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TWISTOR SPINORS ON LORENTZIAN MANIFOLDS, CR-GEOMETRY AND FEFFERMAN SPACES

HELGA BAUM

ABSTRACT. The paper deals with twistor spinors on Lorentzian manifolds. In particular, we explain a relation between a certain class of Lorentzian twistor spinors and the Fefferman spaces of strictly pseudoconvex spin manifolds which appear in CR-geometry.

Let $(M^{n,k}, g)$ be a n-dimensional smooth semi-Riemannian spin manifold of index k with the spinor bundle S, $n \geq 3$. There are two conformally covariant differential operators of first order acting on the spinor fields $\Gamma(S)$, the Dirac operator D and the twistor operator $\mathcal{D}$. The Dirac operator is defined as the composition of the spinor derivative ∇^S with the Clifford multiplication μ

$$D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{\approx} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S),$$

whereas the twistor operator is the composition of the spinor derivative ∇^S with the projection p onto the kernel of the Clifford multiplication μ

$$\mathcal{D} : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{\approx} \Gamma(TM \otimes S) \xrightarrow{p} \Gamma(\operatorname{Ker} \mu).$$

The elements of the kernel of $\mathcal{D}$ are called *twistor spinors*. A spinor field φ is a twistor spinor if and only if it satisfies the *twistor equation*

$$\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$$

for each vector field X. The dimension of the space $\mathcal{T}(M^{n,k}, g)$ of all twistor spinors of $(M^{n,k}, g)$ is conformally invariant, finite and bounded by $d_n := 2 \cdot \operatorname{rang} S = 2^{[\frac{n}{2}]+1}$. For each simply connected, conformally flat semi-Riemannian spin manifold the dimension of the space of twistor spinors equals d_n. On the other hand the maximal dimension d_n of $\mathcal{T}(M^{n,k}, g)$ can only occur if (M, g) is conformally flat.

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Twistor spinors were introduced by R.Penrose in General Relativity (see [Pen67], [PR86], [NW84]). They are related to Killing vector fields in semi-Riemannian supergeometry (see [ACDS98]). In Riemannian geometry the twistor equation first appeared as an integrability condition for the canonical almost complex structure of the twistor space of an oriented four-dimensional Riemannian manifold (see [AHS78]). In the second half of the 80th A. Lichnerowicz and Th. Friedrich started the systematic investigation of twistor spinors on Riemannian spin manifolds from the view point of conformal differential geometry. Nowadays one has a lot of structure results and examples for manifolds with twistor spinors in the Riemannian setting (see e.g. [Lic88b], [Lic88a], [Lic89], [Wan89], [Fri89] [Lic90], [BFGK91], [Hab90], [Bär93], [Hab93], [Hab94], [KR94], [KR96], [KR97b], [KR97a], [KR98]).

In opposite to this, there is not much known about solutions of the twistor equation in the *pseudo*-Riemannian setting. Pseudo-Riemannian manifolds of arbitrary index k with special kinds of twistor spinors were studied e.g. in [Kat96], [Kat97], [Kat98], [BK97]. In the present paper we explain some of the known results concerning the *Lorentzian* twistor equation.

In the following we will consider only *Lorentzian* spin manifolds $(M^{n,1}, g)$ of dimension n (and index $k = 1$). First, let us give some examples of Lorentzian manifolds which admit twistor spinors.

Example 1. Let $M^{n,1} = \mathbb{R}^{n,1}$ be the Minkowski space. Then the spinor fields are the smooth functions from $\mathbb{R}^{n,1}$ to the spinor modul $\Delta_{n,1}$. A smooth function $\varphi \in C^\infty(\mathbb{R}^{n,1}, \Delta_{n,1})$ is a twistor spinor if and only if it is of the form $\varphi(x) = u + x \cdot v$, where u, v are constant spinors in $\Delta_{n,1}$ and $\cdot$ denotes the Clifford multiplication.

Example 2. In [Bau98b] we determined all twistor spinors on the indecomposible Lorentzian symmetric spaces explicitly. For example, let us consider the simply connected solvable Lorentzian symmetric space $M^n_{\underline{\lambda}} = (\mathbb{R}^n, g_{\underline{\lambda}})$, where

$$(g_{\underline{\lambda}})_{(s,t,x)} = 2\, ds\, dt + \sum_{j=1}^{n-2} \lambda_j x_j^2\, ds^2 + \sum_{j=1}^{n-2} dx_j^2$$

and $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n-2})$, $\lambda_i \in \mathbb{R}\backslash\{0\}$, $n \geq 3$. $M_{\underline{\lambda}}$ is Ricci flat iff $\sum_{j=1}^{n-2} \lambda_j = 0$. $M_{\underline{\lambda}}$ is conformally flat iff $\underline{\lambda} = (\lambda, \dots, \lambda)$, $\lambda \in \mathbb{R}\backslash\{0\}$. For the non-conformally flat spaces $M_{\underline{\lambda}}$ the dimension of the space $\mathcal{T}(M_{\underline{\lambda}})$ of twistor spinors is $\frac{1}{4} d_n$ and each twistor spinor is parallel. In particular, contrary to the Riemannian case, there exist Lorentzian manifolds with parallel spinors which are not Ricci flat, and there exist Lorentzian symmetric spaces with parallel spinors which are not flat.

The pseudosphere $S^n_1(r) \subset \mathbb{R}^{n+1,1}$ and the pseudo-hyperbolic space $H^n_1(r) \subset \mathbb{R}^{n+1,2}$ have the maximal possible number d_n of linearely independent twistor spinors.

Example 3. A special kind of twistor spinors are those which are simultaneous eigenspinors for the Dirac operator. These spinor fields satisfy the equation

$$\nabla^S_X \varphi = \lambda X \cdot \varphi \tag{1}$$

for all vector fields X, where λ is a complex number related to the scalar curvature R of the manifold by $R = 4n(n-1)\lambda^2$. Hence, if the scalar curvature is non-zero, the number λ is real and non-zero or purely imaginary and the spinor fields satisfying (1) are called *real Killing spinors* and *imaginary Killing spinors*, respectively.

If φ is a twistor spinor on an Einstein space $M^{n,1}$ of non-zero scalar curvature R, then the spinor fields

$$\psi_{\pm} := \frac{1}{2}\varphi \pm \sqrt{\frac{n-1}{nR}}\, D\varphi$$

are Killing spinors. Therefore, the study of non-Ricci flat Lorentzian Einstein spaces admitting twistor spinors reduces to the study of Lorentzian manifolds admitting Killing spinors. The geometric structure of Lorentzian manifolds admitting real Killing spinors was described by Ch.Bohle in [Boh98]. In particular, all such manifolds are Einstein spaces of positive scalar curvature. Imaginary Killing spinors occur for example on simply connected Lorentzian Einstein Sasaki manifolds. Contrary to the case of real Killing spinors, there exist also Lorentzian manifolds admitting imaginary Killing spinors which are not Einstein spaces. Examples for those are the warped products $(M^{n-1}_{\underline{\lambda}} \times \mathbb{R}^1, g)$ with the warped metric $g := \exp(2t)\, g_{\underline{\lambda}} + dt^2$, where $M^{n-1}_{\underline{\lambda}}$ is a non-Ricci flat symmetric space defined in Example 2.

Now, let $(M^{n,1}, g)$ be a *Lorentzian* spin manifold. For each spinor field $\varphi \in \Gamma(S)$ we define a canonical real vector field V_φ on M by

$$g(V_\varphi, X) := -\langle X \cdot \varphi, \varphi \rangle\,, \qquad X \in TM\,,$$

where $\langle .,. \rangle$ is the indefinit scalar product on S (see [Bau81]). The vector field V_φ has the same zero set as the spinor field φ. If $\varphi \in \Gamma(S)$ is a twistor spinor, the vector field V_φ has some special nice properties. In particular, we have

Proposition 1. *Let φ be a non-trivial twistor spinor on a Lorentzian spin manifold $(M^{n,1}, g)$ and let V_φ be its canonical vector field. Then*

1. *V_φ is a conformal vector field with divergence $div(V_\varphi) = 2\, Re\langle \varphi, D\varphi \rangle$.*
2. *If the dimension n is 4 and φ is a half spinor, then V_φ is an isotropic vector field which satisfies $V_\varphi \cdot \varphi = 0$ and annihilates the Weyl tensor W of $(M^{4,1}, g)$: $V_\varphi \lrcorner\, W = 0$.*

This proposition shows that the existence of a twistor spinor on a 4-dimensional space-time implies that each point of the space-time is of Petrov type N or O. One kind of space times of Petrov type N are the so-called Fefferman spaces which are known in CR-geometry. In 1991 J. Lewandowski proved the following

Proposition 2. ([Lew91])

(1) *Let φ be a twistor spinor on a 4-dimensional space-time $(M^{4,1}, g)$ without zeroes and with non-degenerate rotation $rot(V_\varphi)$. Then $(M^{4,1}, g)$ is locally conformal equivalent to a Fefferman space.*

(2) *There exist local solutions of the twistor equation on each 4-dimensional Fefferman space.*

In the remainder of the paper we want to explain how this result can be generalised to arbitrary even dimension. Detailed proofs of the statements can be found in [Bau97] or [Bau98a]. First we recall some notions from CR-geometry which are necessary to define the Fefferman spaces.

A *CR-manifold* is a pair (M^{2n+1}, T_{10}) consisting of a $(2n+1)$-dimensional smooth manifold M and a complex subbundle T_{10} of the complexified tangent bundle $TM^{\mathbb{C}}$ such that

- $\dim_{\mathbb{C}} T_{10} = n$,
- $T_{10} \bigcap \overline{T_{10}} = \{0\}$ and
- $[\Gamma(T_{10}), \Gamma(T_{10})] \subset \Gamma(T_{10})$.

The complex bundle T_{10} defines a $2n$-dimensional real subbundle $H := Re(T_{10} \oplus \overline{T_{10}})$ of TM with (integrable) almost complex structure $J : H \to H$, $J(U + \overline{U}) = i(U - \overline{U})$. Let (M, T_{10}) be an oriented CR-manifold. A *pseudo-hermitian structure* on (M, T_{10}) is a nowhere vanishing 1-form $\theta \in \Omega^1(M)$ satisfying $\theta(H) = 0$. The hermitian form $L_\theta : T_{10} \times T_{10} \longrightarrow \mathbb{C}$ defined by

$$L_\theta(U, V) := -i\, d\theta(U, \overline{V})$$

is called *Levi form* of (M, T_{10}, θ). In the following we suppose that (M, T_{10}, θ) is *strictly pseudoconvex*, i.e. $L_\theta > 0$. By T we denote the Reeb vector field of the contact form θ, which is determined by $\theta(T) = 1$ and $T \lrcorner\ d\theta = 0$. On the complex bundle T_{10} there exists an uniquely determined covariant derivative $\nabla^W : \Gamma(T_{10}) \longrightarrow \Gamma(T^*M^{\mathbb{C}} \otimes T_{10})$ given by the conditions

- $\nabla^W L_\theta = 0$,
- $\nabla^W_T U = proj_{10}[T, U]$,
- $\nabla^W_{\overline{V}} U = proj_{10}[\overline{V}, U]$,

where $U, V \in \Gamma(T_{10})$. ∇^W is called *Webster connection* and was introduced by Tanaka ([Tan75]). The Levi form and the Webster connection extend to the complexified tangent bundle $TM^{\mathbb{C}}$ using

- $L_\theta(\overline{U}, \overline{V}) := \overline{L_\theta(U, V)}$, $L_\theta(T_{10}, \overline{T_{10}}) := 0$, $L_\theta(T, \cdot) := 0$
- $\nabla^W \overline{U} := \overline{\nabla^W U}$, $\nabla^W T := 0$.

$g_\theta := L_\theta + \theta \circ \theta$ is a Riemannian metric on the strictly pseudoconvex manifold (M, T_{10}, θ). The Webster connection restricted to the tangent bundle TM is a metric connection on the Riemannian manifold (M, g_θ) with torsion.

Let $\mathcal{R}^W$ denote the (4,0)-curvature tensor of the Webster connection

$$\mathcal{R}^W(X, Y, Z, V) := g_\theta(([\nabla^W_X, \nabla^W_Y] - \nabla^W_{[X,Y]})Z, \overline{V}).$$

For a complex 2-form $\omega \in \Lambda^2 M^{\mathbb{C}}$ we define the θ-trace to be

$$Trace_\theta\ \omega := \sum_{\alpha=1}^{n} \omega(Z_\alpha, \overline{Z}_\alpha),$$

where $(Z_1, ..., Z_n)$ is an unitary basis of (T_{10}, L_θ). The *Webster-Ricci-curvature* is the (1,1)-form $Ric^W := Trace_\theta^{(3,4)} \mathcal{R}^W$ and the *Webster-scalar curvature* is the real function $R^W := Trace_\theta\ Ric^W$.

Now, let us come to the definition of the Fefferman spaces. Let

$$K := \Lambda^{n+1,0} M := \{\omega \in \Lambda^{n+1} M^{\mathbb{C}} \mid V \lrcorner\ \omega = 0 \ \ \forall\ V \in \overline{T_{10}}\}$$

be the canonical line bundle of the CR-manifold (M, T_{10}) and let us denote by (F, π, M) the associated S^1-principal bundle. The Fefferman space of the strictly pseudoconvex manifold (M, T_{10}, θ) is the total space F of this S^1-bundle equipped with a Lorentzian metric h_θ of the form $h_\theta := \pi^* L_\theta - i\, c\, \pi^*\theta \circ A_\theta$, where A_θ is a connection on the S^1-bundle (F, π, M) and c is a non-zero real number choosen in such a way that the conformal class $[h_\theta]$ of h_θ does not depend on the pseudo-hermitian form θ. Such a metric can be obtained in the following way (see [Lee86]):

Let A^W be the connection on the S^1-bundle F associated to the covariant derivative on the line bundle K induced by the Webster connection ∇^W. The curvature of A^W is given by the Webster-Ricci curvature: $\Omega^{A^W} = -Ric^W$. Then the connection A_θ is defined as

$$A_\theta := A^W - \frac{i}{2(n+1)} R^W \cdot \theta$$

and the metric h_θ by

$$h_\theta := \pi^* L_\theta - i \frac{4}{n+2} \pi^* \theta \circ A_\theta .$$

The Lorentzian manifold (F, h_θ) is called *Fefferman space* of the strictly pseudo-convex manifold (M, T_{10}, θ).

To obtain *global* solutions of the twistor equation we have to change the topological type of the S^1-bundle (F, π, M). If we suppose the existence of a spin structure on the Riemannian manifold (M, g_θ) this can be done using the following proposition.

Proposition 3. *Let (M, T_{10}, θ) be a strictly pseudoconvex spin manifold. Then each spin structure (Q_M, f_M) of (M, g_θ) defines a square root $\sqrt{F}$ of the canonical S^1-bundle (F, π, M).*

Now, let us consider the analogous construction in this S^1-bundle $(\sqrt{F}, \pi, M)$. We denote by $A^{\sqrt{W}}$ the connection on the S^1-bundle $\sqrt{F}$ defined by the Webster connection ∇^W, consider the modified connection

$$A_\theta^{\surd} := A^{\sqrt{W}} - \frac{i}{4(n+1)} R^W \cdot \theta$$

and define the Lorentzian metric on $\sqrt{F}$ by

$$h_\theta := \pi^* L_\theta - i \frac{8}{n+1} \pi^* \theta \circ A_\theta^{\surd}.$$

The Lorentzian manifold $(\sqrt{F}, h_\theta)$ can be equipped with a spin structure (Q, f) induced by (Q_M, f_M). The Lorentzian spin manifold $((\sqrt{F}, h_\theta), (Q, f))$ is called *Fefferman space of the strictly pseudoconvex spin manifold* $(M, T_{10}, \theta, (Q_M, f_M))$.

The spin structure (Q, f) induces a spin structure of the real vector bundle (H, L_θ). We denote the corresponding spinor bundle by S_H. Then we can prove the following

Proposition 4. *Let (M, T_{10}, θ) be a strictly pseudoconvex spin manifold with the Fefferman space $(\sqrt{F}, h_\theta)$ and the spinor bundle S_H. Then*

(1) *The 2-form $d\theta$ acts by Clifford multiplication as endomorphism on the spinor bundle S_H and has an eigenspace decomposition of the form*

$$S_H = S_{-ni} \oplus S_{-ni+2i} \oplus S_{-ni+4i} \oplus \ldots \oplus S_{ni-2i} \oplus S_{ni},$$

where the subbundles S_{ki} are the eigenspaces of $d\theta$ to the eigenvalue ki which have the rang $\binom{n}{(n+k)/2}$.

(2) *The lifts of the two line bundles S_{-ni} and S_{ni} over M to the Fefferman space $\sqrt{F}$ are trival bundles.*

(3) *The spinor bundle $S_{\sqrt{F}}$ of the Fefferman space can be identified with two copies of the lifted bundle S_H : $S_{\sqrt{F}} = \pi^* S_H \oplus \pi^* S_H$.*

(4) *There exist global sections $\psi_\pm$ in the trivial line bundles $\pi^* S_{\pm ni}$ such that the spinor fields*

$$\phi_\pm = (\psi_\pm, 0)$$

are twistor spinors on the Fefferman space $(\sqrt{F}, h_\theta)$.

Studying the properties of the spinor fields $\phi_\pm$ we obtain

Theorem 1. *Let $(M^{2n+1}, T_{10}, \theta)$ be a strictly pseudoconvex spin manifold and let $(\sqrt{F}, h_\theta)$ be its Fefferman space. Then there exist two linearely independent twistor spinors φ on $(\sqrt{F}, h_\theta)$ with the following properties:*

(1) *The associated vector field V_φ is regular, isotropic and Killing.*

(2) $V_\varphi \cdot \varphi = 0$.

(3) $\nabla^S_{V_\varphi} \varphi = i\, c\, \varphi, \qquad$ *where* $c \in \mathbb{R} \setminus \{0\}$.

On the other hand it holds

Theorem 2. *Let (B^{2n+2}, h) be a Lorentzian spin manifold which admits a non-trivial twistor spinor satisfying*

(1) *The associated vector field V_φ is regular, isotropic and Killing.*

(2) $V_\varphi \cdot \varphi = 0$.

(3) $\nabla^S_{V_\varphi} \varphi = i\, c\, \varphi, \qquad$ *where* $c \in \mathbb{R} \setminus \{0\}$.

Then there exists a strictly pseudoconvex spin manifold $(M^{2n+1}, T_{10}, \theta)$ *such that* (B, h) *is locally isometric to the Fefferman space* $(\sqrt{F}, h_\theta)$ *of* (M, T_{10}, θ).

The proof of Theorem 2 is based on the following characterisation of Fefferman spaces given by Sparling and Graham ([Gra87]):
Let (B^m, h) be a Lorentzian manifold and let us denote by R the scalar curvature, by Ric the Ricci-curvature, by W the Weyl tensor, by K the Schouten tensor

$$K := \frac{1}{m-2}\left\{\frac{1}{2(m-1)} R \cdot h - Ric\right\},$$

and by C the Schouten-Weyl-tensor

$$C(X, Y, Z) := h\Big(X, (\nabla_Y K)(Z) - (\nabla_Z K)(Y)\Big)$$

of (B, h). If V is a regular isotropic Killing field on (B, h) such that $V \lrcorner W = 0$, $V \lrcorner C = 0$ and $K(V, V) = const < 0$, then there exists a strictly pseudoconvex manifold (M, T_{10}, θ) such that (B, h) is locally isometric to the Fefferman space (F, h_θ) of (M, T_{10}, θ).

For each twistor spinor on (B^m, h) the following equations are satisfied (see [BFGK91]):

- $W(\eta) \cdot \varphi = 0$,
- $W(X \wedge Y) \cdot D\varphi = m\, C(X, Y) \cdot \varphi$,
- $\nabla^S_X D\varphi = \frac{m}{2}\, K(X) \cdot \varphi$.

These conditions imply that for each twistor spinor φ the equation $V_\varphi \lrcorner C = 0$ holds. Using in addition the assumptions of Theorem 2 we obtain $V_\varphi \lrcorner W = 0$ and $K(V_\varphi, V_\varphi) = -c^2 < 0$.

Remark. If we proceed in the same way as above in Kählerian geometry we end up with Lorentzian manifolds admitting imaginary Killing spinors: Let (M^{2m}, g, J) be a Kähler-Einstein spin manifold of negative scalar curvature $R_M < 0$. Let us denote by (B, π, M) the S^1-principal bundle associated to the square root of the canonical bundle $K := \Lambda^{m,0}M$ defined by the spin structure of (M, g) and by A the connection on B defined by the Levi-Civita-connection of (M, g). We consider the Lorentzian metric $h := \pi^* g - \frac{16m}{R_M(m+1)} A \circ A$. The manifold (B, h) is a Lorentzian Einstein-Sasaki spin manifold. The spinor bundle S_M of (M, g, J) decomposes into the eigenspaces S_{ki} of the Kähler-form ω to the eigenvalues ki: $S_M = S_{-im} \oplus S_{-im+2i} \oplus S_{-mi+4i} \oplus ... \oplus S_{mi-2i} \oplus S_{mi}$. The spinor bundle S_B of (B, h) is isomorphic to the lift $\pi^* S_M$. There exist global sections ψ_ε in the line bundles $\pi^* S_{\varepsilon mi} \subset S_B$ which are imaginary Killing spinors to the Killing number $\lambda_\varepsilon := (-1)^m \varepsilon^{m+1} \sqrt{\frac{-R_M}{16m(m+1)}}\, i$, $\varepsilon = \pm 1$. This is a special case of an observation due to I. Kath in a more general situation (see [Kat98]).

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Humboldt University, Department of Mathematics,
Ziegelstrasse 13a, 10099 Berlin, GERMANY
E-mail: baum@mathematik.hu-berlin.de