tfrac{1}{\beta}\ \mathcal{A}\,, \\ \mathcal{B}_{xx} = \lambda\ \beta\ \mathcal{A}_y - \tfrac{\gamma_x}{\gamma}\ \mathcal{B}_x\quad \mathcal{A}_{xx} = \lambda\ \gamma\ \mathcal{B}_y - \tfrac{\beta_x}{\beta}\ \mathcal{A}_x\,, \\ \mathcal{B}_{yy} = \tfrac{\beta}{\lambda}\ \mathcal{A}_x - \tfrac{\gamma_y}{\gamma}\ \mathcal{B}_y\quad \mathcal{A}_{yy} = \tfrac{\gamma}{\lambda}\ \mathcal{B}_x - \tfrac{\beta_y}{\beta}\ \mathcal{A}_y\,. \end{gathered} \tag{30}$$

In [4] Demoulin established in a purely geometric manner the existence of Bäcklund transformations for Godeaux-Rozet and Demoulin surfaces and proved the corresponding permutability theorems. Apparently, Demoulin did not formulate his results in terms of analytic expressions. In [6], a Toda lattice connection and the second-order linear problem (30) is used to derive explicitly a Bäcklund transformation for Demoulin surfaces.

Projectively minimal, Godeaux-Rozet and Demoulin surfaces also arise in the theory of envelopes of Lie quadrics associated with the surface M^2. For brevity, we only recall the necessary definitions. The details can be found in [2], [8], [15], etc. Thus, let us consider a point p^0 on the surface M^2 and the x-asymptotic line passing through p^0. Let us take three additional points p^i, $i = 1, 2, 3$ on this asymptotic line close to p^0 and draw three y-asymptotic lines γ^i passing through p^i. The three straight lines which are tangential to γ^i and pass through the points p^i uniquely define a quadric Q containing them as rectilinear generators. As p^i tend to p^0, the quadric Q tends to a limiting quadric, the so-called Lie quadric of the surface M^2 at the point p^0. Even though this construction depends on the initial choice of either the x- or the y-asymptotic line through p^0, the resulting quadric Q is independent of that choice. Thus, we arrive at a two-parameter family of quadrics associated with the surface M^2. In terms of the Wilczynski tetrahedral, the parametric equation for Q is of the form [2, p.311]

$$Q = \boldsymbol{\eta} + \mu r_1 + \nu r_2 + \mu\nu r,$$

where μ, ν are parameters.

Now, in the neighbourhood of a generic point p^0 on M^2, the envelopes of the family of Lie quadrics consist of the surface M^2 itself and four, in general, distinct sheets. The case of projectively minimal surfaces is characterized by the additional requirement that the asymptotic lines on all these sheets correspond to the asymptotic lines of the surface M^2 itself [21]. Moreover, for projectively minimal surface all four sheets of the envelope will be projectively minimal as well [16]. Surfaces of Godeaux-Rozet are characterized by the degenerate case of two distinct sheets while Demoulin surfaces are present if all four sheets coincide. Surfaces of Godeaux-Rozet and Demoulin have been investigated extensively in [4, 11, 18], see also [6].

6. Acknowledgements

I would like to thank A.I.Bobenko, B.G. Konopelchenko, K.R. Khusnutdinova, U. Pinkall and W.K. Schief for their interest and useful remarks.

This research was supported by the RFFI grants 96-01-00166, 96-06-80104, INTAS 96-0770 and the Aleksander von Humboldt Foundation.

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