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OSSERMAN-P SPACES OF IWASAWA TYPE

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ABSTRACT. We show that a solvable Lie group of Iwasawa type, 2-step nilpotent nilradical with a left invariant metric satisfying the Osserman-p condition, $(1 \leq p < dim\ S)$ is either a space of constant negative sectional curvature or a Damek-Ricci space.

Let M^n be a riemannian manifold, R its curvature tensor and R_X the Jacobi operator, defined by $R_X Y = R(Y,X)X$, X a unit tangent vector. We say that M satisfies the Osserman-p condition, (Oss-p, for short), p a natural number, if for any set of p orthonormal vectors $E = \{X_1, X_2, \ldots, X_p\}$ in the tangent bundle, the symmetric operator

$$j_E = R_{X_1} + R_{X_2} + \cdots + R_{X_p}$$

has constant eigenvalues, counting multiplicities.

This notion, introduced in [S-V], generalizes the Osserman condition $(p = 1)$. It is immediate to see that if $p = n$ then M^n has constant scalar curvature. Moreover, if p satisfies $1 \leq p < n$ then the riemannian manifold is Einstein and there exist a duality between p and $n - p$, that is, M^n satisfies the Osserman-p condition if and only if it satisfies the Osserman-$(n - p)$ condition.

We note that spaces of constant sectional curvature satisfy condition Oss-p for all $p \geq 1$. It follows from the curvature formula for spaces of constant curvature c,

$$R(X,Y)Z = c\left(\langle Y,Z\rangle X - \langle Z,X\rangle Y\right) ,$$

that $R_X Y = cX$ for orthogonal tangent vectors X and Y. Hence, if $E = \{X_1, \ldots, X_p\}$ is an orthonormal set, we have that

$$j_E \mid_E = c(p-1) \text{ Id}, \quad j_E \mid_{E^\perp} = cp \text{ Id} ,$$

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In this note we consider the following question raised by P. Gilkey: if a non positively curved homogeneous manifold M^n satisfies condition Oss-p, for some $1 < p < n$ then the curvature is constant. The purpose of the present work is to give an affirmative answer to this question in a subclass of that of homogeneous manifolds of nonpositive curvature. More precisely we prove:

Theorem. *If S is a solvable Lie group of Iwasawa type and dimension n, with $[S,S]$ two step nilpotent and satisfying the Osserman-p condition for some $1 \leq p < n$, then S is either a Damek-Ricci space or a space of constant sectional curvature.*

1. Curvature formulas on Iwasawa type Lie groups

A solvable Lie algebra $\mathfrak{s}$ with inner product $\langle , \rangle$ is a metric Lie algebra of Iwasawa type, if it satisfies the conditions

(i) $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and $\mathfrak{a}$, the orthogonal complement of $\mathfrak{n}$, is abelian.

(ii) All operators ad_H, $H \in \mathfrak{a}$ are symmetric.

(iii) For some $H_0 \in \mathfrak{a}$ $\mathrm{ad}_{H_0}|_{\mathfrak{n}}$ has positive eigenvalues.

The simply connected Lie group S with Lie algebra $\mathfrak{s}$ and left invariant metric induced by the inner product $\langle , \rangle$ will be called of Iwasawa type. The Levi Civita connection and the curvature tensor associated to the metric can be computed by,

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle \\ R(X,Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \end{aligned}$$

for any X, Y, Z in $\mathfrak{s}$.

Let $n = \dim \mathfrak{s}$ and $1 \leq p < n$. We will say that the metric Lie algebra $\mathfrak{s}$ satisfies the Oss-p condition, if the corresponding Lie group S of Iwasawa type satisfies it. As we recalled above, Oss-p spaces with $1 \leq p < n$ are Einstein. Here, and from now on, n will stand for the dimension of S.

Let $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ be a metric Lie algebra of Iwasawa type. Since $\mathfrak{a}$ is abelian, $\mathfrak{a}$ operates on $\mathfrak{n}$ by commuting symmetric maps $\{\mathrm{ad}_H|_{\mathfrak{n}} : H \in \mathfrak{a}\}$ so $\mathfrak{n}$ has an orthogonal decomposition into root spaces $\mathfrak{n}_{\lambda_i}$, where $\lambda_i \in \mathfrak{a}^*$ is such that

$$\mathrm{ad}_H X_i = \lambda_i(H) X_i \text{ for all } X_i \in \mathfrak{n}_{\lambda_i},\ H \in \mathfrak{a}.$$

Let $\mathfrak{z}$ denote the center of $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and let $\mathfrak{v}$ be the orthogonal complement of $\mathfrak{z}$ with respect to the metric $\langle , \rangle$ restricted to $\mathfrak{n}$. Thus $\mathfrak{n}$ decomposes $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ and for all $H \in \mathfrak{a}$, $\mathrm{ad}_H : \mathfrak{z} \to \mathfrak{z}$ hence $\mathrm{ad}_H : \mathfrak{v} \to \mathfrak{v}$ (since ad_H is symmetric). Thus $\mathfrak{z} = \sum \mathfrak{z}_{\mu_i}$ and $\mathfrak{v} = \sum \mathfrak{v}_{\lambda_j}$ (Note that the equality $\mu_i = \lambda_j$ is possible). For each element $Z \in \mathfrak{z}$ we define a skew symmetric linear transformation $j_Z : \mathfrak{v} \to \mathfrak{v}$ by

$$\langle j_Z X, Y\rangle = \langle [X,Y], Z\rangle \text{ for all } X,\ Y \in \mathfrak{v},\ \text{all } Z \in \mathfrak{z}.$$

The transformations j_Z coincides with the usual ones considered in the case of a 2-step nilpotent $\mathfrak{n}$ (see [E]) and play an important role, since the geometry of $\mathfrak{n}$

and $\mathfrak{s}$ seems to be concentrated on their properties. We note that, by using the expression of the connection, one easily obtains

$$\nabla_Z X = \nabla_X Z = -\frac{1}{2} j_Z X, \quad Z \in \mathfrak{z}, \quad X \in \mathfrak{v}\,.$$

Computations. Now we express the curvature operators of the Lie group S in terms of the curvature of N, the Lie group associated to $\mathfrak{n}$, with induced metric from $\mathfrak{s}$ and second fundamental form α with respect to unit normal vectors $H \in \mathfrak{a}$. We will denote by $R^{\mathfrak{n}}$ the curvature tensor of N. We recall that if $\nabla^{\mathfrak{n}}$ denotes the associated connection to the induced metric $\langle , \rangle$ on $\mathfrak{n}$, then the Gauss's formula is

$$\nabla_X Y = \nabla^{\mathfrak{n}}_X Y + \sum_{i=1}^{\dim \mathfrak{a}} \langle \mathrm{ad}_{H_i} X, Y\rangle\, H_i$$

since the second fundamental form α is defined by

$$\langle \alpha(X,Y), H\rangle = \langle -\nabla_X H, Y\rangle = \langle [H,X], Y\rangle, \; H \in \mathfrak{a}\,.$$

Thus, for any X, Y, W in $\mathfrak{n}$ the equation of Gauss yields (see [K-N, Proposition 4.1, Chapter VII)]

$$\begin{aligned} \langle R(Y,X)X, W\rangle &= \langle R^{\mathfrak{n}}(Y,X)X, W\rangle \\ &\quad + \sum_i \langle \mathrm{ad}_{H_i} Y, X\rangle \langle \mathrm{ad}_{H_i} X, W\rangle - \sum_i \langle \mathrm{ad}_{H_i} Y, W\rangle \langle \mathrm{ad}_{H_i} X, X\rangle \end{aligned}$$

where $\{H_1, \ldots, H_r\}$ is an orthonormal basis of $\mathfrak{a}$. Hence

$$\begin{aligned} (1) \qquad R_X Y &= R^{\mathfrak{n}}_X Y + \sum_i^r \langle \mathrm{ad}_{H_i} X, Y\rangle\, \mathrm{ad}_{H_i} X - \sum_i^r \langle \mathrm{ad}_{H_i} X, X\rangle\, \mathrm{ad}_{H_i} Y \\ &\quad + \sum_{i=1}^r \langle R_X Y, H_i\rangle\, H_i\,. \end{aligned}$$

One can easily verify that if $Z, Z^* \in \mathfrak{z}$ and $X \in \mathfrak{v}$ then

$$(2) \qquad R^{\mathfrak{n}}_Z Z^* = 0, \quad R^{\mathfrak{n}}_Z X = -\frac{1}{4} j_Z^2 X\,.$$

Furthermore, in the case when $\mathfrak{n}$ is 2-step nilpotent, we will use the following curvature formulas given in [E]

$$(3) \qquad R^{\mathfrak{n}}_X Z = \frac{1}{4}[X, j_Z X] \text{ and } R^{\mathfrak{n}}_X Y = -\frac{3}{4} j_{[X,Y]} X \quad Z \in \mathfrak{z},\; X, Y \in \mathfrak{v}\,.$$

In the next proposition we summarize the basic curvature formulas that will be used throughout the article.

Proposition 1.1. *If $\mathfrak{s}$ is a Lie algebra of Iwasawa type then the following formulas involving curvatures hold. Assume $\{H_1, \ldots, H_r\}$ is an orthonormal basis of $\mathfrak{a}$.*

(i) $R_H = -ad_H^2$ *for all* $H \in \mathfrak{a}$.

(ii) *If either* $Z \in \mathfrak{z}_\mu$, $|Z| = 1$ *or* $X \in \mathfrak{v}_\lambda$, $|X| = 1$ *then*

$$R_Z H = -\mu(H) \sum_{i=1}^{r} \mu(H_i) H_i \ , \quad R_X H = -\lambda(H) \sum_{i=1}^{r} \lambda(H_i) H_i \,.$$

(iii) *If* $Z \in \mathfrak{z}_{\mu_i}$, $|Z| = 1$, *and* $Z^* \in \mathfrak{z}_{\mu_j}$ *then*

$$R_Z Z^* = \sum_{k=1}^{r} \mu_i(H_k)\mu_j(H_k) \left(\langle Z, Z^* \rangle Z - Z^* \right) \,.$$

(iv) *If* $Z \in \mathfrak{z}_\mu$, $|Z| = 1$, *and* $X \in \mathfrak{v}_\lambda$

$$R_Z X = -\frac{1}{4} j_Z^2(X) - \sum_{i=1}^{r} \mu(H_i)\lambda(H_i) X \,.$$

If in addition $\mathfrak{n}$ *is 2-step nilpotent, then*

(v) *If* $X \in \mathfrak{v}_{\lambda_j}$ *and* $Z \in \mathfrak{z}_\mu$

$$R_X Z = \frac{1}{4}[X, j_Z X] - \sum_{i=1}^{r} \lambda_j(H_i)\mu(H_i) Z \,.$$

(vi) *If* $X \in \mathfrak{v}_{\lambda_j}$ *and* $Y \in \mathfrak{v}_{\lambda_k}$

$$R_X Y = -\frac{3}{4} j_{[X,Y]} X + \sum_{i=1}^{r} \left(\lambda_j(H_i)^2 \langle X, Y \rangle X - \lambda_j(H_i)\lambda_k(H_i) Y \right) \,.$$

When $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$ *with* $|H| = 1$ *and 2-step nilpotent* $\mathfrak{n}$ *one also has*

(vii) $Ric(H) = -trad_H^2$. *If* $Z \in \mathfrak{z}$ *and* $X \in \mathfrak{v}$ *are unit vectors then*

$$Ric(Z) = -\frac{1}{4} tr j_Z^2 - \langle [H, Z], Z \rangle trad_H \ , \ Ric(X) = \frac{1}{2} \langle \sum_{i=1}^{k} j_{Z_i}^2 X, X \rangle - \langle [H, X], X \rangle trad_H \,.$$

Proof. By a direct calculation, using the conexion formula, $\nabla_H = 0$ if $H \in \mathfrak{a}$. Hence, $R_H = -\mathrm{ad}_H^2$ and (i) follows. Moreover, for $Y \in \mathfrak{n}_\gamma$ and using again $\nabla_H = 0$, one obtains $R_Y H = -\nabla_{[H,Y]} Y = -\gamma(H) \nabla_Y Y$. We observe next that $\nabla_Z Z$ and $\nabla_X X$ for $Z \in \mathfrak{z}_\mu$, $X \in \mathfrak{v}_\lambda$ have only components in $\mathfrak{a}$. Hence (ii) follows since

$$\nabla_Z Z = \sum_i \mu(H_i) H_i, \ \nabla_X X = \sum_i \lambda(H_i) H_i \,.$$

The proofs of (iii) and (iv) are immediately obtained by substituting, in each case, identity (2) into identity (1).

We note that $(R_X Z)_{\mathfrak{a}} = (R_X Y)_{\mathfrak{a}} = 0$, for any $Z \in \mathfrak{z}_\mu$, $X \in \mathfrak{v}_{\lambda_j}$ and $Y \in \mathfrak{n}_{\lambda_k}$ since by (ii) above one has

$$\langle R_X Z, H\rangle = \langle Z, R_X H\rangle = 0\,,\ \langle R_X Y, H\rangle = \langle Y, R_X H\rangle = 0\,.$$

Therefore, replacing (3) in (1), the proof of (v) and (vi) follows.

Since $Ric(Y) = \mathrm{tr} R_Y$, using (i) it follows $Ric(H) = -\ \mathrm{tr}\ \mathrm{ad}_H^2$. If $Y \in \mathfrak{n}$, $|Y| = 1$, identity (1) implies

$$Ric(Y) = Ric^{\mathfrak{n}}(Y) - \langle \mathrm{ad}_H Y, Y\rangle\, \mathrm{tr}\, \mathrm{ad}_H\,.$$

Finally, for unit vectors $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$, (2) and (3) give $Ric^{\mathfrak{n}}(Z) = -\frac{1}{4}\mathrm{tr}\ j_Z^2$ and $Ric^{\mathfrak{n}}(X) = \frac{1}{2}\langle \sum_{i=1}^k j_{Z_i}^2 X, X\rangle$ (see [E] for a proof). Thus (vii) is obtained.

2. The Osserman-p condition on solvable Lie algebras of Iwasawa type and rank 1

In this section we obtain various results that hold if one imposes the Oss-p condition to Iwasawa type Lie algebras with $\dim \mathfrak{a} = 1$. We will call this class of algebras of rank one. We observe first

Lemma 2.1. *Let $\mathfrak{s}$ be a solvable Lie algebra of Iwasawa type and $Z \in \mathfrak{z}_\mu$, where μ is a root of $ad_H|_{\mathfrak{z}}$. Then, $j_Z|_{\mathfrak{v}_{\lambda_i}} : \mathfrak{v}_{\lambda_i} \to \mathfrak{v}_{\mu-\lambda_i}$. In particular j_Z^2 preserves $\mathfrak{v}_{\lambda_i}$. Moreover, if $\mathfrak{n}$ is 2-step nilpotent and $X \in \mathfrak{v}_\lambda$ then $[X, j_Z X] \in \mathfrak{z}_\mu$.*

Proof. We first show that $j_Z|_{\mathfrak{v}_{\lambda_i}} : \mathfrak{v}_{\lambda_i} \to \mathfrak{v}_{\mu-\lambda_i}$. Let $X \in \mathfrak{v}_{\lambda_i}$ and assume that $j_Z X \neq 0$. Hence for some $Y \in \mathfrak{v}_{\lambda_j}$ we have, $\langle j_Z X, Y\rangle = \langle [X,Y], Z\rangle \neq 0$ implying that $[X,Y]$ has a non zero component in $\mathfrak{z}_\mu$. Thus, $\lambda_i + \lambda_j = \mu$. For the last assertion, if $[X, j_Z X] \neq 0$, then $[X, j_Z X] \in \mathfrak{n}_\mu$ and $j_Z X \neq 0$. Since $[X, j_Z X] \in \mathfrak{z} \cap \mathfrak{n}_\mu$, it follows that $[X, j_Z X] \in \mathfrak{z}_\mu$.

Proposition 2.2. *If $\mathfrak{s}$ is a solvable Lie algebra of Iwasawa type with 2-step nilpotent $\mathfrak{n}$ and satisfying Oss-p for some $p \geq 1$, then* $\dim \mathfrak{a} = 1$.

Proof. We assume $\mathfrak{n}$ is 2-step nilpotent and $\dim \mathfrak{a} = r > 1$. Let H_1 and H_2 be orthonormal vectors in $\mathfrak{a}$ and

$$E_{s,t} = \{sH_1 + tH_2, H_3, \ldots, H_r, Z_1, \ldots, Z_k, X_1, \ldots, X_l\}$$

with $s^2 + t^2 = 1$ be the orthonormal set of p vectors in $\mathfrak{s}$, obtained after completing with $H_i \in \mathfrak{a}$, $Z_i \in \mathfrak{z}$ and $X_i \in \mathfrak{v}$, eigenvectors of $\mathrm{ad}_{\mathfrak{a}}|_{\mathfrak{n}}$. Then

$$j_{E_{s,t}} = R_{sH_1+tH_2} - \sum_{i=3}^{r} \mathrm{ad}_{H_i}^2 + \sum_{i=1}^{k} R_{Z_i} + \sum_{i=1}^{l} R_{X_i} = -\mathrm{ad}_{sH_1+tH_2}^2 + T$$

where $T = -\sum_{i=3}^r \mathrm{ad}_{H_i}^2 + \sum_{i=1}^k R_{Z_i} + \sum_{i=1}^l R_{X_i}$. Since the eigenspaces $\mathfrak{z}_\mu$ are invariant under T (see the curvature formulas in Proposition 1.1 and previous lemma), by choosing an eigenvector $Z \in \mathfrak{z}_\mu, |\, Z\,| = 1$, of T, we have that Z is also an eigenvector of $j_{E_{s,t}}$ with associate eigenvalue

$$\langle j_{E_{s,t}} Z, Z\rangle = \langle \left(-\mathrm{ad}_{sH_1+tH_2}^2 + T\right) Z, Z\rangle = -\mu(sH_1 + tH_2)^2 + \langle TZ, Z\rangle$$

where $\langle TZ, Z\rangle$ does not depend on (s,t). By using the condition Oss-p, the continuous function from the circle $(s^2+t^2=1)$

$$(s,t) \ \rightarrow \langle j_{E_{s,t}} Z, Z\rangle$$

is constant since it take values in the finite set of eigenvalues of J_E, where E varies on p-dimensional subspaces. Consequently, $\mu(sH_1+tH_2)^2$ is constant, implying $\mu(sH_1+tH_2) = s\mu(H_1)+t\mu(H_2)$ is also constant. Thus $\mu(H_1)=\mu(H_2)$ hence $\mu(H_i)=\mu(H_j)$ for all $i,j=1,\dots,r=\dim\mathfrak{a}$ which is impossible since $\mu\in\mathfrak{a}^*-\{0\}$.

In what follows we set $\dim\mathfrak{a}=1$, choose a unit vector $H\in\mathfrak{a}$, and assume that all eigenvalues of $\mathrm{ad}_H|_{\mathfrak{n}}$ are positive. It follows from the curvature formulas that

$$R_Z H = -\mid [H,Z] \mid^2 H \text{ for any } Z\in\mathfrak{z}\,. \tag{4}$$

In fact, if $X\in\mathfrak{n}$,

$$\langle R_Z H, X\rangle = -\left\langle \nabla_{[H,Z]} Z, X\right\rangle = 0 \ \text{ and } \ \langle R_Z H, H\rangle = -\left|[H,Z]\right|^2 .$$

Moreover, it is immediate from Proposition 1.1 that

$$R_X H = -\lambda^2 H \text{ for } X\in\mathfrak{v}_\lambda \quad \text{and } R_Z H = -\mu^2 H \text{ if } Z\in\mathfrak{z}_\mu\,.$$

The next proposition shows that under the assumption that the Osserman-p condition is satisfied on an Iwasawa type algebra of rank 1, the center of $\mathfrak{n}$ is contained in an eigenspace of $\mathrm{ad}H$.

Proposition 2.3. *If $\mathfrak{s}$ is a solvable Lie algebra of Iwasawa type and* $\dim\mathfrak{a}=1$ *satisfying the Oss-p condition, $p\geq 1$, then $ad_H|_{\mathfrak{z}}=\lambda$ Id. Furthermore, if $\mathfrak{n}$ is not abelian, let $\{\lambda_1>\cdots>\lambda_m\}$ denote the eigenvalues of $ad_H|_{\mathfrak{v}}$, then*

(i) *$\lambda>\lambda_i$ for all $i=1,\dots,m$, in particular $\mathfrak{n}_\lambda=\mathfrak{z}$. Moreover, $\lambda-\lambda_1$ is also an eigenvalue of $ad_H|_{\mathfrak{v}}$, and $\lambda_1\geq\frac{1}{2}\lambda$.*

(ii) $[\mathfrak{v}_{\lambda_1},\mathfrak{v}]\subseteq\mathfrak{z}$.

Proof. Assume there exist Z_1 and Z_2 unit vectors in $\mathfrak{z}_\mu$ and $\mathfrak{z}_\nu$, respectively, with $\mu\neq\nu$ eigenvalues of $\mathrm{ad}_H|_{\mathfrak{z}}$. Let

$$E_{r,s}=\{rZ_1+sZ_2, X_2,\dots,X_p\},\ r^2+s^2=1$$

be an orthonormal set of p vectors in $\mathfrak{s}$, obtained by completing with $X_i\in\mathfrak{n}$, eigenvectors of $\mathrm{ad}_H|_{\mathfrak{n}}$, with eigenvalues γ_i. Then the associated operator $j_{E_{r,s}}$ is

$$j_{E_{r,s}} = R_{rZ_1+sZ_2}+\sum_{i=2}^{p} R_{X_i}\,.$$

From $\sum_{i=2}^p R_{X_i}H = -\left(\sum_{i=2}^p\gamma_i^2\right)H$ and $R_{rZ_1+sZ_2}(H) = -|[H,rZ_1+sZ_2]|^2H$, (see (4)), it follows that H is an eigenvector of $j_{E_{r,s}}$ with associated eigenvalue

$$\langle j_{E_{r,s}}H,H\rangle = -\mid [H,rZ_1+sZ_2]\mid^2 - \sum_{i=2}^p \gamma_i^2\,.$$

Since $[H, rZ_1 + sZ_2] = r\mu Z_1 + s\nu Z_2$, arguing as in the previous proposition, one obtains $r^2\mu^2 + s^2\nu^2$ is a constant (independent of (r,s)). Thus $\mu = \nu$, since all eigenvalues of $\mathrm{ad}_H|_{\mathfrak{n}}$ are positive. We will denote by λ the eigenvalue of $\mathrm{ad}_H|_{\mathfrak{z}}$.

The proof of (i) and (ii) is the same as that of Proposition 1.2 in [D-D]. All is needed is $\mathrm{ad}_H|_{\mathfrak{z}} = \lambda\,\mathrm{Id}$.

We summarize in the next proposition certain restrictions imposed by the Oss-p condition to the algebraic structure of the Lie algebra $\mathfrak{n}$, involving the transformations $\{j_Z : Z \in \mathfrak{z}\}$. We observe first

Remark. If $Z \in \mathfrak{z}_\mu$, then $\ker j_Z$ is invariant by $\mathrm{ad}_H, H \in \mathfrak{a}$.

The above assertion holds without restriction on dim $\mathfrak{a}$, since $j_Z X = 0$ implies

$$\begin{aligned}\langle j_Z([H,X]), Y\rangle &= \langle [[H,X],Y], Z\rangle = -\langle [[X,Y],H],Z\rangle - \langle [[Y,H],X],Z\rangle \\ &= \langle [X,Y],[H,Z]\rangle + \langle j_Z([H,Y]),X\rangle = \langle j_Z X, \mu Y - [H,Y]\rangle = 0\,.\end{aligned}$$

Proposition 2.4. *If $\mathfrak{s}$ is a solvable Lie algebra of Iwasawa type with* $\dim\mathfrak{a} = 1$ *and 2-step nilpotent n, satisfying the Oss-p condition, $p \geq 1$ then*

(i) *If $Z \neq 0$, the linear map $j_Z : \mathfrak{v} \to \mathfrak{v}$ is an isomorphism. Hence, the linear map $\mathfrak{z} \to \mathfrak{v}$, $Z \to j_Z X$, is one to one and $ad_X|_{\mathfrak{v}} : \mathfrak{v} \to \mathfrak{z}$ is onto if $X \in \mathfrak{v}, X \neq 0$.*

(ii) *If λ_i is an eigenvalue then $\lambda - \lambda_i$ is also an eigenvalue. In particular $\lambda - \lambda_1$ is the minimum eigenvalue of $ad_H|_{\mathfrak{v}}$.*

(iii) *If $X \in \mathfrak{v}_{\lambda_i}$ then $j_{\mathfrak{z}} X = \{j_Z X : Z \in \mathfrak{z}\}$ is the orthogonal complement of* $\ker ad_X|_{\mathfrak{v}}$ *in $\mathfrak{v}$. Consequently, $\mathfrak{v} = \ker ad_X|_{\mathfrak{v}} \oplus j_{\mathfrak{z}} X$ and $ad_H|_{j_{\mathfrak{z}}X} = (\lambda - \lambda_i)\, Id$.*

Proof. Assume that for some Z in $\mathfrak{z}$ and $X \in \mathfrak{v}$, unit vectors, $j_Z X = 0$. By the remark above we may assume $X \in \mathfrak{v}_{\lambda_i}$. Computing R_{rZ+sX} with $r^2 + s^2 = 1$ we have,

$$\begin{aligned}R_{rZ+sX}(H) &= r^2 R_Z H + s^2 R_X H + rs\,(R(H,Z)X + R(H,X)Z) \\ &= r^2 R_Z H + s^2 R_X H\end{aligned}$$

since $R(H,Z)X = \frac{1}{2}\lambda j_Z X = 0$ and $R(H,X)Z = \frac{1}{2}\lambda_i j_Z X = 0$. Hence

$$R_{rZ+sX}(H) = -\left(r^2\lambda^2 + s^2\lambda_i^2\right) H\,.$$

Choosing $E_{r,s} = \{H, rZ+sX, X_2, \ldots, X_{p-1}\}$ the p-space completed with orthonormal eigenvectors X_i $(i = 2, \ldots, p)$ of $\mathrm{ad}_H|_{\mathfrak{n}}$, we have that $j_{E_{r,s}} = -\mathrm{ad}_H^2 + R_{rZ+sX} + \sum_{i=2}^{p-1} R_{X_i}$ has H as common eigenvector, with associated eigenvalue

$$\langle j_{E_{r,s}} H, H\rangle = -\left(r^2\lambda^2 + s^2\lambda_i^2\right) + \langle\sum_{i=2}^{p-1} R_{X_i} H, H\rangle\,.$$

Since the last term above is independent of (r,s) and we are assuming that the Oss-p condition holds, $r^2\lambda^2 + s^2\lambda_i^2$ is constant; this is impossible since $\lambda_i < \lambda$. Therefore j_Z is an isomorphism. It is straightforward now to complete the proof of (i), (ii) follows from Lemma 2.1 together with (i) above and finally (iii) follows as in Proposition 1.3 of [D-D].

3. Reduction of the eigenvalues when $\mathfrak{n}$ is 2-step nilpotent

We assume in this section $\mathfrak{n}$ is 2-step nilpotent, non abelian and obtain strong restrictions on the number of eigenvalues.

Proposition 3.1. *Let $V \in \mathfrak{v}_{\lambda_1}$, $|V|=1$. If $\mathfrak{s}$ is a solvable Lie algebra of Iwasawa type with non abelian 2-step nilpotent $\mathfrak{n}$ and satisfying Oss-p then* $\ker ad_V|_{\mathfrak{v}} \subset \mathfrak{v}_{\lambda_1}$.

Proof. We may assume that a unit $X \in \mathfrak{v}_{\lambda_i}$ satisfies $[X,V]=0$ ($\ker \mathrm{ad}_V|_{\mathfrak{v}}$ is ad_H- invariant). We compute, for (r,s) in the circle $r^2+s^2=1$,

$$R_{rV+sX}(H) = r^2 R_V H + s^2 R_X H = -\left(r^2\lambda_1^2 + s^2\lambda_i^2\right) H\,,$$

since $[\mathfrak{v},\mathfrak{v}] \subset \mathfrak{z}$ gives $R(H,X)V = -\nabla_{[H,X]}V = 0$, $R(H,V)X = -\nabla_{[H,V]}X = 0$. Hence, the p-space $E_{r,s} = \{H, rV+sX, X_2, \ldots, X_{p-1}\}$ of orthonormal vectors, where X_i are eigenvectors of $\mathrm{ad}_H|_{\mathfrak{n}}$, has associated operator

$$j_{E_{r,s}} = -\mathrm{ad}_H^2 + R_{rV+sX} + \sum_{i=2}^{p-1} R_{X_i}$$

with H a common eigenvector. Thus the continuous map $(r,s) \to \langle j_{E_{r,s}} H, H\rangle$ is constant. This implies that

$$-\left(r^2\lambda_1^2 + s^2\lambda_i^2\right) + \langle \sum_{i=2}^{p-1} R_{X_i}, H\rangle$$

is constant as function of (r,s), or equivalently, $r^2\lambda_1^2 + s^2\lambda_i^2$ is constant on the circle $r^2+s^2=1$. Therefore $\lambda_i = \lambda_1$, and the assertion follows.

The following theorem shows that if $\mathfrak{v} \neq \{0\}$, only one eigenvalue occurs in $\mathfrak{v}$.

Theorem 3.2. *If $\mathfrak{s}$ is a solvable Lie algebra of Iwasawa type and dimension n with non abelian 2-step nilpotent $\mathfrak{n}$ and satisfying the Oss-p condition, $1 \le p < n$, then the eigenvalues of $ad_H|_{\mathfrak{n}}$ are λ and $\frac{1}{2}\lambda$.*

Proof. It follows from Propositions 2.4 and 3.1 that if $V \in \mathfrak{v}_{\lambda_1}$

$$\mathfrak{v} = \mathfrak{v}_{\lambda_1} \oplus \left(\mathfrak{v}_{\lambda-\lambda_1} \cap \ker \mathrm{ad}_V\right) \oplus j_{\mathfrak{z}} V\,.$$

Then, if $\mathfrak{v}_{\lambda-\lambda_1} \cap \ker\mathrm{ad}_V \neq 0$, we have that $\lambda_1 = \frac{1}{2}\lambda$. If $\mathfrak{v}_{\lambda-\lambda_1} \cap \ker\mathrm{ad}_V = 0$, it follows that $\mathfrak{v}_{\lambda-\lambda_1} = j_{\mathfrak{z}} V$ and $\mathfrak{v}$ has the direct sum decomposition $\mathfrak{v} = \mathfrak{v}_{\lambda_1} \oplus j_{\mathfrak{z}} V$, with $\mathfrak{v}_{\lambda-\lambda_1} = j_{\mathfrak{z}} V$ isomorphic to $\mathfrak{v}_{\lambda_1}$ by (i) of Proposition 2.4. In particular $\dim \mathfrak{v}_{\lambda_1} = \dim \mathfrak{v}_{\lambda-\lambda_1} = \dim \mathfrak{z} = k$. We assume next the last case holds.

Using that the associated riemannian manifold is Einstein and Proposition 1.1, (vii), from $Ric(H) = Ric(Z)$, for any unit $Z \in \mathfrak{z}$ we obtain

$$-\left(k\lambda^2 + k\lambda_1^2 + k(\lambda-\lambda_1)^2\right) = -\frac{1}{4}\mathrm{tr}\; j_Z^2 - 2k\lambda^2\,,$$

or equivalently,

$$-\frac{1}{4}\mathrm{tr}\; j_Z^2 = 2k\lambda_1(\lambda-\lambda_1)\,. \tag{5}$$

Let X be a unit vector in $\mathfrak{v}_{\lambda_1}$. Setting $Ric(H) = Ric(X)$ and using the expressions for the Ricci curvatures obtained in Proposition 1.1, (vii), it follows that

$$k\lambda^2 + k\lambda_1^2 + k(\lambda - \lambda_1)^2 = \frac{1}{2}\sum_{i=1}^{k} \mid j_{Z_i}X \mid^2 + 2k\lambda_1\lambda\,,$$

obtaining

$$(6)\quad \frac{1}{4}\sum_{i=1}^{k} \mid j_{Z_i}X \mid^2 = k(\lambda - \lambda_1)^2, \ \text{hence} \ -\frac{1}{4}\sum_{i=1}^{k} j_{Z_i}^2|_{\mathfrak{v}_{\lambda_1}} = k(\lambda - \lambda_1)^2\mathrm{Id}\,.$$

Similarly, from $Ric(H) = Ric(Y)$ for a unit $Y \in \mathfrak{v}_{\lambda-\lambda_1}$,

$$k\lambda^2 + k\lambda_1^2 + k(\lambda - \lambda_1)^2 = \frac{1}{2}\sum_{i=1}^{k} \mid j_{Z_i}Y \mid^2 + 2k\lambda(\lambda - \lambda_1)\,,$$

gives

$$(7)\qquad \frac{1}{4}\sum_{i=1}^{k} \mid j_{Z_i}Y \mid^2 = k\lambda_1^2\,, \ \text{hence} \ -\frac{1}{4}\sum_{i=1}^{k} j_{Z_i}^2|_{\mathfrak{v}_{\lambda-\lambda_1}} = k\lambda_1^2\mathrm{Id}\,.$$

If $\{Z_1, \dots, Z_k\}$ is an orthonormal basis of $\mathfrak{z}$, then using (5), $\mathrm{tr}\Big(-\frac{1}{4}\sum_{i=1}^{k} j_{Z_i}^2\Big) = 2k^2\lambda_1(\lambda - \lambda_1)$. On the other hand, from (6) and (7)

$$\mathrm{tr}\left(-\frac{1}{4}\sum_{i=1}^{k} j_{Z_i}^2\right) = k^2(\lambda - \lambda_1)^2 + k^2\lambda_1^2$$

gives the equality $2\lambda_1(\lambda - \lambda_1) = (\lambda - \lambda_1)^2 + \lambda_1^2$ with solution $\lambda_1 = \frac{1}{2}\lambda$, as asserted.

4. The Osserman-p condition and Damek Ricci spaces

We recall that S is called a Damek-Ricci space (see for instance [B-T-V]) when the associated metric Lie algebra of Iwasawa type $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$, $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ satisfies the extra conditions, ad_H acts as Id on $\mathfrak{z}$ and $\frac{1}{2}$ Id on $\mathfrak{v}$ and when $\mathfrak{z} \neq \{0\}$, the skew symmetric transformations J_Z, $Z \neq 0$ satisfy $J_Z^2 = -|Z|^2$ Id. It is known that Damek-Ricci spaces are Einstein manifolds of non positive curvature and contain as a subclass that of symmetric spaces of rank one and non compact type.

Lemma 4.1. *Let $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$ be a solvable Lie algebra of Iwasawa type and 2-step nilpotent $\mathfrak{n}$, satisfying the Osseman-p condition. If $\dim \mathfrak{z} \geq p - 1$, then $\mathfrak{n} = \mathfrak{z}$ is abelian. In particular S has constant negative sectional curvature.*

Proof. Let $E = \{H, Z_1, \dots, Z_{p-1}\}$ be an orthonormal set of p-vectors with $Z_i \in \mathfrak{z}$. We consider the operator $j_E = -\mathrm{ad}_H^2 + \sum_{1=1}^{p-1} R_{Z_i}$. First of all we compute,

$$j_E H = \sum_{1=1}^{p-1} R_{Z_i}H = -(p-1)\lambda^2 H\,, \ \ j_E Z_i = -(p-1)\lambda^2 Z_i,\ (i = 1, \dots, p-1)\,.$$

Hence $j_E|_{\mathfrak{z}+\mathbf{R}H} = -(p-1)\lambda^2\,\mathrm{Id}$. Note that since $\mathrm{ad}_H|_{\mathfrak{v}} = \frac{1}{2}\lambda\,\mathrm{Id}$, it follows that $R_{Z_i}|_{\mathfrak{v}} = -\frac{1}{4}j_{Z_i}^2 - \frac{1}{2}\lambda^2\,\mathrm{Id}$ (see Proposition 1.1). Thus $\mathfrak{v}$ is j_E-invariant and consequently the remaining eigenvalues of j_E have to be attained in $\mathfrak{v}$.

Assume there exists $X \in \mathfrak{v}$, with $X \neq 0$ and consider the orthonormal set of p vectors $E^* = \{Z_1, \ldots, Z_{p-1}, X\}$. It follows easily

$$j_{E^*}H = (-(p-1)\lambda^2 - \frac{1}{4}\lambda^2)H = -\frac{1}{4}(4p-3)\lambda^2 H\,,$$

hence $-\frac{1}{4}(4p-3)\lambda^2$ $(\neq -(p-1)\lambda^2)$, is an eigenvalue of j_E that has to be attained in $\mathfrak{v}$. Finally, any unit eigenvector Y of $j_E|_{\mathfrak{v}}$ has associated eigenvalue

$$\begin{aligned}\langle j_E Y, Y\rangle = \langle -\mathrm{ad}_H^2 Y + \sum_{1=1}^{p-1} R_{Z_i}Y, Y\rangle &= -\frac{1}{4}\lambda^2 + \sum_{1=1}^{p-1}\left(\frac{1}{4}\mid j_{Z_i}Y\mid^2 - \frac{1}{2}\lambda^2\right)\\ &= -\frac{1}{4}\left(1+2(p-1)\right)\lambda^2 + \frac{1}{4}\sum_{1=1}^{p-1}\mid j_{Z_i}Y\mid^2\ .\end{aligned}$$

In particular the equality $-\frac{1}{4}(4p-3)\lambda^2 = \langle j_E Y, Y\rangle$ for some $Y \in \mathfrak{v}$ gives

$$\frac{1}{4}(4p-3)\lambda^2 = \frac{1}{4}(2p-1)\lambda^2 - \frac{1}{4}\sum_{1=1}^{p-1}\mid j_{Z_i}Y\mid^2$$

which can not occur. Hence $\mathfrak{v} = 0$ and in this case $\mathfrak{n}$ is abelian. This implies that S has constant sectional curvature $-\lambda^2$.

Theorem 4.2. [1] *If S is a solvable Lie group of Iwasawa type, dimension n and 2-step nilpotent $[S,S]$ satisfying the Osserman-p condition for some $1 \leq p < n$, then S is (up to a re-scaling of the metric) either a Damek-Ricci space or a space of constant sectional curvature -1.*

Proof. Let $\mathfrak{s} = \mathfrak{n} \oplus \mathbf{R}H$ be the Lie algebra of S and let $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$. We may assume that $1 < p \leq \dim \mathfrak{v}$. In fact, if $p = 1$, it follows from [D-D] that S is a symmetric space of non compact type and rank one, hence a Damek-Ricci space. If $p > \dim \mathfrak{v} = n - \dim \mathfrak{z} - 1$, or equivalently $\dim \mathfrak{z} > (n-p) - 1$, it follows from the previous lemma that S has constant sectional curvature since the conditions Oss-p and Oss-$(n-p)$ are equivalent.

Next we show that $j_Z^2 = -\lambda^2 \mid Z\mid^2\mathrm{Id}$, for all nonzero $Z \in \mathfrak{z}$.

Let Z_1 be a unit vector in $\mathfrak{z}$ and let $\{Z_1, Z_2, \ldots, Z_k\}$ be an orthonormal basis of $\mathfrak{z}$. For an arbitrary and unit $X_1 \in \mathfrak{v}$, we consider the p-space $E_{r,s} = \{rZ_1 + sH, Z_2, \ldots, Z_k, X_1, \ldots, X_{p-k}\}$ with $r^2 + s^2 = 1$ and orthonormal vectors $X_i \in \ker \mathrm{ad}_{X_1}$, $i = 1, \ldots, p-k$. The last choice is possible since $\dim\ker\mathrm{ad}_{X_1} = \dim\mathfrak{v} - k \geq p-k$. By using the computation made in Proposition 2.1 of [D-D] (last part of the proof) and the fact that $\mathrm{ad}_H|_{\mathfrak{v}} = \frac{1}{2}\lambda\mathrm{Id}$, we have that

$$R_{rZ_1+sH}|_{\mathfrak{v}} = r^2 R_{Z_1} - s^2\frac{1}{4}\lambda^2\mathrm{Id}\,.$$

[1] Recently the authors have improved this result showing that the only Damek-Ricci space satisfying the Ossermann-p condition, for some $p > 1$ is the real hyperbolic space.

Hence X_1 is an eigenvector of

$$j_{E_{r,s}} = R_{rZ_1+sH} + \sum_{i=2}^{k} R_{Z_i} + \sum_{i=1}^{p-k} R_{X_i} \text{ for all } (r,s)$$

since $\sum_{i=1}^{k} R_{Z_i}|_{\mathfrak{v}} = -\frac{1}{4}k\lambda^2\mathrm{Id}$, (which follows from $-\frac{1}{4}\sum_{i=1}^{k} j_{Z_i}^2|_{\mathfrak{v}} = k\lambda^2$ Id) and $R_{X_i}|_{\ker ad_{X_1}} = -\frac{1}{4}\lambda^2\mathrm{Id}$. Thus the continuous map given by $(r,s) \to \langle j_{E_{r,s}}X_1, X_1\rangle$ must be constant as a function of (r,s). Using the expression

$$\langle j_{E_{r,s}}X_1, X_1\rangle = r^2\left(\frac{1}{4} \mid j_{Z_1}X_1 \mid^2 - \frac{1}{2}\lambda^2\right) - s^2\frac{1}{4}\lambda^2 + \langle\left(\sum_{i=2}^{k} R_{Z_i} + \sum_{i=1}^{p-k} R_{X_i}\right) X_1, X_1\rangle\,,$$

and taking into account that the last term above does not depend on (r,s), we obtain that

$$r^2\left(\frac{1}{4}|j_{Z_1}X_1|^2 - \frac{1}{2}\lambda^2\right) - s^2\frac{1}{4}\lambda^2$$

is independent of (r,s) in the circle $r^2+s^2=1$. Therefore, $|j_{Z_1}X_1|^2 = \lambda^2$. Since Z_1 and X_1 are arbitrary and the eigenvalues of j_Z^2 are of the form $|j_Z X|^2$, it then follows that $j_Z^2 = -\lambda^2|Z|^2\mathrm{Id}$ for any non zero $Z \in \mathfrak{z}$. Since ad_H acts as λ Id on $\mathfrak{z}$ and $\frac{1}{2}\lambda$ Id on $\mathfrak{v}$, re-scaling the metric we obtain that S is a Damek-Ricci space.

Corollary 4.3. *If S is the associated simply connected Lie group to a solvable Lie algebra of Iwasawa type of negative curvature and 2-step nilpotent nilradical satisfying the Osserman-p condition, $1 < p < dim\, S$, then S has constant negative sectional curvature.*

Proof. Since a Damek-Ricci space of negative curvature is a symmetric space of non compact type and rank one (see [D]) the corollary follows immediately after one checks that the real hyperbolic space is the only one satisfying Oss-p for some $1 < p < \dim S$.

If dim $\mathfrak{z} = 1$ and dim $\mathfrak{s} = n = 2r+2$ we may assume $1 < p \leq r+1$. In this case $\mathfrak{s}$ is spanned by the orthonormal basis

$$\{H, Z, X_1, \ldots, X_r, j_Z X_1, \ldots, j_Z X_r\}$$

with $Z \in \mathfrak{z}$, $X_i \in \mathfrak{v}$. Choosing $E = \{H, X_1, \ldots, X_{p-1}\}$ and computing the eigenvalues of j_E one obtains $-(p-1)\frac{1}{4}\lambda^2$, $-(p+3)\frac{1}{4}\lambda^2$ both with multiplicity p and $-p\frac{1}{4}\lambda^2$ with multiplicity $2(r-p+1)$. Next we consider the p-dimensional subspaces $E^* = \{H, X_1, \ldots, X_l, j_Z X_1, \ldots, j_Z X_l\}$ if p is odd and $E^* = \{X_1, \ldots, X_l, j_Z X_1, \ldots, j_Z X_l\}$ if p is even. In both cases X_1 is an eigenvector of j_{E^*} with eigenvalue $-(p+2)\frac{1}{4}\lambda^2$ contradicting constancy of the eigenvalues.

If $\dim \mathfrak{z} = 3$ and $\dim \mathfrak{s} = n = 4r+4$ we may assume $1 < p \leq 2r+2$. We fix an orthonormal basis of $\mathfrak{s}$ as follows

$$\{H, Z_1, Z_2, Z_3, X_1, j_{Z_1}X_1, j_{Z_2}X_1, j_{Z_3}X_1, \ldots, X_r j_{Z_1}X_r, j_{Z_2}X_r, j_{Z_3}X_r\}\,,$$

with $Z_i \in \mathfrak{z}$, $X_j \in \mathfrak{v}$. Choosing $E = \{Z_1, Z_2, X_1, \ldots, X_s, j_{Z_1}X_1, \ldots, j_{Z_1}X_s\}$ if p is even and $E = \{Z_1, X_1, \ldots, X_s, j_{Z_1}X_1, \ldots, j_{Z_1}X_s\}$ if p is odd it is possible to find all eigenvalues of j_E. They are given by $-(p+6)\frac{1}{4}\lambda^2$, $-(p+2)\frac{1}{4}\lambda^2$ and $-p\frac{1}{4}\lambda^2$ with multiplicities 2, $2+4s$, $2(r-s)$ respectively, in the case $p=2s+2$. When $p=2s+1$, the eigenvalues are, $-(p+3)\frac{1}{4}\lambda^2$, $-(p-1)\frac{1}{4}\lambda^2$, $-(p+2)\frac{1}{4}\lambda^2$ and $-p\frac{1}{4}\lambda^2$ with multiplicities 3, 1, $4s$, $4(r-s)$ respectively.

Changing the p-subspaces by $E^* = \{Z_1, Z_2, Z_3, X_2, \dots, X_l, j_{Z_1}X_1, \dots, j_{Z_1}X_l\}$ if p is even and $E^* = \{Z_1, Z_2, Z_3, X_4, \dots, X_l, j_{Z_1}X_1, \dots, j_{Z_1}X_l\}$ if p is odd, one shows that H is an eigenvector with corresponding eigenvalue $-(p+9)\frac{1}{4}\lambda^2$, p even and $-(p+8)\frac{1}{4}\lambda^2$, p odd.

The case $\dim \mathfrak{z} = 7$ follows immediately from Lemma 4.1 since being $\dim \mathfrak{s} = 16$ we may assume $p < 8$.

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