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RIEMANNIAN MANIFOLDS WHOSE SKEW–SYMMETRIC CURVATURE OPERATOR HAS CONSTANT EIGENVALUES II

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ABSTRACT. A Riemannian metric on a manifold is said to be IP if the eigenvalues of the skew-symmetric curvature operator are pointwise constant, i.e. they depend upon the point of the manifold but not upon the particular 2 plane in the tangent bundle at that point. The IP metrics for manifolds of dimension $m \geq 4$ and $m \neq 7, 8$ were classified previously; the cases $m = 7$ and $m = 8$ are exceptional. In this paper, we use somewhat different techniques to classify IP metrics if $m = 8$.

0. Introduction

Let R_g be the skew-symmetric curvature operator of a Riemannian manifold (M, g) of dimension $m \geq 4$. Let $\{X, Y\}$ be an oriented orthonormal basis for an oriented 2 plane π in the tangent space $T_P M$ to M at P. We define a skew-symmetric endomorphism $R_g(\pi) := R(X, Y)$ of $T_P M$ which independent of the particular orthonormal basis chosen for π. We say that the metric g is IP at a point $P \in M$ if the eigenvalues of R_g are constant on the Grassmanian $Gr_2^+(T_P M)$ of oriented 2 planes in $T_P M$; we say that the metric g is globally IP if g is IP at every point of M;the eigenvalues in question may vary with the point $P \in M$.

0.1 Examples of metrics which are globally IP.

(1) Let g be a metric of constant sectional curvature C on a manifold M. The local isometries of (M, g) act transitively on the set of all 2 planes in the tangent bundle of M so g is globally IP.
(2) Let $M = I \times N$ be a product manifold where I is a subinterval of $\mathbb{R}$ and where ds_N^2 is a metric of constant sectional curvature K on N. Give M the metric $ds_M^2 := dt^2 + f(t) ds_N^2$ where $f(t) := (\mathcal{K} t^2 + At + B)/2 > 0$ and where $A^2 - 4B\mathcal{K} \neq 0$. Then g is globally IP and is not a metric of constant sectional curvature. We refer to [3, 5] for further details.

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It is convenient to study a purely algebraic problem. An algebraic curvature tensor R is a 4 tensor which satisfies the usual curvature symmetries:

$$R_{ijkl} = -R_{jikl},\ R_{ijkl} = R_{klij},\ \text{and}\ R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

where R is expressed relative to an orthonormal basis for $\mathbb{R}^m$. If $R = R_g$ at a point P, then R is an algebraic curvature tensor. Conversely, given an algebraic curvature tensor R and a metric g defined at a single point P of M, there exists an extension of g to all of M with the given curvature tensor at the point in question. We say that algebraic curvature tensor R is IP if $R(\pi)$ has constant eigenvalues on $Gr_2^+(T_PM)$. We refer to [5] for the proof of the first assertion and to [3] for the proof of the second two assertions in the following classification result. We note the situation if $m = 3$ is quite different; see [5] for details.

0.2 Theorem.

(1) *Let g be a globally IP metric on a 4 dimensional manifold M. Then the metric g on an open dense subset of M is locally isometric to one of the metrics given in* Example 0.1.
(2) *Let R be an algebraic IP curvature tensor. If $m = 5$, if $m = 6$, or if $m \geq 9$, then* $\operatorname{rank}(R) \leq 2$.
(3) *Let g be a globally IP metric on a manifold M of dimension at least* 5. *Assume that* $\operatorname{rank}(R_g) \leq 2$ *at each point of M. Then the metric g is locally isometric to one of the metrics given in* Example 0.1.

We note that assertion (2) fails if $m = 4$; there exists an algebraic IP curvature tensor R of rank 4 if $m = 4$ see [5, 4] for details; the non-zero entries of R up to the obvious symmetries be given by:

$$\begin{aligned}
&R_{1221} = 1,\ R_{1243} = -2,\ R_{3421} = -2,\ R_{3443} = 1,\\
&R_{1331} = -2,\ R_{1342} = -1,\ R_{2431} = -1,\ R_{2442} = -2,\\
&R_{1441} = -2,\ R_{1432} = 1,\ R_{2341} = 1,\ R_{2332} = -2.
\end{aligned}$$

The discussion in [5] uses a Singer-Thorpe basis and applies the methods of classical differential geometry. One classifies the IP algebraic curvature tensors and then uses the second Bianchi identities to derive the required integrality results necessary to complete the proof. The discussion in [3] uses algebraic topology; the proof of (2) is quite topological in flavor. The authors then classify the algebraic IP curvature tensors of rank 2 and proceed to the proof of (3).

The cases $m = 7$ and $m = 8$ are left unsettled by the results of [3, 5]. In this paper we extend Theorem 0.2 (2) to the case $m = 8$; it then follows that any globally IP metric in dimension m is locally isometric to one of the metrics given in Example 0.1. We also obtain some partial results concerning the possible eigenvalue structures of algebraic IP curvature tensors if $m = 7$. The exceptional geometry of dimensions 7 and 8 play a crucial role in our arguments; the octonians are always lurking in the background.

We introduce the following notational conventions. Let $\mathfrak{so}(m)$ be the space of skew-symmetric $m \times m$ real matrices; this is the Lie algebra of the special orthogonal group $SO(m)$. A continuous map R from $Gr_2^+(m)$ to $\mathfrak{so}(m)$ is said to be *admissible* if $R(-\pi) = -R(\pi)$ and if R has constant rank. We showed [3] that if R is admissible and non-trivial, then $\operatorname{rank}(R) = 2$ if $m \neq 4$, $m \neq 7$, and $m \neq 8$. As noted above, this fails if $m = 4$. Let $R : Gr_2^+(m) \to \mathfrak{so}(m)$ be admissible. Let

$$W(\lambda_i) := \{(\pi, \xi) \in Gr_2^+(m) \times \mathbb{R}^m : R(\pi)^2\xi = \lambda_i\xi\}; \tag{0.3}$$

since the eigenvalues of $R(\pi)$ are constant and $R(\pi)^2$ is self-adjoint, the $W(\lambda_i)$ are vector bundles over $Gr_2^+(m)$ and we may decompose the trivial bundle $m \cdot 1$ in the form $m \cdot 1 = \oplus_i W(\lambda_i)$. We assume that $\lambda_0 = 0$; this eigenspace may be trivial. We order the remaining eigenvalues so that $0 < \dim(W_1) \leq \dim(W_2)$.... Let $\vec{n} := (n_i)$ where $n_i := \dim(W_i)$. The main result of this paper is the following:

0.4 Theorem.

(1) *Let $R : Gr_2^+(m) \to \mathfrak{so}(m)$ be admissible with* $\operatorname{rank}(R) > 2$.
  1-a) *If $m = 7$, then $\vec{n} = (1, 6)$, $\vec{n} = (1, 2, 4)$, or $\vec{n} = (3, 4)$.*
  1-b) *If $m = 8$, then $\vec{n} = (0, 8)$, $\vec{n} = (0, 2, 6)$, or $\vec{n} = (2, 6)$.*
  1-c) *There exist admissible maps with these eigenvalue structures.*

(2) *If M is an algebraic IP curvature tensor and if $m = 8$, then* $\operatorname{rank}(R) \leq 2$.

(3) *If R is an algebraic IP curvature tensor and if $m = 7$, then R does not have the eigenvalue structure* $(1, 6)$.

In §1, we will use methods of algebraic topology to prove Theorem 0.4 (1). We will also use the exceptional geometries of $\mathbb{R}^7$ and $\mathbb{R}^8$ to construct maps R which have these eigenvalue structures; these maps will not satisfy the other curvature symmetries of course. In §2, we eliminate the eigenvalue structure $(0, 8)$. In §3, we will establish some technical results we shall need to deal with the remaining possibilities. In §4, we eliminate the eigenvalue structures $(2, 6)$ and $(0, 2, 6)$, and in §5, we eliminate the eigenvalue structure $(1, 6)$.

The maps constructed in Theorem 0.4 (1) are multilinear. There are similar investigations in the literature dealing with an analogous problem. Let $L(m, n, k)$ be the dimension of the largest linear subspace V in $\operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ so that the rank of $0 \neq v \in V$ is k. Various authors have tried to bound L for given k where k is relatively large. Adams [1] determines $L(m, m, m)$. Lam and Yiu [6] determine $L(m, m, m-1)$, $L(m, m-1, m-2)$, and $L(m, m, m-2)$. There is an extensive literature on the subject; see the bibliographies in Lam and Yiu [6] and in Meshulam [7] for further references.

1. The proof of Theorem 0.4 (1)

We begin by recalling some facts concerning real projective space $\mathbb{RP}^k$. If U is a real vector bundle of dimension r over $\mathbb{RP}^k$, let $[U]$ denote the corresponding element of the K theory group $KO(\mathbb{RP}^k)$ and let $w(U) = 1 + w_1(U) + ... + w_r(U)$ be the total Stiefel-Whitney class in $\mathbb{Z}_2$ cohomology $H^*(\mathbb{RP}^k; \mathbb{Z}_2)$. Recall that $w(U) = 1$ if U is trivial, that $w(U \oplus V) = w(U)w(V)$, and that $w(L) = 1 + x$ where L is the classifying real line bundle over $\mathbb{RP}^k$ and where x generates $H^1(\mathbb{RP}^k; \mathbb{Z}_2)$.

1.1 Lemma. *Let $k = 5$ or $k = 6$.*

(1) *We have $H^*(\mathbb{RP}^k;\mathbb{Z}_2) = \mathbb{Z}_2[x]/x^{k+1} = 0$ is a truncated polynomial ring.*
(2) *We have $KO(\mathbb{RP}^k) = [1]\cdot\mathbb{Z}\oplus([L]-[1])\cdot\mathbb{Z}_8$.*
(3) *Expand $[U] = \dim(U)[1] + a(U)([L]-[1])$. Then $a(U)$ is well defined mod 8 and $w(U) = (1+x)^{a(U)}$.*
(4) *If U is isomorphic to $U\otimes L$, then $2a(U)\equiv\dim(U)$ mod 8.*

Proof. The first assertion is well known; the second follows from the work of Adams [1], and the third follows from the second. If U is isomorphic to $U\otimes L$, then

$$\begin{aligned}[U] &= \dim(U)[1] + a(L)([L]-[1])\\ &= [U\otimes L] = \dim(U)[L] + a(L)([1]-[L])\\ &= \dim(U)[1] + (\dim(U) - a(L))([L]-[1])\end{aligned}$$

so by assertion (2), $a(U)\equiv\dim(U) - a(U)$ mod 8; the final assertion follows. $\square$

Suppose that $R : Gr_2^+(m)\to\mathfrak{so}(m)$ is admissible. The eigenbundles of R^2 define a decomposition of the trivial bundle over $Gr_2^+(m)$ of the form $m\cdot 1 = \oplus_i W_i$ where the W_i are as defined in equation (0.3). We have $R(\pi)$ preserves the W_i and $W_0 = \ker(R)$. Since $R^2(-\pi) = R^2(\pi)$, these bundles descend to bundles V_i over the unoriented Grassmanian $Gr_2(m)$. Since $\pi_1(Gr_2(m)) = \mathbb{Z}_2$, there is only one non-trivial real line bundle over $Gr_2(m)$; denote this bundle by L_1. Since $R(-\pi) = -R(\pi)$, R descends to define isomorphisms between V_i and $V_i\otimes L_1$ for $i > 0$. Let $\{e_i\}$ be the standard basis for $\mathbb{R}^m$. The map $\xi\mapsto\operatorname{Span}\{\xi, e_m\}$ defines a natural embedding of S^{m-2} in $Gr_2^+(m)$ and $\mathbb{RP}^{m-2}$ in $Gr_2(m)$. Let U_i be the restriction of the bundles V_i to $\mathbb{RP}^{m-1}$. Note that the restriction of L_1 to $\mathbb{RP}^{m-1}$ is L. We summarize our discussion so far. There are bundles U_i over $\mathbb{RP}^{m-1}$ so

(1.2) $\quad m\cdot 1 = \oplus_i U_i,\ \vec{n} = (\dim U_0, \dim U_1, \ldots)$, and $U_i = U_i\otimes L$ for $i > 0$.

To prove Theorem 0.4 1-a), we must show that the eigenvalue decompositions $\vec{n} = (3,2,2)$ and $\vec{n} = (1,2,2,2)$ are impossible in equation (1.2) if $m = 7$. If one of these cases occurs, we could combine eigenspaces if necessary to construct a decomposition $7\cdot 1 = U_0\oplus U_1\oplus U_2$ over $\mathbb{RP}^5$ with $U_1 = U_1\otimes L$, $U_2 = U_2\otimes L$, and $\dim(U_1) = \dim(U_2) = 2$. We use Lemma 1.1 to see $2a(U_1)\equiv 2$ mod 8. If $a(U_1)\equiv 5$ mod 8, then $w(U_1) = (1+x)^5$ in $\mathbb{Z}_2[x]/x^6$ which is impossible for dimensional reasons. Thus $a(U_1)\equiv 1$ mod 8; similarly $a(U_2)\equiv 1$ mod 8. Thus $w(U_1) = w(U_2) = (1+x)$ so $1 = w(U_0)w(U_1)w(U_2) = w(U_0)(1+x)^2$. This implies $w(U_0) = (1+x)^6$; this is impossible for dimensional reasons.

To prove Theorem 0.4 1-b), we must show that the eigenvalue decompositions $\vec{n} = (2,2,4)$, $\vec{n} = (0,2,2,4)$, $\vec{n} = (4,4)$, $\vec{n} = (0,4,4)$, $\vec{n} = (2,2,2,2)$, and $\vec{n} = (0,2,2,2,2)$ are impossible in equation (1.2) if $m = 8$. In these 6 cases, we could combine eigenspaces to construct a decomposition of the form $8\cdot 1 = U_0\oplus U_1$ over $\mathbb{RP}^6$ where $\dim(U_1) = 4$ and where $U_1 = U_1\otimes L$. We use Lemma 1.1 to see $2a(U_1)\equiv 4$ mod 8. If $a(U_1)\equiv 2$ mod 8, then $w(U_1) = (1+x)^2$ so $w(U_0) = (1+x)^6$ which is impossible for dimensional reasons. If $a(U_1)\equiv 6$

mod 8, then $w(U_1) = (1+x)^6$ which is impossible for dimensional reasons. This completes the proof of Theorem 0.4 i-b).

To prove Theorem 0.4 i-c), we construct admissible maps R with the given eigenvalue decompositions. We refer to Atiyah, Bott, and Shapiro [2] for details concerning Clifford algebras and spinors. Recall that the Clifford algebra $\mathrm{Clif}(m)$ is the universal real unital algebra generated by $\mathbb{R}^m$ subject to the commutation relations $v*w+w*v = -2(v,w)1$. Let $\mathrm{Spin}(m)$ be the subset of $\mathrm{Clif}(m)$ generated by all products of the form $\omega = v_1 * ... * v_k$ where the v_i are unit vectors in $\mathbb{R}^m$ and k is even. This is a smooth submanifold of $\mathrm{Clif}(m)$ which has the structure of a Lie group under Clifford multiplication. If $\{v_1, v_2\}$ is an oriented orthonormal basis for a 2 plane π, let $\sigma(\pi) := v_1 * v_2 \in \mathrm{Spin}(m)$; this is independent of the particular oriented orthonormal basis for π and defines a smooth embedding $\sigma : Gr_2^+(m) \mapsto \mathrm{Spin}(m)$. There is a natural representation $\rho : \mathrm{Spin}(m) \mapsto \mathrm{SO}(m)$ that exhibits $\mathrm{Spin}(m)$ as the universal cover of $\mathrm{SO}(m)$ for $m \geq 3$; $\rho(\sigma(\pi))$ is -1 on π and $+1$ on $\pi^\perp$.

Suppose $m = 8$. The spin representation c gives a representation of $\mathrm{Clif}(8)$ on $\mathbb{R}^{16}$. There is a decomposition $\mathbb{R}^{16} = \mathbb{R}^8_+ \oplus \mathbb{R}^8_-$ so that Clifford multiplication by a vector $v \in \mathbb{R}^8$ interchanges the two summands, i.e. $c(v) : \mathbb{R}^8_\pm \to \mathbb{R}^8_\mp$. Clifford multiplication by an element $\omega \in \mathrm{Spin}(m)$ preserves the two summands; thus c restricts to define representations, called the half spin representations, $c_\pm$ of $\mathrm{Spin}(8)$ on $\mathbb{R}^8_\pm$. Let $R(\pi) = c_+(\sigma(\pi))$; R is an admissible map from $Gr_2^+(8)$ to $\mathfrak{so}(8)$ which has constant rank 8 since $R(\pi)^2 = -1$ and $R(-\pi) = -R(\pi)$. This shows the eigenvalue decomposition $(0,8)$ is possible. Fix $0 \neq x_- \in \mathbb{R}^8_-$. If $\{v_1, v_2\}$ is an orthonormal basis for an oriented 2 plane π, let

$$E_0(\pi) := \mathrm{Span}\{c(v_1)x_-, c(v_2)x_-\};$$

$E_0(\pi)$ and $E_0\pi)^\perp$ are invariant subspaces of $R(\pi)$. Decompose $R = R_0 \oplus R_2$ where R_0 is the restriction of R to E_0 and R_2 is the restriction of R to $E_0^\perp$; $-R_0^2$ is orthogonal projection on E_0 and $-R_2^2$ is orthogonal projection on $E^\perp$. The map $C_0 R_1 \oplus R_2$ is an admissible map of $Gr_2^+(8)$ to $\mathfrak{so}(8)$ with $\vec{n} = (2,6)$ if $C_1 = 0$ and $\vec{n} = (0,2,6)$ if $C_1 \neq 0$.

Suppose $m = 7$. Let $\{e_i\}$ be the standard orthonormal basis for $\mathbb{R}^8$. The inclusion $\mathbb{R}^7 \subset \mathbb{R}^8$ embeds $Gr_2^+(7)$ in $Gr_2^+(8)$. Let $E_0(\pi)$ be as above and let

$$E_1(\pi) := \mathrm{Span}\{c(\pi)c(e_8)x_-, c(e_8)x_-\} \subset \mathbb{R}^8_+.$$

If $\pi \subset Gr_2^+(7)$, then $E_0(\pi)$ and $E_1(\pi)$ are orthogonal subspaces invariant under $R(\pi)$. Decompose $R = R_0 \oplus R_1 \oplus R_2$ where R_0 and R_1 are the restriction of R to E_0 and to E_1 and where R_2 is the restriction of R to $(E_0 \oplus E_1)^\perp$. The map $c_1 R_1 + R_2$ preserves $c(e_8)x_-^\perp$ which we may identify with $\mathbb{R}^7$; this gives an admissible map of $Gr_2^+(7)$ to $\mathfrak{so}(7)$ with $\vec{n} = (1,6)$ if $C_1 = 1$, $\vec{n} = (1,2,4)$ if $C_1 = 2$, and $\vec{n} = (3,4)$ if $C_1 = 0$. This completes the proof of Theorem 0.4 i-c).

Remark. In joint work with Semmelman [4], we have given an alternate description of the construction in dimension 7 using the exceptional Lie group G_2 and have shown that it does not lead to an algebraic IP curvature tensor of higher rank in dimension 7. It is still, however, an open question of whether there exist algebraic IP curvature tensors of higher rank in dimension 7 which do not arise from the spin representation.

2. The eigenvalue structure $(0, 8)$

We wish to prove that there does not exist an algebraic IP curvature tensor on $\mathbb{R}^8$ with an eigenvalue structure $(0, 8)$. Suppose the contrary. By rescaling R, we may suppose $R(\xi, \eta)^2 = -1$ for any orthonormal set $\{\xi, \eta\}$. Fix a unit vector $e_1 \in \mathbb{R}^8$. Let $\phi(\xi) := R(e_1, \xi)e_1$. The identity $(R(e_1, \xi)e_1, \eta) = (R(e_1, \eta)e_1, \xi)$ shows ϕ is symmetric. If $\xi \perp e_1$, then $|\phi(\xi)| = |\xi|$ so ϕ is an isometry of $e_1^\perp$. We diagonalize ϕ to find an orthonormal basis e_i for $\mathbb{R}^8$ so that $R(e_1, e_i)e_1 = \epsilon_i e_i$ for $i \geq 2$ for suitably chosen signs $\epsilon_i = \pm 1$. Let $c(\xi) = R(e_1, \xi)$ for $\xi \perp e_1$. Then $c(\xi)^2 = -|\xi|^2$. Thus if $2 < i \leq j$, $c(e_i)c(e_j) + c(e_j)c(e_i) = -2\delta_{ij}$ so these endomorphisms satisfy the Clifford commutation relations discussed in §1. We use the Bianchi identity to establish the desired contradiction by computing

$$\begin{aligned}
1 =& |R(e_2, e_3)e_1| = |R(e_3, e_1)e_2 + R(e_1, e_2)e_3| \\
=& |\epsilon_2 R(e_3, e_1)R(e_1, e_2)e_1 + \epsilon_3 R(e_1, e_2)R(e_1, e_3)e_1| \\
=& |\{-\epsilon_2 c(e_3)c(e_2) + \epsilon_1 c(e_2)c(e_3)\}e_1| \\
=& |(\epsilon_1 + \epsilon_2)c(e_2)c(e_3)e_1| = |\pm 1 \pm 1| = 0 \text{ or } 2. \qquad \square
\end{aligned}$$

3. Technical preliminaries

In this section, we establish some technical results we will use in §4 and §5 to complete the proof of Theorem 0.4. For the remainder of this paper, we assume that R is an algebraic IP curvature tensor with two distinct eigenvalues λ_1 and λ_2 on $\mathbb{R}^m$ for $m = 7$ or $m = 8$. Let W_u be the corresponding eigen bundles. We suppose $\dim W_2 = 6$ and normalize the choice of R so that $\lambda_1 = -1$. If $\lambda_1 = 0$, then $W_1 = \dim \ker R$; this is a slight change of notation from previous sections.

It is convenient to distinguish two copies of $\mathbb{R}^m$. We shall for the most part use Greek letters ξ, η, etc. for vectors which are arguments of the curvature operator $R(\cdot, \cdot)$. We shall use Roman letters e, f, etc. for vectors upon which the curvature operator R acts. If $\{\xi, \eta\}$ is an orthonormal set, we let $W_u(\xi, \eta)$ be the eigenspace for the eigenvalue λ_u defined by $R(\xi, \eta)$. If $\{\xi_i\}$ is an orthonormal set, we let $r_{ij} := R(\xi_i, \xi_j)$ and $W_{u,ij} := W_u(\xi_i, \xi_j)$.

3.1 Lemma.

(1) *Let e be a unit vector. We have $(r_{12}^2 e, e) \in [\min(\lambda_1, \lambda_2), \max(\lambda_1, \lambda_2)]$. We have $e \in W_{u,12}$ if and only if $(r_{12}^2 e, e) = \lambda_u$. If $\lambda_1 = 0$, then $r_{12}^3 = -r_{12}$.*

(2) *Let $\mathcal{E} := r_{12}r_{13} + r_{13}r_{12}$. If $e \in W_{u,12}$, then $(\mathcal{E}e, e) = 0$ and $r_{12}e \perp r_{13}e$.*

(3) *If $e \in W_{u,12} \cap W_{u,13}$, then $\mathcal{E}e = 0$.*

Proof. Since r_{12}^2 is symmetric and has exactly two eigenvalues, the first assertion follows. Let $r(\theta) := R(\xi_1, \cos\theta\xi_2 + \sin\theta\xi_3)$ and let $e \in W_{u,12}$. Expand $(r(\theta)^2 e, e)$

in a Taylor series about $\theta = 0$:

$$\begin{aligned}(r(\theta)^2 e, e) =& \lambda_u \cos^2\theta + 2\cos\theta\sin\theta(\mathcal{E}e, e) + \sin^2\theta(r_{13}^2 e, e)\\ =& \lambda_u + 2\theta(\mathcal{E}e, e) + O(\theta^2).\end{aligned}$$

By assertion (1), λ_u is an extremal value. This implies that $(\mathcal{E}e, e) = 0$ and hence $r_{12}e \perp r_{13}e$ which proves assertion (2). If $e \in W_{u,12} \cap W_{u,13}$, then $r(\theta)^2 e = \lambda_u e + \mathcal{E}e$. Since $(\mathcal{E}e, e) = 0$, $(r(\theta)^2 e, e) = \lambda_u$ so $r(\theta)^2 e = \lambda_u e$ for all θ and hence $\mathcal{E}e = 0$. □

3.2 Lemma. *Let ξ_1, η_2, η_3 be unit vectors with $\xi_1 \perp \eta_2$ and $\xi_1 \perp \eta_3$. The following assertions are equivalent: (1) $\eta_2 = \pm\eta_3$, (2) $W_1(\xi_1, \eta_2) = W_1(\xi_1, \eta_3)$, and (3) $W_2(\xi_1, \eta_2) = W_2(\xi_1, \eta_3)$.*

Proof. Clearly assertion (1) implies assertion (2). Since $W_2^\perp = W_1$, assertion (2) is equivalent to assertion (3). Suppose that assertion (3) holds but that assertion (1) fails; we argue for a contradiction. Let $\xi_2 = \eta_2$ and choose ξ_3 so that $\{\xi_i\}$ is an orthonormal set and so that $\eta_3 = \cos\theta\xi_2 + \sin\theta\xi_3$ where $\sin\theta \neq 0$. Let $e \in W_2(\xi_1, \xi_2)$. By Lemma 3.1, $(\mathcal{E}(\xi_1, \xi_3)e, e) = 0$. Since $W_2(\xi_1, \eta_2) = W_2(\xi_1, \eta_3)$,

$$-1 = (R(\xi_1, \eta_3)^2 e, e) = -\cos\theta^2 + \sin^2\theta(r_{13}^2 e, e).$$

Since $\sin\theta \neq 0$, $(r_{13}^2 e, e) = -1$. By Lemma 3.1, $e \in W_{2,13}$. Thus $W_{2,12} \subset W_{2,13}$. Since $\dim W_{2,12} = \dim W_{2.13}$, $W_{2,12} = W_{2,13}$. Let $\tau(i) := r_{12}$ and $\tau(j) := r_{13}$. Then $\tau(i)^2 = \tau(j)^2 = -1$ on $W_2(\xi_1, \eta_1)$; Lemma 3.1 implies $\tau(i)\tau(j) + \tau(j)\tau(i) = 0$ on $W_2(\xi_1, \eta_1)$. Thus τ defines a unital representation of the quaternions $\mathbb{H}$ on $W_2(\xi_1, \eta_1)$. This implies $\dim W_2(\xi_1, \eta_1) \equiv 0 \bmod 4$ which is false. □

3.3 Lemma. *There exists an orthonormal basis $\{e_i\}$ for $\mathbb{R}^m$ so that:*

1. *We have $r_{12}e_1 = e_2$, $r_{12}e_2 = -e_1$, $r_{12}e_3 = -e_4$, $r_{12}e_4 = e_3$, $r_{13}e_1 = e_3$, $r_{13}e_2 = e_4$, $r_{13}e_3 = -e_1$, $r_{13}e_4 = -e_2$, and $r_{12}r_{13} = -r_{13}r_{12}$ on the space $S := \mathrm{Span}\{e_1, e_2, e_3, e_4\} \subset W_{2,12} \cap W_{2,13}$.*
2. *We have $e_6 = r_{12}e_5$, $e_7 = r_{13}e_5$, $W_{2,12} = \mathrm{Span}\{e_1, e_2, e_3, e_4, e_5, e_6\}$, and $W_{2,13} = \mathrm{Span}\{e_1, e_2, e_3, e_4, e_5, e_7\}$.*
3. *If $m = 7$, then $W_{1,12} = \mathrm{Span}\{e_7\}$ and $W_{1,13} = \mathrm{Span}\{e_6\}$.*
4. *If $m = 8$, then $W_{1,12} = \mathrm{Span}\{e_7, e_8\}$, $W_{1,13} = \mathrm{Span}\{e_6, e_8\}$. Furthermore, $r_{12}r_{13}e_1 = \pm|\lambda_1|^{1/2}e_8$.*

Proof. Let $V := W_{2,12} \cap W_{2,13}$. Since $\dim W_{2,12} = \dim W_{2,13} = 6$ and since $m \leq 8$, we have the inequality $\dim V \geq 6 + 6 - 8 = 4$. Since r_{12} preserves $W_{2,12}$ and since r_{12} is non-singular on $W_{2,12}$, we have $r_{12}V$ is a 4 dimensional subspace of $W_{2,12}$. Thus we have the inequality $\dim(V \cap r_{12}V) \geq 4 + 4 - 6 = 2$. Choose a unit vector e_1 in $V \cap r_{12}V$. Let $e_2 := r_{12}e_1$, let $e_3 := r_{13}e_1$, and let $e_4 := r_{12}r_{13}e_1$. We use Lemma 3.1 to prove the first assertion:

1. Since $e_1 \in V$ and $e_2 \in V$, $(r_{12}r_{13} + r_{13}r_{12})e_1 = 0$, $(r_{12}r_{13} + r_{13}r_{12})e_2 = 0$, $r_{12}^2 e_1 = -e_1$, $r_{13}^2 e_1 = -e_1$, $r_{12}^2 e_2 = -e_2$, and $r_{13}^2 e_2 = -e_2$.

(2) Since $r_{12}^2e_3 = r_{12}^2r_{13}e_1 = -r_{12}r_{13}r_{12}e_1 = -r_{12}r_{13}e_2 = r_{13}r_{12}e_2 = -e_3$, and $r_{13}^2e_3 = r_{13}^3e_1 = -r_{13}e_1 = -e_3$, we have $e_3 \in V$ so $(r_{12}r_{13} + r_{13}r_{12})e_3 = 0$.
(3) Since $r_{12}^2e_4 = r_{12}^2r_{12}e_3 = -r_{12}e_3 = -e_4$ and similarly $r_{13}^2e_4 = -e_4$, $e_4 \in V$ and $(r_{12}r_{13} + r_{13}r_{12})e_4 = 0$.

Since r_{12} and r_{13} are skew-symmetric and preserve S, the orthogonal complement T is preserved by r_{12} and r_{13} as well. Let $\tilde W_{2,12} := W_{2,12} \cap T$ and $\tilde W_{2,13} := W_{2,13} \cap T$. Since $S \subset W_{2,12}$ and $S \subset W_{2,13}$, we can decompose

$$W_{2,12} = S \oplus \tilde W_{2,12} \text{ and } W_{2,13} = S \oplus \tilde W_{2,13}.$$

We have $\dim W_{2,12} = \dim W_{2,13} = 2$. Suppose there exists a unit vector e_5 in $\tilde W_{2,12} \cap \tilde W_{2,13}$. Let $e_6 := r_{12}e_5$ and $e_7 := r_{13}e_5$. Then $\tilde W_{2,12} = \operatorname{Span}\{e_5, e_6\}$ and $\tilde W_{2,13} = \operatorname{Span}\{e_5, e_7\}$. By Lemma 3.1, $(r_{12}r_{13} + r_{13}r_{12})e_5 = 0$; it now follows that $\{e_5, e_6, e_7\}$ is an orthonormal subset of T. Since $\{e_i\}_{i\le 6}$ is an orthonormal basis for $W_{2,12}$, it follows that $e_7 \in W_{1,12}$ so $e_7 \in W_{1,12} \cap W_{2,13}$; similarly $e_6 \in W_{2,12} \cap W_{1,13}$. The second assertion of the Lemma now follows if $m = 7$. If $m = 8$, we may choose e_8 to complete the orthonormal basis. We then have $e_8 \perp e_i$ for $i \le 6$ so $e_8 \in W_{1,12}$; similarly $e_8 \in W_{1,13}$ so $e_8 \in W_{1,12} \cap W_{1,13}$. Since $r_{12}r_{13}e_5 = r_{12}e_7$ and since $e_7 \in W_{1,12}$, we have $r_{12}e_7 \in W_{1,12}$. Since $r_{12}e_7 \perp e_7$, $r_{12}e_7 = \epsilon e_8$. Since $r_{12}^2e_7 = \lambda_1 e_7$, we have $\epsilon^2 = |\lambda_1|$ and the final assertion of the Lemma follows. Thus to prove the Lemma, we must show

$$U := \tilde W_{2,12} \cap \tilde W_{2,13} \neq \{0\}. \tag{3.4}$$

If $m = 7$, we have $\dim\{\tilde W_{2,12} \cap \tilde W_{2,13}\} \ge 2 + 2 - 3 \ge 1$ and equation (3.4) follows. We may therefore assume that $m = 8$.

Suppose first that $\lambda_1 \neq 0$. If f is a unit vector in $W_{1,12}$, then $\{f, r_{12}f\}$ spans $W_{1,12}$. Since $r_{13}f \perp f$ and $r_{13}f \perp r_{12}f$, we have $r_{13}f \in W_{2,12}$. Thus $r_{13}W_{1,12} \subset \tilde W_{2,12}$ and $r_{13}\tilde W_{2,12} \subset W_{1,12}$ so r_{13}^2 preserves $\tilde W_{2,12}$. Let μ_i be the two eigenvalues of r_{13}^2 on $\tilde W_{2,12}$; each μ_i is either λ_1 or λ_2. If $\mu_i = \lambda_2$ for $i = 1$ or $i = 2$, then $\tilde W_{2,12} \cap \tilde W_{2,13} \neq \{0\}$ as desired. We therefore suppose $\mu_1 = \mu_2 = \lambda_1$ or equivalently that $\tilde W_{2,12} = W_{1,13}$. We have r_{12} and $r_{13}/\sqrt{-\lambda_1}$ define rotations of $\pm\pi/2$ on $\tilde W_{2,12}$. Hence $r_{12} = \pm r_{13}/\sqrt{-\lambda_1}$ on this 2 dimensional subspace. Choose θ so that $\tan\theta = \mp\sqrt{-\lambda_1}$. Then $R(\xi_1, \cos\theta\xi_2 + \sin\theta\xi_3) = \cos\theta r_{12} + \sin\theta r_{13} = 0$ on $\tilde W_{2,12}$. This is false as $R(\xi, \eta)$ is invertible for any orthonormal set.

Suppose $\lambda_1 = 0$ so $r_{ij}^3 = -r_{ij}$. If $e \in \tilde W_{2,12}$, then $\tilde W_{2,12} = \operatorname{Span}\{e, r_{12}e\}$. Thus $r_{13}\tilde W_{2,12} \subset W_{1,12}$ and $\ker(r_{13}) = W_{1,13}$. Suppose $\tilde W_{2,12} \cap W_{1,13} = \{0\}$. Then $r_{13}e$ and $r_{13}r_{12}e$ are linearly independent so $W_{1,12} = \operatorname{Span}\{r_{13}e, r_{13}r_{12}e\}$. We compute

$$(r_{13}^2e, r_{13}e) = -(r_{13}^3e, e) = -(r_{13}e, e) = 0, \text{ and}$$
$$(r_{13}^2e, r_{13}r_{12}e) = -(r_{13}^3e, r_{12}e) = (r_{13}e, r_{12}e) = 0.$$

This shows that $r_{13}^2e \in \tilde W_{2,12}$ so r_{13}^2 preserves $\tilde W_{2,12}$. Since $r_{13}e \neq 0$, $r_{13}^2e \neq 0$ and thus $\tilde W_{2,12} = \tilde W_{2,13}$ and $W_{2,12} = W_{2,13}$ which contradicts Lemma 3.2.

Thus we may suppose $\lambda_1 = 0$ and $\tilde{W}_{2,12} \cap W_{1,13} \neq \{0\}$; let e_6 be a unit vector in this intersection. Let $e_5 = -r_{12}e_6 \in \tilde{W}_{2,12}$; $r_{12}e_5 = e_6$ so $r_{13}r_{12}e_5 = r_{13}e_6 = 0$. Since $r_{13}e_5 \perp e_5$ and $r_{13}e_5 \perp e_{12}e_5 = e_6$, we have $r_{13}e_5 \in W_{1,12}$ so $r_{12}r_{13}e_5 = 0$ as well. Let $r(\theta) = R(\xi_1, \cos\theta\xi_2 + \sin\theta\xi_3)$; $r(\theta)^3 = -r(\theta)$. We compute:

$$\begin{aligned} r(\theta)^3 e_5 = &\{\cos^3\theta r_{12}^3 + \cos^2\theta\sin\theta(r_{12}^2 r_{13} + r_{12}r_{13}r_{12} + r_{13}r_{12}^2) \\ &+ \cos\theta\sin^2\theta(r_{12}r_{13}^2 + r_{13}r_{12}r_{13} + r_{13}^2 r_{12}) + \sin^3\theta r_{13}^3\}e_5 \\ = &\{-\cos^3\theta r_{12} - \cos^2\theta\sin\theta r_{13} + \cos\theta\sin^2\theta r_{12}r_{13}^2 - \sin^3\theta r_{13}\}e_5 \\ = &\{-r(\theta) + \cos\theta\sin^2\theta(r_{12}r_{13}^2 + r_{12})\}e_5 = -r(\theta)e_5. \end{aligned}$$

This shows that $r_{12}r_{13}^2 e_5 = r_{12}e_5$. Since $|r_{12}e_5| = 1$ and since r_{12} is a linear operator of norm 1, we conclude $|r_{13}^2 e_5| \geq 1$. Since 1 is an extremal value, we have $e_5 \in W_{2,13}$ and hence $W_{2,13} \cap W_{2,13} \neq \{0\}$. □

3.5 Lemma. *If $|\xi| = 1$, let the axis $\mathfrak{A}_u(\xi) := \cap_{\eta\perp\xi,|\eta|=1} W_u(R(\xi,\eta))$.*

(1) *If ξ is any unit vector and if $\xi \in \mathfrak{A}_u(\xi)$, then $\lambda_u = 0$.*
(2) *We have $\cap_\xi \mathfrak{A}_u(\xi) = \{0\}$.*

Proof. Suppose assertion (1) is false. Thus $\lambda_u \neq 0$ and we may choose a unit vector $\xi_1 \in \mathfrak{A}_1(\xi_1)$. Then if $\{\xi_1, \eta\}$ is an orthonormal set, we have $\xi_1 \in W_u(R(\xi_1,\eta))$ so $|R(\xi_1,\eta)\xi_1|^2 = -\lambda_u$. Thus the map $\phi : \eta \mapsto R(\xi_1,\eta)\xi_1/\sqrt{-\lambda_u}$ is an isometry from $\xi_1^\perp$ to itself. The curvature symmetry $R(\xi_1,\eta_1,\xi_1,\eta_2) = R(\xi_1,\eta_2,\xi_1,\eta_1)$ implies that $(\phi(\eta_1),\eta_2) = (\phi(\eta_2),\eta_1)$. Thus ϕ is a symmetric isometry. We can therefore choose a basis ξ_i for $\xi_1^\perp$ so that $\phi\xi_i = \epsilon_i\xi_i$ where $\epsilon_i = \pm 1$, i.e. $r_{1i}\xi_1 = \epsilon_i\sqrt{-\lambda_u}\xi_i$ for $i \geq 2$. By Lemma 3.1, $(r_{1i}r_{1j} + r_{1j}r_{1i})\xi_1 = 0$ for $i \neq j$. Let $\eta_{ij} := r_{1i}r_{1j}e_1$. Let $k \geq 2$ be distinct from i and j. We use the Bianchi identity to compute:

$$\begin{aligned} 0 =&(\sqrt{-\lambda_u})^2\{(r_{1i}\xi_j,\xi_k) + (r_{1j}\xi_k,\xi_i) + (r_{1k}\xi_i,\xi_j)\} \\ =&\epsilon_j\epsilon_k(r_{1i}r_{1i}\xi_1, r_{1k}\xi_1) + \epsilon_k\epsilon_i(r_{1j}r_{1k}\xi_1, r_{1i}\xi_1) + \epsilon_i\epsilon_j(r_{1k}r_{1i}\xi_1, r_{1k}\xi_1) \\ =&(\epsilon_j\epsilon_k + \epsilon_k\epsilon_i + \epsilon_i\epsilon_j)(r_{1i}r_{1j}\xi_1, r_{1k}\xi_1). \end{aligned}$$

Since $(\pm 1 \pm 1 \pm 1) \neq 0$, $\eta_{ij} \perp r_{1k}\xi_1$ so $\eta_{ij} \perp \xi_k$. We have $\eta_{ij} \perp r_{1i}e_1$ and $\eta_{ij} \perp r_{1j}e_1$ so $\eta_{ij} \perp \xi_i$ and $\eta_{ij} \perp \xi_j$. Since $e_1 \in W_{u,1i} \cap W_{u,1j}$, $\eta_{ij} \perp e_1$ by Lemma 3.1. Thus $\eta_{ij} \perp \xi_k$ for all k so $\eta_{ij} = 0$ for all i,j. Since $\lambda_u \neq 0$ and $e_1 \in W_{u,1i}$, $f_i := r_{1i}e_1 \neq 0$ for $2 \leq i$. Furthermore $f_i \perp f_j$ for $i \neq j$. Since $\{f_3, f_4, f_5\} \subset \ker r_{12}$, we see $\dim W_{1,12} \geq 3$ which is false.

Suppose assertion (2) fails. Choose a unit vector ξ_1 in $\cap_\xi\mathfrak{A}_u(\xi)$. Then in particular, $\xi_1 \in \mathfrak{A}_1(\xi_1)$ and hence $\lambda_u = 0$ by assertion (1). This implies $u = 1$. Let $\{\xi_i\}$ be an orthonormal basis for $\mathbb{R}^m$. Since $\xi_1 \in W_1(\xi_i,\xi_j)$, we have $R(\xi_i,\xi_j)\xi_1 = 0$ for $i \neq j$; trivially $R(\xi_i,\xi_i)\xi_1 = 0$. Then: $(R(\xi_1,\xi_2)\xi_j,\xi_k) = (R(\xi_j,\xi_k)\xi_1,\xi_2) = 0$ and thus $R(\xi_1,\xi_2)\xi_j = 0$ so $R(\xi_1,\xi_2)$ vanishes identically which is false. □

We can now show $\mathfrak{A}_2$ is non-trivial if $m = 7$ and that $\mathfrak{A}_1$ is non-trivial if $m = 8$.

3.6 Lemma.

(1) *Suppose that $m = 8$. Let $\{\xi_1,\xi_2,\xi_3,\xi_4\}$ be an orthonormal set. Then*
  1-a) *If $W_1(\xi_1,\xi_2) \cap W_1(\xi_1,\xi_3) \cap W_1(\xi_1,\xi_3) \neq \{0\}$, then $\dim\mathfrak{A}_1(\xi_1) = 1$.*

1-b) *If* ξ *is any unit vector, then* $\dim \mathfrak{A}_1(\xi) = 1$.
1-c) *The map* $\Phi : \xi \mapsto \mathfrak{A}_1(\xi)$ *is a continuous map from* S^7 *to* $\mathbb{RP}^7$.
1-d) *We have* $W_1(\xi_1, \xi_2) \cap W_1(\xi_1, \xi_3) \cap W_1(\xi_2, \xi_3) = \{0\}$.

(2) *Suppose that* $m = 7$.
2-a) *If* ξ *is a unit vector, then* $\dim \mathfrak{A}_2(\xi) = 1$.
2-b) *The map* $\Phi : \xi \mapsto \mathfrak{A}_2(\xi)$ *is a continuous map from* S^6 *to* $\mathbb{RP}^6$.
2-c) *There exists a unit vector* ξ *so* $\xi \in \mathfrak{A}_2(\xi)$.

Proof. Suppose there exists a unit vector f_1 in $W_{1,12} \cap W_{1,13} \cap W_{1,14}$. Let $\{f_1, f_i\}$ be an orthonormal basis for $W_{1,1i}$ for $i = 2, 3, 4$. We use Lemma 3.3 to see:

$$f_1 \in W_{1,12} \cap W_{1,13} \cap W_{1,14},\ f_2 \in W_{1,12} \cap W_{2,13} \cap W_{2,14},$$
$$f_3 \in W_{2,12} \cap W_{1,13} \cap W_{2,14},\ f_4 \in W_{2,12} \cap W_{2,13} \cap W_{1,14}.$$

Thus $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is an orthonormal set. Extend this set to an orthonormal basis $\{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8\}$ for $\mathbb{R}^8$. By Lemma 3.3, $\dim(W_{1,1i} \cap W_{1,1j}) = 1$ for $i \neq j$. Choose unit vectors $g_{ik} = a_{ik} f_1 + b_{ik} f_i \in W_{1,1i} \cap W_{1,1k}$ for $2 \le i \le 4 < k$. If all the b_{ik} are non-zero, then the g_{ik} are 3 linearly independent vectors in $W_{1,1k}$; this is not possible as $\dim W_{1,1k} = 2$. Thus some $b_{ik} = 0$ so $f_1 \in W_{1,1k}$. Thus $f_1 \in W_{1,1i}$ for $2 \le i$. Let $\eta = \Sigma_{i \ge 2} u_i \xi_i$ be a unit vector perpendicular to ξ_1. We complete the proof of assertion 1-a) by showing $f_1 \in W_2(\xi_1, \eta)$. By Lemma 3.1:

$$R(\xi_1, \eta)^2 f_1 = \Sigma_i u_i^2 r_{1i}^2 f_1 + \Sigma_{i<j} u_i u_j (r_{1i} r_{1j} + r_{1j} r_{1i}) f_1 = \lambda_1 f_1.$$

Suppose that assertion 1-b) fails. Since $\dim W_1(\xi, \eta) = 2$, $\dim \mathfrak{A}_1(\xi) \le 2$. If $\dim \mathfrak{A}_1(\xi) = 2$, then $\mathfrak{A}_1(\xi) = W_1(\xi, \eta)$ for any $\eta \perp \xi$; this contradicts Lemma 3.2. Thus assertion 1-b) fails implies $\mathfrak{A}_1(\xi) = \{0\}$ for some $\xi = \xi_1$. Let $\{\xi_1, \xi_2, \xi_3, \eta\}$ be an orthonormal set. By assertion 1-a), $W_{1,12} \cap W_{1,13} \cap W_1(\xi_1, \eta) = \{0\}$. For $i = 2, 3$, let $\{f_1, f_i\}$ be an orthonormal basis for $W_{1,1i}$; $\{f_1, f_2, f_3\}$ is an orthonormal set. Let $g_i = a_i f_1 + b_i f_i$ be a unit vector in the 1 dimensional space $W_{1,1i} \cap W_1(R(\xi_1, \eta))$. If $g_2 = \pm g_3$, then $g_i = \pm f_1$ and $W_{1,12} \cap W_{1,13} \cap W_1(R(\xi_1, \eta))$ is non-trivial which is false. Thus g_2 and g_3 are linearly independent so

$$W_1(R(\xi_1, \eta)) = \operatorname{Span}\{g_2, g_3\} \subset W_{1,12} + W_{1,13}.$$

Note that $\dim(W_{1,12} + W_{1,13}) = 3$. We use the equation above to see that the map $\Phi : \eta \mapsto W_1(R(\xi_1, \eta))$ is a smooth injective map from $\mathbb{RP}\{\operatorname{Span}\{\xi_1, \xi_2, \xi_3)^\perp\}$ to $Gr_2(W_{1,12} + W_{1,13})$; this is not possible as $\dim \mathbb{RP}\{\operatorname{Span}\{\xi_1, \xi_2, \xi_3)^\perp\} = 4$ and $\dim Gr_2(W_{1,12} + W_{1,13}) = 2$. Since $\mathfrak{A}_1(\xi_1) = W_1(\xi_1, \xi_2) \cap W_1(\xi_1, \xi_3)$ for any orthonormal set $\{\xi_1, \xi_2, \xi_3\}$, and since the map $(\xi, \eta) \mapsto W_1(\xi, \eta)$ is continuous, assertion 1-c) follows.

Suppose assertion (1-d) fails. Let f be a unit vector in $W_{1,12} \cap W_{1,13} \cap W_{1,23}$. Then $f \in \mathfrak{A}_1(\xi_1) \cap \mathfrak{A}_1(\xi_2)$. Let $\{\xi_1, \xi_2, \eta\}$ be an orthonormal set. Then f belongs to $W_1(\xi_1, \eta) \cap W_1(\xi_2, \eta)$ implies f belongs to $\mathfrak{A}_1(\eta)$. Given a unit vector ζ, we can choose η_1 and η_2 so that $\{\xi_1, \xi_2, \eta_1, \eta_2\}$ is an orthonormal set and so $\{\eta_1, \eta_2, \zeta\}$ is an orthonormal set. Since $\{\xi_1, \xi_2, \eta_i\}$ is an orthonormal set, the previous argument shows that f belongs to $\mathfrak{A}_1(\eta_1) \cap \mathfrak{A}_1(\eta_2)$. Since $\{\eta_1, \eta_2, \zeta\}$ is an orthonormal set, the same argument also implies that $f \in \mathfrak{A}_1(\zeta)$. Thus there is a common axis f which contradicts Lemma 3.5.

Suppose $m = 7$. Let $\{\xi_i\}$ be an orthonormal set. Let f_i be unit vectors spanning $W_{1,1i}$ for $2 \le i \le 7$. We use Lemma 3.3 to see $f_i \perp f_j$ for $i \ne j$ so $\{f_2, ..., f_7\}$ is an orthonormal set. Complete this set to an orthonormal basis $\{f_1, ..., f_7\}$ for $\mathbb{R}^7$. Then $f_1 \perp f_i$ implies $f_1 \perp W_{1,1i}$ so $f_1 \in W_{2,1i}$ for $2 \le i \le 7$. Thus $r_{1i}^2 f = -f$ and $(r_{1i}r_{1j} + r_{1j}r_{1i})f = 0$ for i and j distinct indices between 2 and 7. If $\eta \perp \xi_1$, we may expand $\eta = \Sigma_i u_i \xi_i$ and compute $R(\xi_1, \eta) f_1 = \Sigma_{i>1} i u_i^2 r_{1i}^2 f_1 = -f_1$ so f_1 belongs to $W_2(\xi_1, \eta)$ for any unit vector η perpendicular to e_1. This proves assertion 2-a). Let $\Phi : \xi \mapsto \mathfrak{A}_2(\xi) \in \mathbb{RP}^6$; Φ is a well defined map from S^6 to $\mathbb{RP}^6$. On a small open neighborhood of ξ in S^6, we can choose the $f_i = f_i(\xi)$ described above to vary continuously with ξ; it now follows that Φ is continuous which proves assertion 2-b). Since $\mathfrak{A}_2(-\xi) = \mathfrak{A}_2(\xi)$, Φ descends to a continuous map from $\mathbb{RP}^6$ to $\mathbb{RP}^6$. The Lefschetz fixed point formulas imply any map of $\mathbb{RP}^6$ to itself has a fixed point; this means that there exists ξ so $\xi \in \mathfrak{A}_2(\xi)$. $\square$

Let $m = 8$. By Lemma 3.6, $\dim \mathfrak{A}_1(\xi) = 1$ for any unit vector ξ. The map $\xi \mapsto \mathfrak{A}_1(\xi)$ defines a smooth non-constant map from S^7 to $\mathbb{RP}^7$. We use covering space theory to see that we may lift this map to take values in S^7. This defines a smooth map β on S^7 so that $\beta(\xi) \in \mathfrak{A}_1(\xi)$ for all unit vectors ξ. Note that the distinguished vector e_8 of Lemma 3.3 is $\pm\beta(\xi_1)$.

3.7 Lemma.

(1) *Let $m = 8$. There exist universal choices of sign $K = \pm\sqrt{-\lambda_1}$ and $\epsilon = \pm 1$ so that if $\{\xi_1, \xi_2, \xi_3\}$ is any orthonormal set:*
  1-a) *We have $\{\beta(\xi_1), \beta(\xi_2)\}$ is an orthonormal basis for $W_1(\xi_1, \xi_2)$.*
  1-b) *We have $R(\xi_1, \xi_2)\beta(\xi_1) = K\beta(\xi_2)$.*
  1-c) *We have $\beta(\xi_3) = \epsilon R(\xi_1, \xi_2) R(\xi_1, \xi_3) \beta(\xi_2)$.*
  1-d) *The map β extends to a linear isometry of $\mathbb{R}^8$.*

(2) *Let $m = 7$. There exists a linear map $\beta : \Lambda^2\mathbb{R}^7 \to \mathbb{R}^7$ so that if $\{\xi_i\}$ is any orthonormal set, then*
  2-a) *We have $\beta(\xi_1, \xi_2)$ is a unit vector which spans $W_{1,12}$.*
  2-b) *We have $\beta(\xi_1, \xi_2) = \pm R(\xi_1, \xi_3) R(\xi_1, \xi_2) \beta(\xi_1, \xi_3)$.*

Proof. Let $m = 8$. $\beta_i := \beta(\xi_i)$ where $i = 1, 2, 3$. Let $\{\beta_1, e_i\}$ be an orthonormal basis for $W_{1,1i}$ where $i = 2, 3$. Expand $\beta_i = a_i\beta_1 + b_i e_i$ where $i = 2, 3$. If $\beta_2 = \pm\beta_3$, since $\{\beta_1, e_2, e_3\}$ is an orthonormal set we have $b_i = 0$ so $\beta_2 = \pm\beta_1$. This would imply that $\beta_1 \in W_{1,12} \cap W_{1,13} \cap W_{1,23}$ and would contradict Lemma 3.6. Thus the β_i are linearly independent and $b_i \ne 0$. Since $\beta_i \in W_{1,23}$ for $i = 2, 3$, we have

$$W_{1,23} = \operatorname{Span}\{\beta_2, \beta_3\} \subset W_{1,12} + W_{1,13}. \tag{3.8}$$

Since $b_i \ne 0$, $W_{1,12} = \operatorname{Span}\{\beta_1, e_2\} = \operatorname{Span}\{a_2\beta_1 + b_2 e_2, -b_2\beta_1 + a_2 e_2\}$. Since $\beta_2 = a_2\beta_1 + b_2 e_2 \in \mathfrak{A}(\xi_2)$, we have $a_2\beta_1 + b_2 e_2 \in W_{1,23}$. We use Lemma 3.3 to see the complementary vector satisfies $-b_2\beta_1 + a_2e_2 \perp W_{1,23}$. Similarly we have $-b_3\beta_1 + a_3e_3 \perp W_{1,23}$. Since $\dim W_{1,23} = 2$, $\dim\{(W_{1,12} + W_{1,13}) \cap W_{1,23}^{\perp}\} = 1$ by equation (3.8). Consequently $-b_2\beta_1 + a_2e_2 = \pm - b_3\beta_1 + a_3e_3$. That implies that $a_2 = a_3 = 0$ so $\beta_2 = e_2$ and $\beta_3 = e_3$. Assertions 1-a,b) now follow from Lemma 3.3; continuity shows the sign is in assertion 1-b) is universal.

We use the decomposition of Lemma 3.3. We have $\beta_1 = \pm e_8$. Since $\{\beta_1, \beta_2\}$ is an orthonormal basis for $W_{1,12}$ and $\{\beta_1, \beta_3\}$ is an orthonormal basis for $W_{1,13}$, we have that $\beta_2 = \pm e_7 = \pm r_{13}e_5$ and $\beta_3 = \pm e_6 = \pm r_{12}e_5$. This implies that $r_{12}r_{13}\beta_2 = \pm r_{12}r_{13}r_{13}e_5 = \pm r_{12}e_5 = \pm\beta_3$. The sign in question is universal and independent of the choice of ξ_i; assertion 1-c) now follows.

Let $\{\eta_1, \eta_2\}$ be an orthonormal set. Let $a_1^2 + a_2^2 = 1$. Choose ξ_i so $\{\xi_1, \xi_2, \eta_1, \eta_2\}$ is an orthonormal set. Then

$$\begin{aligned}\beta(a_1\eta_1 + a_2\eta_2) =& \epsilon R(\xi_1, \xi_2)R(\xi_1, a_1\eta_1 + a_2\eta_2)\beta(\xi_2)\\ =& a_1\epsilon R(\xi_1, \xi_2)R(\xi_1, \eta_1)\beta(\xi_2) + a_2\epsilon R(\xi_1, \xi_2)R(\xi_1, \eta_2)\beta(\xi_2)\\ =& a_1\beta(\eta_1) + a_2\beta(\eta_2).\end{aligned}$$

Let $m = 7$. The map $\{\xi, \eta\} \to W_1(\xi, \eta)$ defines a continuous map from $Gr_2(\mathbb{R}^7)$ to $\mathbb{RP}^6$ and which lifts to a continuous map from $Gr_2^+(\mathbb{R}^7)$ to S^6. If $\{\xi_1, \xi_2, \xi_3\}$ is an orthonormal set, we consider the adapted orthonormal frame of Lemma 3.3. Then $\beta(\xi_1, \xi_2) = \pm r_{13}e_5$ and $\beta(\xi_1, \xi_2) = \pm r_{12}e_5$ so $\beta(\xi_1, \xi_2) = \pm r_{13}r_{12}\beta(\xi_1, \xi_3)$. It now follows that β can be extended bilinearly and the desired conclusion follows.□

4. The eigenvalue structures $(2, 6)$ and $(0, 2, 6)$

4.1 Normal form for R. Throughout §4, let R be an algebraic IP curvature tensor on $\mathbb{R}^8$ with an eigenvalue structure of $(2, 6)$ or $(0, 2, 6)$. Let β be the linear isometry of $\mathbb{R}^8$ defined in Lemma 3.7. If $\{\xi, \eta\}$ is an orthonormal set, let

$$p(\xi, \eta) : e \mapsto (\beta(\xi), e)\beta(\eta) - (\beta(\eta), e)\beta(\xi).$$

Since $\{\beta(\xi), \beta(\eta)\}$ is an orthonormal basis for $W_1(\xi, \eta)$, p is a rotation through an angle of $\frac{\pi}{2}$ on $W_1(\xi, \eta)$ and vanishes on $W_2(\xi, \eta)$. Thus there exists a universal constant $K = \pm\sqrt{-\lambda_1}$ so that $R = Kp$ on W_1. Let

$$c(\xi, \eta) := R(\xi, \eta) + (1 - K)p(\xi, \eta).$$

Then $c(\xi, \eta)$ agrees with R on $W_2(\xi, \eta)$ and with p on $W_1(\xi, \eta)$. Consequently p and c commute so $\beta(\eta) = p(\xi, \eta)\beta(\xi) = c(\xi, \eta)\beta(\xi)$, $c(\xi, \eta)^2 = -1$, and $R = c + (K-1)p$. Thus $c(\xi_1, \eta_1)c(\xi_1, \eta_2) + c(\xi_1, \eta_2)c(\xi_1, \eta_2) = -2(\eta_1, \eta_2)$ if $\xi_1 \perp \eta_i$ so c satisfies the Clifford commutation relations.

4.2 Lemma. *Let R be an algebraic IP curvature tensor on $\mathbb{R}^8$ with an eigenvalue structure of type $(0, 2, 6)$ or $(0, 6)$. If $|\xi| = 1$, let $\sigma := \xi + (K - 1)(\xi, \beta(\xi))\beta(\xi)$ and let $Z(\xi) := \{\eta : (\xi, \eta) = 0,\ (\beta(\xi), \eta) = 0,\ (\xi, \beta(\eta)) = 0\}$.*

(1) *If $\beta(\xi) \neq \pm\xi$, there an orthonormal set $\{\xi, \Xi\}$ so that $\beta(\xi) = \{a + bc(\xi, \Xi)\}\xi$ and so that $Z(\xi) = \{\eta : (\xi, \eta) = 0,\ (\beta(\xi), \eta) = 0,\ (\Xi, \eta) = 0\}$.*

(2) *If* $\beta(\xi) = \pm\xi$, *then* $Z(\xi) = \{\eta : (\xi, \eta) = 0\}$.
(3) *If* $\eta \in Z(\xi)$, *then* $R(\xi, \eta)\xi = c(\xi, \eta)\sigma$.
(4) *We have* $\phi : \eta \mapsto R(\xi, \eta)\xi$ *is a symmetric map preserving* $Z(\xi)$.
(5) *We have* $\sigma(\xi) = 0$.

Proof. If $\xi \neq \pm\beta(\xi)$, expand $\beta(\xi) = a\xi + b\zeta$ where $\{\xi, \zeta\}$ is an orthonormal set, where $a^2 + b^2 = 1$ and where $b \neq 0$. The map $\xi \to c(\xi, \Xi)\xi$ is an isometry of $\xi^\perp$ and hence we can choose Ξ so that $c(\xi, \Xi)\xi = \zeta$. We prove assertion (1) by computing:

a) If $\xi \perp \eta$, then $c(\xi, \eta)c(\xi, \Xi) + c(\xi, \Xi)c(\xi, \eta) = -2(\Xi, \eta)$.
b) If $\xi \perp \eta$, then $(\beta(\eta), \xi) = (c(\xi, \eta)\beta(\xi), \xi) = (c(\xi, \eta)\{a + bc(\xi, \Xi)\}\xi, \xi)$ $= \frac{b}{2}(\{c(\xi, \Xi)c(\xi, \eta) + c(\xi, \eta)c(\xi, \Xi)\}\xi, \xi) = -b(\Xi, \eta)(\xi, \xi) = -b(\Xi, \eta)$.
c) We have $(\xi, \eta) = 0$ and $(\xi, \beta(\eta)) = 0$ if and only if $(\xi, \eta) = 0$ and $(\eta, \Xi) = 0$.

Suppose next that $\beta(\xi) = \pm\xi$. If $\eta \perp \xi$, then $\beta(\eta) = c(\xi, \eta)\xi \perp \xi$ so $\eta \in Z(\xi)$. We prove assertion (3) by computing:

$$\begin{aligned}
R(\xi, \eta)\xi =&(K - 1)\{(\xi, \beta(\xi))\beta(\eta) - (\xi, \beta(\eta))\beta(\xi)\} + c(\xi, \eta)\xi \\
=&(K - 1)(\xi, \beta(\xi))c(\xi, \eta)\beta(\xi) + c(\xi, \eta)\xi \\
=&c(\xi, \eta)\{\xi + (K - 1)(\xi, \beta(\xi))\beta(\xi)\}.
\end{aligned}$$

The curvature symmetry $(R(\xi, \eta_1)\xi, \eta_2) = (R(\xi, \eta_2)\xi, \eta_1)$ shows that ϕ is symmetric. If $\xi = \pm\beta(\xi)$, then $R(\xi, \eta)\xi \perp \xi$ so ϕ preserves $Z(\xi)$. We therefore suppose $\xi \neq \pm\beta(\xi)$. Let $\eta \in Z(\xi)$. Since R is skew-symmetric, $(R(\xi, \eta)\xi, \xi) = 0$. We show that $R(\xi, \eta)\xi \perp \beta(\xi)$ by using assertion (3) to compute

$$\begin{aligned}
0 =&(\xi, \beta(\eta)) = (\xi, c(\xi, \eta)\beta(\xi)) = -(c(\xi, \eta)\xi, \beta(\xi)) \\
0 =&(c(\xi, \eta)\xi, \beta(\xi)) + (\xi, \beta(\xi))(K - 1)(c(\xi, \eta)\beta(\xi), \beta(\xi)) \\
=&(c(\xi, \eta)\sigma, \beta(\xi)) = (R(\xi, \eta)\xi, \beta(\xi)).
\end{aligned}$$

Since $R(\xi, \eta)\xi \perp \xi$ and $R(\xi, \eta)\xi \perp \beta(\xi)$, by assertion (1) to show $R(\xi, \eta)\xi \in Z(\xi)$, we need only show that $R(\xi, \eta)\xi \perp \Xi$. Since $b \neq 0$,

$$\begin{aligned}
&\mathrm{Span}\{\xi, \beta(\xi)\} = \mathrm{Span}\{\xi, c(\xi, \Xi)\xi\} = \mathrm{Span}\{\beta(\xi), c(\xi, \Xi)\beta(\xi)\}, \\
&\eta \perp \xi, \eta \perp \beta(\xi), \eta \perp c(\xi, \Xi)\xi, \text{ and } \eta \perp c(\xi, \Xi)\beta(\xi) = \beta(\Xi).
\end{aligned}$$

We complete the proof of assertion (4) by computing:

$$\begin{aligned}
0 =&(K - 1)\{(\beta(\xi), \xi)(\beta(\Xi), \eta) - (\beta(\xi), \eta)(\beta(\Xi), \xi)\} + (c(\xi, \Xi)\xi, \eta) \\
=&(R(\xi, \Xi)\xi, \eta) = (R(\xi, \eta)\xi, \Xi).
\end{aligned}$$

Suppose assertion (5) fails so that $\sigma \neq 0$. For any unit vector $\eta \in Z(\xi)$, we have $|R(\xi, \eta)\xi| = |c(\xi, \eta)\sigma| = |\sigma|$. Thus $\phi/|\sigma|$ is a symmetric isometry of $Z(\xi)$ and we can diagonalize ϕ. This decomposes $Z = Z_+ \oplus Z_-$ where $\phi = \pm|\sigma|$ on $Z_\pm$. Choose a unit vector $\eta_1 \in Z_\pm$. Then $\phi\eta_1 = \pm|\sigma|\eta$ i.e. $c(\xi, \eta_1)\sigma = \pm|\sigma|\eta_1$. Since

$\dim(Z) \geq 5$, at least one of the eigenspaces has dimension 3 so we can choose a unit vector η_2 in Z so $(\eta_1, \eta_2) = 0$, $(\beta(\eta_1), \eta_2) = 0$ and $c(\xi, \eta_2)\sigma = \pm|\sigma|\eta_2$. Then

$$\begin{aligned}(\eta_1, \beta(\eta_2)) &= \pm|\sigma|^{-1}(c(\xi,\eta_1)\sigma, c(\xi,\eta_2)\beta_1)\\ &= \mp\,|\sigma|^{-1}(c(\xi,\eta_2)\sigma, c(\xi,\eta_1)\beta_1) = \mp(\eta_2, \beta(\eta_1)) = 0.\end{aligned}$$

To summarize: we have an orthonormal set $\{\xi, \eta_1, \eta_2\}$ so that

$$\begin{aligned}&\xi \perp \beta(\eta_1),\ \xi \perp \beta(\eta_2),\ R(\eta_1,\eta_2)\xi = c(\eta_1,\eta_2)\xi,\\ &\eta_1 \perp \beta(\xi),\ \eta_1 \perp \beta(\eta_2),\ R(\xi,\eta_2)\eta_1 = c(\xi,\eta_2)\eta_1,\ \eta_1 = \pm|\sigma|^{-1}c(\xi,\eta_1)\sigma\\ &\eta_2 \perp \beta(\xi),\ \eta_2 \perp \beta(\eta_1),\ R(\xi,\eta_1)\eta_2 = c(\xi,\eta_1)\eta_2,\ \eta_2 = \pm|\sigma|^{-1}c(\xi,\eta_2)\sigma.\end{aligned}$$

Since $c(\eta_2, \xi)$ anti-commutes with $c(\eta_1, \xi)$, $R(\eta_1, \eta_2)\xi = -R(\eta_2, \xi)\eta_1 - R(\xi, \eta_1)\eta_2$ by the Bianchi identities so $c(\eta_1, \eta_2)\xi = |\sigma|^{-1}\{\pm c(\eta_2, \xi)c(\xi, \eta_1) \pm c(\xi, \eta_1)c(\xi, \eta_2)\}\sigma$. We take norms to see $1 = |\pm 1 + \pm 1|$ which is false. □

Proof of Theorem 0.4 (2)**.** We use Lemma 4.2 to see that $\sigma(\xi) = 0$ for every unit vector ξ. This means that: $\xi + (K-1)(\xi, \beta(\xi))\beta(\xi) = 0$. Consequently $\beta(\xi) = \pm\xi$ and $K = 0$ so $\lambda_1 = 0$. By replacing β by $-\beta$ if necessary, we may suppose that $\beta(\xi) = \xi$ for all ξ. Let $\{\xi_i\}$ be an orthonormal set. Adopt the notation of Lemma 3.3. Since $\{\beta_1, \beta_2\}$ is an orthonormal basis for $W_{1,12}$ and since $\{\beta_1, \beta_3\}$ is an orthonormal basis for $W_{1,13}$, we have $\beta_1 = \pm e_8$, $\beta_2 = \pm e_7$, and $\beta_3 = \pm e_6$. Consequently $r_{13}\beta_2 = \pm r_{12}\beta_3 = \pm e_5$. Similarly, $r_{23}\beta_1 = \pm r_{12}\beta_3$. Thus $0 = r_{12}\xi_3 + r_{23}\xi_1 + r_{31}\xi_2 = \pm r_{12}\beta_3 \pm r_{13}\beta_2 \pm r_{23}\beta_1 = (\pm 1 \pm 1 \pm 1)e_5$ by the Bianchi identities; this yields the final contradiction. □

5. The proof of Theorem 0.4 (3)

Proof of Theorem 0.4 (3)**.** Choose $\xi_1 \in S^6$ so that $\xi_1 \in \mathfrak{A}_2(\xi_1)$. If $\eta \perp \xi_1$, then ξ_1 belongs to $W_2(\eta, \xi)$ so $|R(\xi_1, \eta)\xi_1| = 1$. Since $R(\xi_1, \eta)\xi_1 \perp \xi_1$ and since we have the curvature symmetry $(R(\xi_1, \eta_1)\xi_1, \eta_2) = (R(\xi_1, \eta_2)\xi_1, \eta_1)$, the map $\phi : \eta \mapsto R(\xi_1, \eta)\xi_1$ is a self-adjoint isometry of $\xi_1^\perp$. We may therefore decompose $\xi_1^\perp = V_+ \oplus V_-$ into the ± 1 eigenspaces of ϕ. By replacing R by $-R$, we may suppose $\dim V_+ \leq \dim V_-$. If $V_+ \neq \{0\}$, choose a unit vector $\xi_2 \in V_+$; otherwise choose $\xi_2 \in V_-$. The map β discussed in Lemma 3.7 is bilinear. The decomposition $V_+ \oplus V_-$ is orthogonal. We have $\dim V_- \geq 3$; if $\xi_2 \in V_-$, then $\dim V_- \geq 4$. Therefore we may choose a unit vector $\xi_3 \in V_-$ so $(\xi_3, \xi_2) = 0$ (this is automatic if $\xi_2 \in V_+$), so $(\beta(\xi_2, \xi_3), \xi_1) = 0$, and so $(\beta(\xi_1, \xi_2), \xi_3) = 0$. We complete the proof:

(1) We have $\{\xi_1, \xi_2, \xi_3\}$ is an orthonormal set.
(2) Since $(\beta(\xi_1, \xi_2), \xi_3) = 0$, $\xi_3 \in W_{2,12}$ so $|r_{12}\xi_3| = 1$.
(3) Since $(\beta(\xi_2, \xi_3), \xi_1) = 0$, $\xi_1 \in W_{2,23}$ so $|r_{23}\xi_1| = 1$.
(4) Since $\xi_1 \in W_1(\xi_1)$, $\xi_1 \in W_{1,12} \cap W_{1,13}$ so $(r_{12}r_{13} + r_{13}r_{12})\xi_1 = 0$.
(5) The Bianchi identities imply $-r_{23}\xi_1 = r_{21}\xi_3 + r_{13}\xi_2$.
(6) Since $\xi_i \in V_\pm$ for $i = 2, 3$, $\xi_i = \pm r_{1i}\xi_i$.
(7) We have $1 = |r_{23}\xi_1| = |r_{13}\xi_2 +_{12} \xi_3)| = |(\pm r_{13}r_{12}\xi_1 \pm r_{12}r_{13})\xi_1|$
$= |\pm 1 \pm 1| \cdot |r_{12}r_{13}\xi_1| = |\pm 1 \pm 1| \cdot |r_{12}\xi_3| = |\pm 1 \pm 1|$ which is false. □

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