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THE MODULI SPACE $\mathcal{M}(M)$ OF CERTAIN NON–ORIENTABLE MANIFOLDS OF EVEN DIMENSION

EGIDIO BARRERA-YANEZ

Abstract. Let M be an even dimensional manifold with $\pi_1(M) = \mathbb{Z}_\ell$ for $\ell = 2^\nu \geq 2$. We assume that the universal cover $\tilde{M}$ is *spin*. We shall define $N(M) = \tilde{M} \times \tilde{M}/\mathbb{Z}_{2\ell}$ and establishe a relation between the eta invariant of M and the eta invariant of $N(M)$. We use this to study the moduli space of metrics of positive scalar curvature on the manifold M.

1.0 Introduction. We shall use the eta invariant (which is an analytical invariant) and equivariant spin bordism (which is a topological invariant) to study the space of metrics of positive scalar curvature on a manifold (this is a geometrical question).

Let $\ell := 2^\nu \geq 2$ henceforth; we are only interested in 2 primary phenomena. Let $g_\ell = e^{2\pi\sqrt{-1}/\ell}$ be the canonical generator of the cyclic group $\mathbb{Z}_\ell = \{\lambda \in \mathbb{C} : \lambda^\ell = 1\}$. Let $\rho_s(g_\ell) := e^{2\pi s\sqrt{-1}/\ell}$ define a linear representation of $\mathbb{Z}_\ell$.

Let M be a closed manifold of dimension $m \geq 5$, let $\tilde{M}$ be the universal cover of M, assume that $\tilde{M}$ has a given *spin* structure. Let $\mathcal{R}(M)$ be the space of metrics on M with positive scalar curvature, let $\mathcal{M}(M) = \mathcal{R}(M)/Diff(M)$ be the associated moduli space. A homomorphism $\sigma : \pi_1(M) \to \mathbb{Z}_\ell$ defines a map f_σ (which is unique up to homotopy) from M to the classifying space $B\mathbb{Z}_\ell$ of $\mathbb{Z}_\ell$. If $\pi_1(M) = \mathbb{Z}_\ell$, then the natural homomorphism σ is the identity map. In 1.1 and 1.2 we define the equivariant spin bordism groups, in 1.3 we present the eta invariant which is the analytical invariant that we use in the study of the moduli space $\mathcal{M}(M)$. In 1.4 we define a twisted product $N(M)$ of dimension $2m$ associated to M which is non-orientable and which has a canonical pin^c structure. In 1.8 we shall show that the moduli space of metrics of positive scalar curvature on certain non-orientable manifolds has an infinite number of arc components; see Theorem 1.10 for details.

1.1 Equivariant spin bordism. Let ξ be a real vector bundle over $B\mathbb{Z}_\ell$. The equivariant spin bordism groups $MSpin_m(B\mathbb{Z}_\ell, \xi)$ are the equivalence classes of triples (M, f, s) where M is a closed manifold of dimension m which need not be connected, f is a $\mathbb{Z}_\ell$ structure on M or equivalently a representation of the

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fundamental group $\pi_1(M)$, and s is a *spin* structure on $T(M) \oplus f^*\xi$; we define the relation $(M, f, s) \sim 0$ in $MSpin_m(B\mathbb{Z}_\ell, \xi)$ if there exists a compact manifold X with boundary M so that the structures f and s extend over X. Let M be a manifold with $\pi_1(M) = \mathbb{Z}_\ell$ whose universal cover $\tilde{M}$ admits a *spin* structure. Let ρ be a representation of $\mathbb{Z}_\ell$. There exists bundles and suitable structures s so that $[(M, f, s)] \in MSpin_m(B\mathbb{Z}_\ell, \xi)$. Only two Stiefel Whitney classes ω_i for $i = 1, 2$ play a role. There are 4 cases to consider:

(1) We have $\omega_1(\xi_0) = 0$ and $\omega_2(\xi_0) = 0$. We may take ξ_0 to be the trivial line bundle and identify $MSpin_m(B\mathbb{Z}_\ell, \xi_0)$ with the ordinary bordism groups $MSpin_m(B\mathbb{Z}_\ell)$; such a manifold admits a canonical *spin* structure s.

(2) We have $\omega_1(\xi_1) = 0$ and $\omega_2(\xi_1) \neq 0$. We may take ξ_1 to be the underlying real 2 plane bundle of the complex line bundle defined by the representation ρ_1. Such a manifold M admits a canonical $spin^c$ structure s with determinant line bundle given by ρ_1.

(3) We have $\omega_1(\xi_2) \neq 0$ and $\omega_2(\xi_2) = 0$. We may take the real line bundle define by $\rho_{\ell/2}$. Such a manifold M admits a canonical *pin* structure s.

(4) We have $\omega_1(\xi_3) \neq 0$ and $\omega_2(\xi_3) \neq 0$. We may take $\xi_3 = \xi_1 \oplus \xi_2$. Such a manifold M admits a canonical pin^c structure s with determinant line bundle given by ρ_1.

1.2 Geometric equivariant spin bordism. Let $\tau := R_{ijji}$ be the scalar curvature of a Riemannian metric g; we consider quadruples (M, f, s, g) where (M, f, s) are as in §1.1, and g is a metric of positive scalar curvature on M; necessarily $m \geq 2$. The geometric equivariant spin bordism groups ${}^+MSpin_m(B\mathbb{Z}_\ell, \xi)$ are defined similarly to the bordism groups $MSpin_m(B\mathbb{Z}_\ell, \xi)$. For $m \geq 2$ we consider quadruples (M, f, s, g) as above and say that $(M, f, s, g) \sim 0$ if there exists a compact manifold X with boundary M so that the structures s and f extend over X and so that the metric g extends over X as a metric of positive scalar curvature which is the product near the boundary M.

1.3 The eta invariant. Let M be a compact Riemannian manifold of dimension m. If m is even, we assume that M admits a pin^c structure s; if m is odd, we assume that M admits a $spin^c$ structure s; see Barrera-Yanez [3] and Gilkey [5] for further details. We assume that M is equipped with a $\mathbb{Z}_\ell$ structure f. If ρ is a representation of $\mathbb{Z}_\ell$, let D_ρ be the Dirac operator defined by a metric g and the $(s)pin^c$ structures with coefficients in the locally flat bundle of spinors $\mathcal{S}_\rho$ over M defined by ρ. The operator D_ρ has a discrete spectral resolution. Let $\{\lambda_n\}$ be the eigenvalues of D_ρ where each eigenvalue is repeated according to multiplicity; we have $|\lambda_n| \to \infty$. The eta function of Atiyah, Patodi, and Singer [1] is defined by

$$\eta(D_\rho)(z) := \textstyle\sum_{\lambda_n \neq 0} \operatorname{sign}(\lambda_n)|\lambda_n|^{-z} + \dim\ker D_\rho \,.$$

This sum converges to define a holomorphic function of z for $\Re(z) \gg 0$; it has a meromorphic extension to $\mathbb{C}$ which is regular at $z = 0$ see [5] for details. The eta invariant is a measure of the spectral asymmetry of D_ρ which is defined by

$$\eta(M)(\rho) := \frac{1}{2}\{\eta(D_\rho)(z)\}_{z=0}.$$

The eta invariant is additive with respect to direct sums and extends to the group representation ring $RU(\mathbb{Z}_\ell)$.

1.4 Twisted Products. Assume that M has fundamental group $\mathbb{Z}_\ell$, that the universal cover $\tilde{M}$ of M is *spin*, and that M is connected. If m is odd, assume that M is orientable; if m is even, assume that M is non-orientable. Thus the deck group action $\tilde{x} \to g_\ell \tilde{x}$ preserves the orientation of $\tilde{M}$ if m is odd and reverses the orientation if m is even. Give $\tilde{M} \times \tilde{M}$ the product *spin* structure. We define a fixed point free action of $\mathbb{Z}_{2\ell}$ on $\tilde{M} \times \tilde{M}$ by $g_{2\ell} : (\tilde{x}, \tilde{y}) \to (g_\ell \tilde{y}, \tilde{x})$. Let

$$N := N(M) := \tilde{M} \times \tilde{M}/\mathbb{Z}_{2\ell}$$

be the resulting quotient manifold. If m is odd, the flip reverses the orientation and g_ℓ preserves the orientation. If m is even the flip preserves the orientation and g_ℓ reverses the orientation. Thus the action of $g_{2\ell}$ on $\tilde{M} \times \tilde{M}$ reverses the orientation so N is not orientable. We refer to [2, 3] for the proof of the following Lemma:

1.5 Lemma. *Let m be even, with the notation established above, we have:*

(1) *If $\ell = 2$ and if m is even, then $[N(M)] \in MSpin_{2m}(B\mathbb{Z}_4, \xi_3)$ admits a canonical pin^c structure.*
(2) *If $\ell \geq 4$ and if m is even, then $[N(M)] \in MSpin_{2m}(B\mathbb{Z}_{2\ell}, \xi_2)$ admits a canonical pin^- structure.*
(3) *If m is odd, then $[N(M)] \in M_{2m}(B\mathbb{Z}_{2\ell}, \xi_i)$ for $i = 2, 3$ admits a canonical pin^- or pin^c structure.*

We use these structures to define Dirac operators on M and N with coefficients in the line bundle defined by ρ_s and let $\eta(M)(\rho_s)$ and $\eta(N)(\rho_s)$ be the resulting eta invariants, see [3,6] for details. Let $b = 0$ if M admits a $(s)pin^-$ structure s so that the associated complex line bundle $\det(s)$ is trivial, let $b = 1$ if M admits a $(s)pin^c$ structure s so that the associated complex line bundle $\det(s)$ is given by the representation ρ_1. Then $\det(s) = \rho_b$. If m is odd, Gilkey proved [6] that:

1.6 Theorem. *Let m be odd and let M and N be as above. If $m \equiv 3 \mod 4$, let $\beta = 0$; if $m \equiv 1 \mod 4$, let $\beta = \ell/2$.*

(1) *If $u = 2s - b + \beta$, then $\eta(N)(\rho_u) = \eta(M)(\rho_s) + \eta(M)(\rho_{s-\ell/2})$ in $\mathbb{R}/\mathbb{Z}$.*
(2) *If $u = 2s - b + \beta + 1$, then $\eta(N)(\rho_s) = 0$ in $\mathbb{R}/\mathbb{Z}$.*
(3) *If there are no harmonic spinors on $\tilde{M}$, the equalities above hold in $\mathbb{R}$ not just $\mathbb{R}/\mathbb{Z}$.*

If m is even we have the following Theorem see [2,3] for details.

1.7 Theorem. *Let m be even and let M and N be as above.*

(1) *If $\ell = 2$, then we have:*
    1-a) *If $u = 2s - b + m/2$ then $\eta(N)(\rho_u) = \eta(M)(\rho_s)$ in $\mathbb{R}/\mathbb{Z}$.*
    1-b) *If $u = 2s - b + 1 + m/2$, then $\eta(N)(\rho_u) = \eta(M)(\rho_s)$ in $\mathbb{R}/\mathbb{Z}$.*
(2) *If $\ell > 2$, then we have:*

2-a) *If* $u = 2s - b + m/2 + \ell/4$, *then* $\eta(N)(\rho_u) = \eta(M)(\rho_s + \rho_{s+\ell/4})$ *in* $\mathbb{R}/\mathbb{Z}$.

2-b) *If* $u = 2s - b + m/2 + \ell/4 + 1$, *then* $\eta(N)(\rho_u) = 0$ *in* $\mathbb{R}/\mathbb{Z}$.

(3) *If there are no harmonic spinors on* $\tilde{M}$, *then these equalities hold in* $\mathbb{R}$.

1.8 Moduli space of metrics of positive scalar curvature. Let $\mathcal{R}(M)$ be the space of metrics of positive scalar curvature on a manifold M and let $\mathcal{M}(M)$ be the moduli space of metrics of positive scalar curvature. We say that two metrics g_i for $i = 1, 2$ are geometrically bordant if $[(M, f, s, g_1)] = [(M, f, s, g_2)]$ in ${}^+MSpin(B\mathbb{Z}_\ell, \xi)$. We shall need the following result due to Botvinnik and Gilkey [4] that uses work of Gromov and Lawson [7], Rosenberg [8], Rosenberg and Stolz [9] and Schoen and Yau [10].

1.9 Theorem. *Let* M *be a closed connected manifold of dimension* $m \geq 5$ *with fundamental group* $\mathbb{Z}_\ell$ *so that* $[(M, f, s)] \in MSpin_m(B\mathbb{Z}_\ell, \xi_i)$. *If* m *is odd, assume* $i = 0, 1$; *if* m *is even, assume* $i = 2, 3$. *Let* $\rho \in RU(\mathbb{Z}_\ell)$.

(1) *If* $[(M, f, s)] = [(M_1, f_1, s_1)]$ *in* $MSpin_m(B\mathbb{Z}_\ell, \xi)$ *where* M_1 *admits a metric of positive scalar curvature* g_1, *then* M *admits a metric of positive scalar curvature* g *such that* $[(M, f, s, g)] = [(M_1, f_1, s_1, g_1)]$ *in* ${}^+MSpin_m(B\mathbb{Z}_\ell, \xi)$.
(2) *If* $[(M_2, f_2, s_2, g_2)] = 0$ *in* ${}^+MSpin_m(B\mathbb{Z}_\ell, \xi)$, *then* $\eta(M_2)(\rho) = 0$ *in* $\mathbb{R}$.
(3) *If* M *admits a metric of positive scalar curvature and if there exists any element* $[(M_3, f_3, s_3, g_3)]$ *in* ${}^+MSpin_m(B\mathbb{Z}_\ell, \xi)$ *with* $\eta(M_3)(\rho) \neq 0$ *in* $\mathbb{R}$, *then* $\mathcal{M}(M)$ *has an infinite number of components and there exists a countable family of metrics* g_j *of positive scalar curvature on* M *which are not geometrically bordant.*

We can now state the main theorem of this note:

1.10 Theorem. *Let* X *be an even dimensional non-orientable manifold of dimension* $m \geq 6$ *which admits a metric of positive scalar curvature. Assume that the fundamental group of* X *is* $\mathbb{Z}_{2\ell}$ *and that the universal cover of* X *is spin. Assume that one of the following cases given below holds.*

(1) *We have that* $\ell = 2$, $m \equiv 0 \mod 4$ *and* $\omega_2(X) \neq 0$.
(2) *We have that* $\ell = 2$, $m \equiv 2 \mod 4$ *and* $\omega_2(X) = 0$.
(3) *We have that* $\ell = 4$, $m \equiv 0 \mod 4$ *and* $\omega_2(X) \neq 0$.
(4) *We have that* $\ell \geq 4$, $m \equiv 2 \mod 4$ *and* $\omega_2(X) = 0$.
(5) *We have that* $\ell \geq 8$, $m \equiv 4 \mod 8$, $m \geq 12$ *and* $\omega_2(X) = 0$.

Then the moduli space $\mathcal{M}(X)$ *of metrics of positive scalar curvature on* X *has an infinite number of components and there exists a countable family of metrics* g_j *on* X *which have positive scalar curvature and which are not geometrically bordant.*

Proof. We apply Theorem 1.9. We must only construct a manifold N so that N admits a metric of positive scalar curvature, so that $N \in MSpin_{2m}(B\mathbb{Z}_{2\ell}, \xi)$, and so that $\eta(N)(\rho) \neq 0$ for some ρ. We take $N = N(M)$ where (M, ξ) are as given by:

(1) Let $m = 4k$, $\xi = \xi_3$ and $M := \mathbb{RP}^{4k}$.
(2) Let $m = 4k + 2$, $\xi = \xi_2$ and $M := \mathbb{RP}^{4k+2}$.
(3) Let $m = 4k$, $\xi = \xi_3$ and $M := (S^{2k} \times S^{2k})/\mathbb{Z}_4$.

(4) Let $m = 4k+2$, $\xi = \xi_2$ and $M := (S^{2k+1} \times S^{2k+1})/\mathbb{Z}_\ell$.
(5) Let $m = 8k+4$, $\xi = \xi_2$ and $M := ((S^{2k+1} \times S^{2k+1}) \times (S^{2k+1} \times S^{2k+1}))/\mathbb{Z}_\ell$.

We use the formulas in Theorems 1.6 and 1.7 to show that the eta invariant of $N(M)$ is non-trivial. □

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Departamento de Matematicas CINVESTAV-IPN
Apartado Postal 14-740, 07000 Mexico, D. F. MEXICO
E-mail: ebarrera@math.cinvestav.mx