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CLASSICAL PARTICLE PHASE SPACE IN GENERAL RELATIVITY

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Abstract. The geometric structure of phase space in the Einstein general relativistic framework is studied. Namely, the jet space of time-like submanifolds of space-time is introduced and the associated connections, jet–contact and contact structure are analysed. Moreover, the electromagnetic field is incorporated in the gravitational structures. A rigorous mathematical treatment of units of measurement is included as well.

The above setting is regarded as a framework aimed at extending to the Einstein case a recent quantisation procedure in the Galilei case, based on jets, connections and co–symplectic forms. According to this purpose, a structural comparison between Einstein's and Galilei's frameworks is performed.

Introduction

Recently, a covariant geometric approach to general relativistic quantum mechanics based on jets, general connections and co–symplectic forms has been presented in a Galilei general relativistic background (i.e. in a curved space–time with absolute time). The case of a scalar particle has been studied in [4], [5]; later, this formulation has been extended to a spin particle in [1].

The authors started from Galilei's general relativistic model just as first step toward Einstein's case. Actually, several interesting techniques developed in the Galilei case can be fruitfully used also in the Einstein case, after a certain adjustment. This paper is aimed at transferring to the Einstein space–time the basic geometrical techniques developed in the Galilei case, in view of a covariant formulation of quantum mechanics in the Einstein framework by an approach analogous to [1], [4], [5].

The Galilei space–time is assumed to be a fibred manifold equipped with a vertical Riemannian metric and a fibre preserving linear connection. On the other

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hand, the Einstein space time is assumed to be a time orientable Lorentzian manifold. A fundamental role in the classical and quantum theory is played by the classical phase space, namely by the first jets of sections in the Galilei case and by the first jets of time–like submanifolds in the Einstein case.

Essentially, the goal of the present paper is to show how typical geometrical constructions arising from the fibring in the Galilei case can be recovered through the Lorentzian metric in the Einstein case. The price we pay consists in the fact the several objects living on space–time in the Galilei case have their counterpart living on the jet space in the Einstein case. Actually, this fact encodes one of the main difficulties for a covariant quantum theory in the Einstein background.

Summarising the main results of the paper, we obtain the Einstein analogues of the differential of the absolute time function, of the tangent sequence of space–time fibred manifold, of its splitting over the jet space, of the first and second order connections on the jet bundle and of the co–symplectic form. These objects have a key role in the Galilei's case, hence we expect they will be an essential tool for pursuing the programme toward a covariant quantum mechanics in Einstein's case.

Our approach yields a structural comparison between Galilei's and Einstein's settings.

Throughout the paper all manifolds and maps are assumed to be smooth.

We shall be involved with non–linear connections on fibred manifolds. So, we recall a few basic notions and results (see [14]).

Let us consider a fibred manifold $q : F \to B$. We denote a fibred chart of F by (x^φ, y^i) and the induced chart of TF by $(x^\varphi, y^i, \dot{x}^\varphi, \dot{y}^i)$.

We recall that a (general) *connection* can be defined as a tangent valued form

$$C : F \to T^*B \underset{F}{\otimes} TF$$

projectable over id_{TB}. The connection C is also characterised by the vertical projection, or, equivalently, by the vertical valued form

$$\nu_C : TF \to VF\,, \qquad \nu_C : F \to T^*F \underset{F}{\otimes} VF\,.$$

We have the coordinate expressions

$$C = d^\varphi \otimes (\partial_\varphi + C_\varphi{}^i \partial_i)\,, \qquad \nu_C = (d^i - C_\varphi{}^i d^\varphi) \otimes \partial_i\,, \qquad C_\varphi{}^i \in C^\infty(F)\,.$$

The curvature and the torsion of C can be expressed in a unified way by means of the Frölicher-Nijenhuis bracket $[\,,\,]$. The torsion requires the choice of a vertical valued 1–form

$$\Phi : F \to T^*B \underset{F}{\otimes} VF\,.$$

Actually, in most cases, the geometrical structure of $q : F \to B$ exhibits a distinguished Φ; for instance, the standard torsion of linear connections on a manifold M are taken with respect to the basic form $id_M : M \to T^*M \otimes_M TM$. Curvature and torsion fulfill generalised first and second Bianchi's identities, which are again expressed through the Frölicher-Nijenhuis bracket.

Thus, the *curvature* and the *torsion* with respect Φ are defined, respectively, as the 2–forms

$$R_C = \frac{1}{2}[C, C] : F \to \wedge^2 T^* B \underset{F}{\otimes} VF\,, \qquad T_C = [C, \Phi] : F \to \wedge^2 T^* B \underset{F}{\otimes} VF\,,$$

with coordinate expressions, respectively,

$$R_C = (\partial_\lambda C_\mu{}^i + C_\lambda{}^j\, \partial_j C_\mu{}^i)\, d^\lambda \wedge d^\mu \otimes \partial_i\,, \tag{0.1}$$

$$T_C = (\partial_\lambda \Phi_\mu{}^i + C_\lambda{}^j\, \partial_j \Phi_\mu{}^i - \Phi_\mu{}^j\, \partial_j C_\lambda{}^i)\, d^\lambda \wedge d^\mu \otimes \partial_i\,. \tag{0.2}$$

We assume the following fundamental unit spaces [4]:
(1) the oriented 1–dimensional vector space $\mathbb{T}$ over $I\!R$ of *time intervals*,
(2) the positive 1–dimensional semi–vector space $\mathbb{L}$ over $I\!R^+$ of *lengths*,
(3) the positive 1–dimensional semi–vector space $\mathbb{M}$ over $I\!R^+$ of *masses*.
A *time unit of measurment* is defined to be an oriented basis of $\mathbb{T}$ or its dual

$$u_0 \in \mathbb{T}\,, \qquad u^0 \in \mathbb{T}^*\,.$$

Further details about the mathematical formalism on units of measurement can be found in [1], [5].

We shall be involved with the following construction. Let M be a manifold and consider the vector bundle $\pi_M : \mathbb{T}^* \otimes TM \to M$. The projection $T\pi_M : T(\mathbb{T}^* \otimes TM) \to TM$ can be regarded as vector valued 1-form

$$\sigma : \mathbb{T}^* \otimes TM \to T^*(\mathbb{T}^* \otimes TM) \underset{\mathbb{T}^* \otimes TM}{\otimes} TM\,;$$

its coordinate expression is

$$\sigma = d^\varphi \otimes \partial_\varphi\,. \tag{0.3}$$

1. Galilei's case

In this section we briefly recall the basic geometric constructions on Galilei's general relativistic space–time introduced in [4], [5], along with some additional results on the correspondence between space–time connections and connections on the phase space and the 'universal' electric and magnetic fields.

1.1. Gravitational structures

We start with the phase space and the gravitational structures in the Galilei framework.

Assumption G.1. We assume *space–time* to be a 4-dimensional oriented fibred manifold

$$t : E \to \boldsymbol{T}\,,$$

over a 1-dimensional oriented affine space $\boldsymbol{T}$ (time), associated with the vector space $\mathbb{T}$, equipped with a scaled Riemannian metric on the fibres, i.e. with a *scaled vertical Riemannian metric*,

$$g : E \to \mathbb{L}^2 \otimes V^*E \underset{E}{\otimes} V^*E \,. \quad \square$$

We observe that g can be regarded as a non-scaled Riemannian metric on the vector bundle $\mathbb{L}^* \otimes VE \to E$.

The typical space–time coordinate charts, adapted to the fibring, to a time unit of measurement u_0 and to the space–time orientation, will be denoted by (x^0, x^i). Throughout this section, the index 0 will refer to the base space, Latin indices $i, j, p, \ldots = 1, 2, 3$ will refer to the fibres, while Greek indices $\lambda, \mu, \varphi, \dots = 0, 1, 2, 3$ will refer both the base space and the fibres.

In coordinates

$$g = g_{ij} \check{d}^i \otimes \check{d}^j \,, \qquad g_{ij} : E \to \mathbb{L}^2 \otimes I\!R \,, \qquad |g| \equiv \det(g_{ij}) > 0 \,,$$

where the check '$\check{\ }$' denotes vertical restriction.

A *motion* in E is defined to be a section $s : \boldsymbol{T} \to E$.

The *phase space* is defined to be the first jet space of sections $\pi^1_0 : JE \equiv J_1E \to E$. A relevant feature of the Galilei case is due to the affine structure of the bundle $JE \to E$. A first consequence is the fibred isomorphism over JE

$$V_E JE \simeq JE \underset{E}{\times} \mathbb{T}^* \otimes VE \,.$$

We deal with the natural complementary jet–contact maps

$$д : JE \times \mathbb{T} \to TE \,, \quad \vartheta : JE \underset{E}{\times} TE \to VE \,,$$

or equivalently

$$д : JE \to \mathbb{T}^* \otimes TE \,, \quad \vartheta : JE \to T^*E \underset{E}{\otimes} VE \,,$$

which split the natural exact sequence

$$0 \longrightarrow VE \longrightarrow TE \xrightarrow{dt} E \times \mathbb{T} \longrightarrow 0 \,, \tag{1.1}$$

through the exact sequence over JE

$$0 \longrightarrow JE \times \mathbb{T} \xrightarrow{д} JE \underset{E}{\times} TE \xrightarrow{\vartheta} JE \underset{E}{\times} VE \longrightarrow 0 \,. \tag{1.2}$$

We have the coordinate expressions

$$д = u^0 \otimes д_0 = u^0 \otimes (\partial_0 + x^i_0 \partial_i) \,, \qquad \vartheta = \vartheta^i \otimes \partial_i = (d^i - x^i_0 d^0) \otimes \partial_i \,, \tag{1.3}$$

where (x^0, x^i, x^i_0) is the induced coordinate chart on JE.

A connection K on the bundle $TE \to E$ can be expressed, equivalently, by a tangent valued form, or by a vertical valued form

$$K : TE \to T^*E \underset{TE}{\otimes} TTE \,, \qquad \nu_K : TE \to T^*TE \underset{TE}{\otimes} TE \,,$$

respectively, with coordinate expressions

$$K = d^\varphi \otimes (\partial_\varphi + K_\varphi{}^\mu \dot{\partial}_\mu)\,, \quad \nu_K = (\dot{d}^\mu - K_\varphi{}^\mu d^\varphi)\otimes \partial_\mu\,, \tag{1.4}$$

$K_\varphi{}^\mu \in C^\infty(TE)$, where $(x^\varphi, \dot{x}^\varphi)$ is the induced coordinate chart on TE.

The connection K is linear if and only if its coordinate expression is of the type

$$K_\varphi{}^\lambda = K_\varphi{}^\lambda{}_\psi \dot{x}^\psi\,, \qquad K_\varphi{}^\lambda{}_\psi \in C^\infty(E)\,. \tag{1.5}$$

If the connection K is linear, then we can define the vertical projection $\nu_{\Lambda^*\otimes K} : T(\mathbb{T}^*\otimes TE) \to \mathbb{T}^*\otimes TE$, where Λ is the canonical linear flat connection on $\mathbb{T}$. For the sake of brevity, we shall use the same symbol and write $\nu_K \equiv \nu_{\Lambda^*\otimes K}$. In the induced coordinate chart $(x^\varphi, \dot{x}^\varphi_0)$ on $\mathbb{T}^*\otimes TE$, we have

$$\nu_K = \nu_{\Lambda^*\otimes K} = u^0 \otimes (\dot{d}^\mu_0 - K_\varphi{}^\mu{}_\psi \dot{x}^\psi_0 d^\varphi)\otimes \partial_\mu\,. \tag{1.6}$$

Analogously, a connection Γ on the affine bundle $JE \to E$ can be expressed, equivalently, by a tangent valued form, or by a vertical valued form

$$\Gamma : JE \to T^*E \underset{JE}{\otimes} TJE\,, \qquad \nu_\Gamma : JE \to \mathbb{T}^*\otimes T^*JE \underset{JE}{\otimes} VE\,,$$

respectively, with coordinate expressions

$$\Gamma = d^\varphi \otimes (\partial_\varphi + \Gamma_\varphi{}^i_0 \partial^0_i)\,, \quad \nu_\Gamma = u^0 \otimes (d^i_0 - \Gamma_\varphi{}^i_0 d^\varphi)\otimes \partial_i\,, \tag{1.7}$$

$\Gamma_\varphi{}^i_0 \in C^\infty(JE)$.

The connection Γ is affine if and only if its coordinate expression is of the type

$$\Gamma_\varphi{}^i_0 = \Gamma_\varphi{}^{i}{}_0{}^0_j x^j_0 + \Gamma_\varphi{}^{i}{}_0{}^0_0\,, \qquad \Gamma_\varphi{}^{i}{}_0{}^0_\alpha \in C^\infty(E)\,.$$

Theorem 1.1. *For any linear connection K on TE, the map*

$$\nu_\Gamma = \vartheta \circ \nu_K \circ T\text{д}$$

given by the following diagram

$$\begin{tikzcd}
TJE \arrow[r, "\nu_\Gamma"] \arrow[d, "(\pi_{JE}, T\text{д})"'] & VJE \arrow[r, "\simeq"] & JE \underset{E}{\times} (\mathbb{T}^*\otimes VE) \\
JE \underset{E}{\times} T(\mathbb{T}^*\otimes TE) \arrow[rr, "(\mathrm{id}_{JE}\times \nu_K)"] & & JE \underset{E}{\times} (\mathbb{T}^*\otimes TE) \arrow[u, "(\mathrm{id}_{JE}\times\vartheta)"']
\end{tikzcd}$$

turns out to be a connection on the bundle $JE \to E$ with coordinate expression

$$\Gamma_\varphi{}^i_0 = K_\varphi{}^i{}_j x^j_0 + K_\varphi{}^i{}_0 - x^i_0(K_\varphi{}^0{}_j x^j_0 + K_\varphi{}^0{}_0)\,. \tag{1.8}$$

Thus, we have obtained a map

$$\chi : K \mapsto \Gamma\,.$$

Proof. It can be proved in coordinates by using (1.3) and (1.4). $\boxed{QED}$

A linear connection K is said to be *time–preserving* if $\nabla^K(dt) = 0$; in coordinates it reads $K^0_{\varphi\psi} = 0$.

Corollary 1.1. *If K is time–preserving, then the corresponding Γ is affine. Moreover, the correspondence between time–preserving linear connections on TE and affine connections on JE is one–to–one,* [5]. □

Corollary 1.2. *If K is time–preserving and torsion free, then the corresponding Γ is torsion free, with respect to the scaled vertical valued form ϑ on JE.* □

The curvatures (see Introduction), of a connection K on TE and of a connection Γ on JE are, respectively, the 2–forms

$$R_K = \frac{1}{2}[K, K] : TE \to \wedge^2 T^*E \underset{E}{\otimes} TE \,, \quad R_\Gamma = \frac{1}{2}[\Gamma, \Gamma] : JE \to \mathbb{T}^* \wedge^2 T^*E \underset{E}{\otimes} VE \,,$$

whose coordinate expressions can be easily computed by (0.1).

Theorem 1.2. *If Γ is the connection on JE induced by a linear connection K on TE, then the we have*

$$R_\Gamma = \vartheta \circ R_K \circ \text{д} \,,$$

according to the following commutative diagram

$$\begin{array}{ccc}
JE \underset{E}{\times} \mathbb{T}^* \otimes TE & \xrightarrow{\mathrm{id}_{JE} \times R_K} & JE \underset{E}{\times} (\wedge^2 T^*E \underset{E}{\otimes} \mathbb{T}^* \otimes TE) \\
{\scriptstyle (\mathrm{id}_{JE}, \text{д})} \big\uparrow & & \big\downarrow {\scriptstyle \vartheta} \\
JE & \xrightarrow{\qquad R_\Gamma \qquad} & \wedge^2 T^*E \underset{E}{\otimes} \mathbb{T}^* \otimes VE
\end{array}$$

i.e. in coordinates

$$(R_\Gamma)_{\lambda\mu}{}^i_0 = (R_K)_{\lambda\mu}{}^i{}_j x^j_0 + (R_K)_{\lambda\mu}{}^i{}_0 - x^i_0 \big((R_K)_{\lambda\mu}{}^0{}_j x^j_0 + (R_K)_{\lambda\mu}{}^0{}_0\big) \,.$$

Proof. It can be proved by using (1.8) and the coordinate expressions of R_K and R_Γ. $\boxed{QED}$

We observe that a connection K on TE and a Riemannian metric h on E would induce the scaled 2–form $\Omega(h, K)$ on $\mathbb{T}^* \otimes TE$ given by $\nu_K \bar{\wedge} \sigma$, where σ is defined in the Introduction and $\bar{\wedge}$ denotes the wedge product followed by the contraction through the metric h. But, in our framework, we cannot define $\nu_K \bar{\wedge} \sigma$ since our metric g is only vertical; we need a projection of TE on VE. Actually, we can use the projection $\vartheta : JE \times_E TE \to VE$.

Thus, we set

$$\Omega(g, K) := (\vartheta \circ \nu_K) \bar{\wedge} (\vartheta \circ \sigma) : JE \underset{E}{\times} \mathbb{T}^* \otimes TE \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \wedge^2 T^*(\mathbb{T}^* \otimes TE) \,.$$

In coordinates

$$\Omega(g, K) = g_{ij} u^0 \otimes \big(\dot{d}^i_0 - x^i_0 \dot{d}^0_0 - (K_\varphi{}^i - x^i_0 K_\varphi{}^0) d^\varphi\big) \wedge \vartheta^j \,. \tag{1.9}$$

On the other hand, a connection Γ on JE and the vertical metric g yield the natural contact scaled 2–form $\Omega(g, \Gamma)$ on JE, [5],

$$\Omega(g, \Gamma) = \nu_\Gamma \bar{\wedge} \vartheta : JE \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \wedge^2 T^* JE$$

with coordinate expression

$$\Omega(g, \Gamma) = g_{ij} u^0 \otimes (d^i_0 - \Gamma_{\varphi}{}^{i}{}_{0} d^{\varphi}) \wedge \vartheta^j \,. \tag{1.10}$$

Theorem 1.3. *If Γ is the connection on JE associated with a linear connection K on TE, then*

$$\Omega(g, \Gamma) = (\mathrm{id}_{JE}, д)^* \Omega(g, K) \,.$$

PROOF. The pullback of the form $\Omega(g, K)$ with respect to $(id_{JE}, д)$ is the form

$$(id_{JE}, д)^* \Omega(g, K) : JE \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \wedge^2 T^* JE \,.$$

The identification of $\Omega(g, \Gamma)$ and $(id_{JE}, д)^* \Omega(g, K)$ follows from (1.3) (1.8), (1.9) and (1.10). $\boxed{QED}$

In [7] all 2–forms on JE generated naturally by a vertical Riemannian metric and by a time–preserving linear connection have been classified by using naturality methods of differential geometry, [8], [9]. It has been proved that all such forms are constant multiples of the form $\Omega(g, \Gamma)$.

The form $\Omega(g, \Gamma)$ is non-degenerate, namely

$$dt \wedge \Omega(g, \Gamma) \wedge \Omega(g, \Gamma) \wedge \Omega(g, \Gamma) : JE \to \mathbb{T}^{*2} \otimes \mathbb{L}^6 \otimes \wedge^7 T^* JE$$

is a scaled volume form on JE.

Theorem 1.4. *In the case when K is torsion free and time–preserving, the 2–form $\Omega(g, \Gamma)$ is closed if and only if* (see [4], [5])

$$\nabla g = 0, \qquad R^i{}_{\lambda}{}^{j}{}_{\mu} = R^j{}_{\mu}{}^{i}{}_{\lambda} \,. \quad \boxed{QED}$$

Assumption G.2. Space–time is assumed to be equipped with a torsion free time–preserving connection $K^{\natural}$, called *gravitational*; moreover, we postulate the following condition as a field equation for g and $K^{\natural}$

$$d\Omega(g, \Gamma^{\natural}) = 0 \,,$$

where $\Gamma^{\natural} := \chi \circ K^{\natural}$. □

Thus, under assumptions (G.1) and (G.2), the form

$$\Omega^{\natural} := \Omega(g, \Gamma)$$

turns out to be "co–symplectic". This form encodes the main information arising from the classical space–time background and is the source of the quantisation procedure presented in [4], [5].

We recall that a second order connection on E is defined as a section

$$\gamma : JE \to J_2 E \,,$$

where $J_2 E$ is the space of second order jets of sections of $E \to \boldsymbol{T}$.

Moreover, we recall that $\text{д}_2 : J_2E \hookrightarrow \mathbb{T}^*\otimes TJE$ turns out to be exactly the fibred submanifold over JE, which makes the following diagram commutative

$$\begin{tikzcd}
J_2E \arrow[r, "\text{д}_2"] \arrow[d, "\pi^2_1"'] & \mathbb{T}^*\otimes TJE \arrow[d, "id\otimes T\pi^1_0"'] \\
JE \arrow[r, "\text{д}"] \arrow[d, "\pi^1"'] & \mathbb{T}^*\otimes TE \arrow[d, "id\otimes Tt"] \\
\boldsymbol{T} \arrow[r, "1_{\mathbb{T}}"] & \mathbb{T}^*\otimes\mathbb{T}
\end{tikzcd}$$

where д_2 denotes the second order jet–contact map.

Therefore, by considering the inclusion д_2, each second order connection γ can be characterised as a scaled vector field

$$\gamma : JE \to \mathbb{T}^*\otimes TJE\,,$$

which makes the following diagram commutative

$$\begin{tikzcd}
JE \arrow[r, "\gamma"] \arrow[d, "id"] & \mathbb{T}^*\otimes TJE \arrow[d, "id\otimes T\pi^1_0"] \\
JE \arrow[r, "\text{д}"] \arrow[d, "\pi^1"'] & \mathbb{T}^*\otimes TE \arrow[d, "id\otimes Tt"] \\
\boldsymbol{T} \arrow[r, "1_{\mathbb{T}}"] & \mathbb{T}^*\otimes\mathbb{T}
\end{tikzcd}$$

In particular, a second order connection turns out to be a (first order) connection on the fibred manifold $JE \to \boldsymbol{T}$.

Now, for each time–preserving connection K, the map [5],

$$\gamma := \text{д} \lrcorner \Gamma : JE \to \mathbb{T}^*\otimes TE$$

turns out to be a second order connection; moreover, it is uniquely characterised by

$$\gamma \lrcorner \Omega(g, \Gamma) = 0\,.$$

In particular, we are involved with the *gravitational* second order connection

$$\gamma := \text{д} \lrcorner \Gamma^\natural\,,$$

which provides the equation of inertial motion of particles.

1.2. Electromagnetic field and geometric structure

So far, we have assumed on space–time the gravitational structure associated with the vertical Riemannian metric g and the space–time connection K. Next, we introduce the electromagnetic field.

Assumption G.3. We assume the *electromagnetic field* to be a closed scaled 2-form on E

$$F : E \to (\mathbb{L}^{1/2}\otimes\mathbb{M}^{1/2})\otimes\wedge^2 T^*E\,. \quad \square$$

We denote the local potential of F by $A : E \to (\mathbb{L}^{1/2}\otimes\mathbb{M}^{1/2})\otimes T^*E$. Thus, by definition, we set

$$2dA = F\,.$$

We define the *'universal' electric* and *'universal' magnetic fields* to be the scaled forms on JE

$$E := -д \lrcorner F : JE \to (\mathbb{T}^{*2}\otimes\mathbb{L}^{1/2}\otimes\mathbb{M}^{1/2})\otimes T^*E\,,$$
$$B := F + 2dt \wedge E : JE \to (\mathbb{L}^{1/2}\otimes\mathbb{M}^{1/2})\otimes\wedge^2 T^*E\,.$$

Thus, we can write

$$F = -2dt \wedge E + B.$$

Hence, the *electric* and *magnetic fields* associated with an observer $o : E \to JE$ are defined as the pullback forms

$$o^*E : E \to (\mathbb{T}^*\otimes\mathbb{L}^{1/2}\otimes\mathbb{M}^{1/2})\otimes T^*E\,, \quad o^*B : E \to (\mathbb{L}^{1/2}\otimes\mathbb{M}^{1/2})\otimes\wedge^2 T^*E\,.$$

We stress that the universal electric field carries the full information of the electromagnetic field.

Remark 1.1. We have the coordinate expressions

$$F = 2F_{0j}d^0 \wedge d^j + F_{ij}d^i \wedge d^j = -2u_0 E_j d^0 \wedge \vartheta^j + B_{ij}\vartheta^i \wedge \vartheta^j\,,$$

and

$$E = E_j d^j + E_0 d^0 = E_i \vartheta^i\,, \qquad B = B_{ij}d^i \wedge d^j + 2B_{0j}d^0 \wedge d^j = B_{ij}\vartheta^i \wedge \vartheta^j\,,$$

where

$$E_j = -u^0(F_{0j} + F_{hj}x^h_0) \qquad E_0 = -u^0 F_{h0}x^h_0\,,$$
$$B_{ij} = F_{ij} \qquad B_{0j} = -F_{hj}x^h_0\,.$$

Therefore, in a chart adapted to an observer o, the coordinate expressions of the observed electric and magnetic fields are

$$o^*E = -u_0 F_{0j}d^j \qquad o^*B = F_{ij}d^i \wedge d^j\,. \quad \square$$

Next, we show that the electromagnetic field can be naturally incorporated into the gravitational structures of the phase spase.

Namely, let us consider the following gravitational objects on the phase space JE:

$$\Gamma^{\natural} : JE \to T^*E \underset{E}{\otimes} TJE \,, \qquad \gamma^{\natural} : JE \to \mathbb{T}^* \otimes TJE \,,$$
$$\Omega^{\natural} : JE \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \wedge^2 T^* JE \,,$$

which fulfill the following structural relations

$$\gamma^{\natural} = д \lrcorner \Gamma^{\natural} \,, \qquad \Omega^{\natural} = \nu_{\Gamma^{\natural}} \bar{\wedge} \vartheta \,, \qquad \gamma^{\natural} \lrcorner \Omega^{\natural} = 0 \,, \qquad d\Omega^{\natural} = 0 \,.$$

We are looking for *total* objects obtained correcting the gravitational objects by an electromagnetic term, in such a way to preserve the above relations.

For this purpose we need a suitable coupling constant. So, we consider a particle with a given mass and charge

$$m \in \mathbb{M} \,, \qquad q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \,,$$

and refer to the coupling constant

$$\frac{q}{m} \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{*1/2} \,.$$

We start from the obvious coupling of the electromagnetic field with the gravitational co–symplectic 2–form on JE.

Accordingly, we define the *total* co–symplectic form to be

$$\Omega := \Omega^{\natural} + \frac{q}{2m} F : JE \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \wedge^2 T^* JE \,.$$

Here, the factor $\frac{1}{2}$ is chosen in such a way to recover the standard normalization in practical formulas.

Of course, we obtain

$$d\Omega = 0 \,.$$

We have

$$dt \wedge \Omega \wedge \Omega \wedge \Omega = dt \wedge \Omega^{\natural} \wedge \Omega^{\natural} \wedge \Omega^{\natural} \,.$$

So, the electromagnetic field does not contribute to the total volume form on the phase space.

Moreover, in [5] the following result has been proved.

Theorem 1.5. *There is a unique torsion free affine connection* Γ *and a unique second order connection on the bundle* $JE \to E$

$$\Gamma = \Gamma^{\natural} + \Gamma^{e} \,, \qquad \gamma = \gamma^{\natural} + \gamma^{e} \,,$$

such that

$$\gamma = д \lrcorner \Gamma \,, \qquad \Omega = \nu_{\Gamma} \bar{\wedge} \vartheta \,, \qquad \gamma \lrcorner \Omega = 0 \,.$$

Namely, γ^e *turns out to be the* Lorentz force

$$\gamma^e = \frac{q}{m} g^\sharp \circ \check{E} : JE \to \mathbb{T}^{*2} \otimes VE$$

and Γ^e *the* electromagnetic soldering form

$$\Gamma^e = -\frac{q}{2m} g^{\sharp 2} \circ (F - 2dt \wedge E) : JE \to T^*E \underset{E}{\otimes} (\mathbb{T}^* \otimes VE) ,$$

where $g^{\sharp 2}$ *denotes the metric isomorphism on the second component after vertical restriction.*

Moreover, the above objects fulfill the following equalities

$$\gamma^e = д \lrcorner \Gamma^e , \qquad -\Gamma^e \bar{\wedge} \vartheta = \Omega^e .$$

We have the coordinate expressions

$$\gamma^e = -\frac{q}{m} (F_0{}^i + F_h{}^i x^h_0) u^0 \otimes \partial^0_i , \quad \Gamma^e = -\frac{q}{2m} u^0 \big((F_h{}^i x^h_0 + 2F_0{}^i) d^0 + F_j{}^i d^j \big) \otimes \partial^0_i ,$$

hence

$$\Gamma_h{}^i{}_{0k}^{0} = \Gamma^\natural{}_h{}^i{}_{0k}^{0} , \qquad \Gamma_0{}^i{}_{0k}^{0} = \Gamma^\natural{}_0{}^i{}_{0k}^{0} + \frac{q}{2m} u^0 F^i{}_k , \qquad \Gamma_0{}^i{}_{00}^{0} = \Gamma^\natural{}_0{}^i{}_{00}^{0} + \frac{q}{2m} u^0 F^i{}_0 . \ \square$$

Hence, the above total Γ induces a total space–time connection K.

2. Einstein's case

In this section, we study the geometric structures of phase space in the Einstein general relativistic framework.

We start with the basic assumptions on space–time.

Asssumption E.1. We assume *space–time* to be a 4–dimensional oriented and time–oriented manifold M equipped with a scaled Lorentzian metric of signature $(+---)$

$$g : M \to \mathbb{L}^2 \otimes T^*M \underset{M}{\otimes} T^*M . \ \square$$

Local coordinate charts on M will be denoted by (x^φ), $\varphi = 0, 1, 2, 3$. The coordinate expression of g is then

$$g = g_{\varphi\psi} d^\varphi \otimes d^\psi , \quad g_{\varphi\psi} : M \to \mathbb{L}^2 \otimes I\!R .$$

In what follows we shall use local coordinate charts such that the vector ∂_0 is time–like and time oriented and $\partial_1, \partial_2, \partial_3$ are space–like; hence $g_{00} > 0, g_{11}, g_{22}, g_{33} < 0$.

Latin indices $i, j, p, \ldots$ will span space–like coordinates, while Greek indices $\lambda, \mu, \varphi, \ldots$ will span space–time coordinates.

Obviously, in the Einstein case we have no time fibring t, i.e. no absolute time $\boldsymbol{T}$. However we still have a vector space $\mathbb{T}$ which now describes the proper time intervals. We stress that this fact is in full agreement with the standard interpretation of general relativity.

Asssumption E.2. We assume the *light velocity* to be a positive element

$$c \in \mathbb{T}^* \otimes \mathbb{L} \,. \quad \square$$

We observe that g and g/c^2 can be regarded as non-scaled Lorentzian metrics on the vector bundles $\mathbb{L}^* \otimes TM \to M$ and $\mathbb{T}^* \otimes TM \to M$, respectively.

2.1. Jets of 1–dimensional submanifolds

In order to describe velocities of motions in the Einstein case we need jets of submanifolds. Let us recall the basic notions (for further details see [15]).

A k–jet of 1–dimensional submanifolds of M at $x \in M$ is defined to be an equivalence class of 1–dimensional submanifolds touching each other at x with a contact of order k. The k–jet of a 1-dimensional submanifold $s \subset M$ at $x \in s$ is denoted by $j_k s(x)$. The set of all k–jets of 1–dimensional submanifolds of M can be equipped, in a natural way, with a smooth structure. The corresponding manifold is denoted by $J_k(M,1)$.

For $p > q$ we have the canonical projection

$$\pi^p_q : J_p(M,1) \to J_q(M,1) \; : \; j_p s(x) \mapsto j_q s(x),$$

which makes $J_p(M,1)$ a bundle over $J_q(M,1)$; in particular, $\pi^k_0 : J_k(M,1) \to J_0(M,1) = M$ is a bundle. We set

$$\underline{\phi} = \pi^k_0(\phi) \,, \quad \phi \in J_k(M,1).$$

We stress that, unlike the case of jets of sections of a fibred manifold, the bundle $J_p(M,1) \to J_{p-1}(M,1)$ is not affine.

Given a 1–dimensional submanifold $s \subset M$ and an integer $k \geq 0$, we have the map

$$j_k s : s \to J_k(M,1) : x \mapsto j_k s(x) \,.$$

Clearly $j_k s(s) \subset J_k(M,1)$ is a 1–dimensional submanifold.

We have the canonical fibred isomorphism over M of the first jet bundle with the Grassmannian bundle of dimension 1

$$J(M,1) \equiv J_1(M,1) \to \mathrm{Grass}(M,1) : \phi \mapsto L_\phi,$$

where $L_\phi \subset T_{\underline{\phi}}M$ is the tangent space at $\underline{\phi}$ of 1–dimensional submanifolds generating ϕ.

A local chart on M is said to be *divided* if the set of its coordinate functions is divided into two subsets of 1 and $(\dim M - 1)$ elements. Our typical notation for a divided chart will be

$$(x^0, x^i), \qquad 1 \leq i \leq \dim M - 1 \,.$$

A divided chart and a 1–dimensional submanifold $s \subset M$ are said to be *related* if the submanifold s can be expressed locally by formulas of the type

$$x^i = s^i(x^0) \,;$$

i.e. , more precisely $x^i|s = s^i \circ x^0|s$, with $s^i : I\!R \to I\!R$.

Every divided chart on M determines canonically a local fibred chart $(x^0, x^i; x^i_0)$ on $J(M,1)$. We shall always refer to such charts.

Thus we can write

$$x^i_0 \circ j_1 s = \partial_0 s^i \equiv (Ds^i) \circ (x^0|s)\,.$$

If $\phi \in J(M,1)$, then the subspace $L_\phi \subset T_{\underline{\phi}} M$ is the span $L_\phi = \langle \partial_0 + \partial_0 s^i \partial_i \rangle\,.$

2.2. Phase space

Next we introduce the phase space as the velocity space of the manifold M and exhibit the 'jet-contact structure' induced by the Lorentz metric.

A *motion* in M is defined to be a 1–dimensional time–like submanifold $s \subset M$.

Let us consider a motion s. By definition, for each $x \in s$, $T_x s$ lies inside the light cone. The 1-jet prolongation $js \equiv j_1 s : s \to J(M,1)$ is said to be the *velocity* of s. The length of a vector $v \in Ts$ is defined to be $\|v\| = \sqrt{|g(v,v)|} \in \mathbb{L} \otimes I\!\!R$.

Lemma 2.1. *Given a motion s we have the canonical isomorphism*

$$u_s : Ts \to s \times \mathbb{T} : v \in T_x s \mapsto (x, \frac{\pm\|v\|}{c}) \in s \times \mathbb{T}\,,$$

where we choose $+$ or $-$ according to the time orientation of v. □

We define the *phase space*

$$UM \equiv U_1 M \subset J(M,1)$$

to be the subspace of all 1–jets of motions. In other words, $\phi = js(\underline{\phi}) \in J(M,1)$ belongs to UM if and only if $L_\phi = T_{\underline{\phi}} s$ lies inside the light cone.

We stress that the bundle $UM \to M$ is not affine (even in the Minkowski case); its fibres are just Riemannian manifolds. This fact encodes one of the main differences between the Galilei and the Einstein cases.

Lemma 2.2. *For each motion $\iota_s : s \hookrightarrow M$, there is a unique linear map over $UM \to M$*

$$\text{Д}_s \; : UM \underset{s}{\times} Ts \to TM\,,$$

which makes the following diagram commutative

$$\begin{tikzcd}
UM \underset{s}{\times} Ts \arrow[rr, "\text{Д}_s"] & & TM \\
& Ts \arrow[ul, "(js\circ\pi_s, \mathrm{id}_{Ts})"] \arrow[ur, "T\iota_s"'] &
\end{tikzcd}$$ □

The above Lemmas, which are based on the Lorentzian structure of space–time, allow us to recover the jet–contact structure in the Einstein case.

Proposition 2.1. *The above maps u_s and д_s, for all motions s, yield the fibred morphism over M*

$$\text{д} : UM \to \mathbb{T}^* \otimes TM \,,$$

with coordinate expression

$$\text{д} = c\,\alpha\,\text{д}_0 = c\,\alpha\,(\partial_0 + x_0^i \partial_i)\,, \tag{2.1}$$

where

$$\alpha = 1/\|\text{д}_0\| = 1/\sqrt{g_{00} + 2 g_{0j} x_0^j + g_{ij} x_0^i x_0^j} \in \mathbb{L}^* \,. \quad \square$$

Remark 2.1. We have

$$g \circ (\text{д}, \text{д}) = c^2 \,.$$

Thus, the map д makes $UM \subset \mathbb{T}^* \otimes TM$ a fibred submanifold over M. $\square$

If s is a motion, then $\text{д} \circ js : s \to \mathbb{T}^* \otimes TM$ is the scaled vector field representing the velocity of s.

The metric allows us to recover the Einstein analogue of the Galilei form dt (but, indeed, not of t).

Proposition 2.2. *We have the scaled* 1*-form*

$$\tau := \frac{g^\flat}{c^2} \circ \text{д} : UM \to \mathbb{T} \otimes T^* M \,,$$

with coordinate expression

$$\tau \equiv \tau_\lambda\, d^\lambda = \frac{\alpha}{c}\,(g_{0\lambda} + g_{i\lambda} x_0^i) d^\lambda \,. \quad \square$$

Remark 2.2. We have $\text{д} \lrcorner \tau = 1$, i.e. , in coordinates,

$$c\,\alpha\,(\tau_0 + \tau_h x_0^h) = 1 \,. \quad \square \tag{2.2}$$

2.3. Orthogonal splitting

The Lorentzian metric yields the standard splitting of each space–time vector into the parallel and orthogonal components with respect to any given time–like direction. This standard construction allows us to recover the Einstein analogues of the Galilei horizontal and vertical bundles and of the corresponding exact tangent sequence. In this context we establish coordinate formulas which will be largely used in the sequel.

We consider the parallel and orthogonal subspaces, with respect to the map д, of the space–time tangent and cotangent bundles pullbacked over the phase space.

Namely we define the vector bundles over UM

$$T^{\|} M := \{(\phi, X) \in UM \underset{M}{\times} TM \mid X \in L_\phi\} \,,$$
$$T^{\perp} M := \{(\phi, X) \in UM \underset{M}{\times} TM \mid X \in L_\phi^\perp\} \,,$$

and

$$T^*_{||}M := \{(\phi, \omega) \in UM \underset{M}{\times} T^*M \mid \langle \omega, L^{\perp}_{\phi} \rangle = 0\} ,$$
$$T^*_{\perp}M := \{(\phi, \omega) \in UM \underset{M}{\times} T^*M \mid \langle \omega, L_{\phi} \rangle = 0\} .$$

Remark 2.3. We have the mutually inverse linear fibred isomorphism over UM

$$(id, д) : UM \times \mathbb{T} \to T^{||}M \qquad (id, \tau) : T^{||}M \to UM \times \mathbb{T} . \ \square$$

Thus, the bundles

$$T^{||}M \simeq UM \times \mathbb{T} \to UM , \qquad T^{\perp}M \to UM$$

will play here, respectively, the roles played by the bundles $\boldsymbol{T} \times \mathbb{T} \to \boldsymbol{T}$, $VE \to E$ in the Galilei case.

Proposition 2.3. *We have the linear fibred splittings over UM*

$$UM \underset{M}{\times} TM = T^{||}M \underset{UM}{\oplus} T^{\perp}M , \qquad UM \underset{M}{\times} T^*M = T^*_{||}M \underset{UM}{\oplus} T^*_{\perp}M . \ \square$$

Remark 2.4. By restriction of the metric, we obtain the scaled metrics

$$g_{||} : UM \to \mathbb{L}^2 \otimes T^*_{||}M \underset{UM}{\otimes} T^*_{||}M , \qquad g_{\perp} : UM \to \mathbb{L}^2 \otimes T^*_{\perp}M \underset{UM}{\otimes} T^*_{\perp}M$$

and the linear fibred isomorphisms over UM

$$g^{\flat}_{||} : T^{||}M \to \mathbb{L}^2 \otimes T^*_{||}M , \qquad g^{\flat}_{\perp} : T^{\perp}M \to \mathbb{L}^2 \otimes T^*_{\perp}M ,$$
$$g^{\sharp}_{||} : T^*_{||}M \to \mathbb{L}^{*2} \otimes T^{||}M , \qquad g^{\sharp}_{\perp} : T^*_{\perp}M \to \mathbb{L}^{*2} \otimes T^{\perp}M . \ \square$$

Lemma 2.3. *The following mutually dual local bases of vector fields and forms are adapted to the above splittings*

$$д_0 \equiv \partial_0 + x^i_0 \partial_i , \qquad b_i \equiv \partial_i - c\alpha\tau_i д_0 = (\delta^j_i - c\alpha\tau_i x^j_0)\partial_j - c\alpha\tau_i \partial_0 ,$$
$$\lambda^0 \equiv d^0 + c\alpha\tau_i \vartheta^i = c\alpha\tau_{\varphi} d^{\varphi} , \qquad \vartheta^i \equiv d^i - x^i_0 d^0 .$$

The inverse transition maps are

$$\partial_0 = c\alpha\tau_0 д_0 - x^i_0 b_i , \qquad \partial_i = b_i + c\alpha\tau_i д_0 ,$$
$$d^0 = \lambda^0 - c\alpha\tau_i \vartheta^i , \qquad d^i = (\delta^i_j - c\alpha\tau_j x^i_0)\vartheta^j + x^i_0 \lambda^0 . \ \square$$

Lemma 2.4. *We have*

$$g^{\flat} \circ д_0 = \frac{1}{\alpha^2}\lambda^0 \qquad g^{\flat} \circ b_i = g^{\perp}_{ij}\vartheta^j = (g_{i\mu} - c^2\tau_i\tau_\mu)d^\mu ,$$
$$g^{\sharp} \circ \lambda^0 = \alpha^2 д_0 , \qquad g^{\sharp} \circ \vartheta^i = g^{ij}_{\perp} b_j = (g^{i\mu} - x^i_0 g^{0\mu})\partial_\mu ,$$

where we have introduced the mutually inverse matrices

$$g^{\perp}_{ij} := b_i \cdot b_j = g_{ij} - c^2\tau_i\tau_j ,$$
$$g^{ij}_{\perp} := \vartheta^i \cdot \vartheta^j = g^{ij} - g^{i0}x^j_0 - g^{j0}x^i_0 + g^{00}x^i_0 x^j_0 . \ \square$$

We shall be involved with further useful identies.

Lemma 2.5. *We have*

$$(g^{i\mu} - x^i_0 g^{0\mu})\tau_\mu = 0 \tag{2.3}$$

$$(g_{i\mu} - c^2 \tau_i \tau_\mu) d^\mu = (g_{ij} - c^2 \tau_i \tau_j)\vartheta^j \tag{2.4}$$

$$(g^{i\mu} - x^i_0 g^{0\mu})(g_{i\nu} - c^2 \tau_i \tau_\nu) = \delta^\mu_\nu - c^2 \tau_\nu \tau^\mu \,. \tag{2.5}$$

Proof. Formula (2.3) follows from

$$g \circ (\vartheta^i, \tau) = 0 \,.$$

Formula (2.4) follows from (2.2), which gives

$$-(g_{ij} - c^2 \tau_i \tau_j) x^j_0 = (g_{i0} - c^2 \tau_i \tau_0) \,.$$

Formula (2.5) follows from

$$\begin{aligned}(g^{i\mu} - x^i_0 g^{0\mu})(g_{i\nu} - c^2 \tau_i \tau_\nu) &= g^{i\mu} g_{i\nu} - c^2 g^{i\mu} \tau_i \tau_\nu - g^{0\mu} g_{i\nu} x^i_0 + c^2 g^{0\mu} \tau_i \tau_\nu x^i_0 = \\ = g^{i\mu} g_{i\nu} - c^2 g^{i\mu} \tau_i \tau_\nu - g^{0\mu} g_{i\nu} x^i_0 &+ g^{0\mu}(g_{0\nu} + g_{j\nu} x^j_0) - c^2 g^{0\mu} \tau_0 \tau_\nu = \delta^\mu_\nu - c^2 \tau_\nu \tau^\mu \,.\end{aligned}$$

$\boxed{QED}$

The parallel and the orthogonal projections induced by the Lorentzian metric will provide the Einstein analogues of the exact tangent sequence (1.1) and of its splitting over the jet space (1.2) in the Galilei case, respectively.

Proposition 2.4. *We have the linear exact sequence over* UM

$$0 \to T^\perp M \to UM \underset{M}{\times} TM \xrightarrow{\lambda} T^{||} M \to 0 \,,$$

and its splitting

$$0 \to T^{||} M \to UM \underset{M}{\times} TM \xrightarrow{\vartheta} T^\perp M \to 0 \,,$$

where the parallel and orthogonal projections

$$\lambda = \tau \otimes д : UM \underset{M}{\times} TM \to T^{||} M \,, \qquad \vartheta = 1_M - \tau \otimes д : UM \underset{M}{\times} TM \to T^\perp M$$

have the coordinate expressions

$$\lambda = \lambda^0 \otimes д_0 = c^2 \tau_\nu \tau^\mu d^\nu \otimes \partial_\mu = c\alpha \tau_\nu (\delta^\mu_0 + \delta^\mu_i x^i_0) d^\nu \otimes \partial_\mu \tag{2.6}$$

and

$$\begin{aligned}\vartheta &= \vartheta^i \otimes b_i = (\delta^\mu_\nu - c^2 \tau_\nu \tau^\mu) d^\nu \otimes \partial_\mu \\ &= (\delta^\mu_\nu - c\alpha \tau_\nu (\delta^\mu_0 + \delta^\mu_i x^i_0)) d^\nu \otimes \partial_\mu = (g^{i\mu} - x^i_0 g^{0\mu})(g_{i\nu} - c^2 \tau_i \tau_\nu) d^\nu \otimes \partial_\mu \,. \; \square\end{aligned} \tag{2.7}$$

Corollary 2.1. *We have the linear exact sequence over UM*

$$0 \to T^*_{\|} M \to UM \underset{M}{\times} T^*M \xrightarrow{\vartheta^*} T^*_{\perp} M \to 0\,,$$

and its splitting

$$0 \to T^*_{\perp} M \to UM \underset{M}{\times} T^*M \xrightarrow{\lambda^*} T^*_{\|} M \to 0\,,$$

where the parallel and orthogonal projections

$$\lambda^* = д \otimes \tau : UM \underset{M}{\times} T^*M \to T^*_{\|} M\,, \qquad \vartheta^* = 1^*_M - д \otimes \tau : UM \underset{M}{\times} T^*M \to T^*_{\perp} M$$

have the coordinate expressions

$$\lambda^* = д_0 \otimes \lambda^0\,, \qquad \vartheta^* = b_i \otimes \vartheta^i\,. \quad \square$$

The physical interpretation of our maps can be achieved by reading the splitting of velocity of motions in our notation.

An *observer* is defined to be a section

$$o : M \to UM\,.$$

Then $д \circ o : M \to \mathbb{T}^* \otimes TM$ is the scaled vector field representing the velocity of the observer o.

By using the parallel and the orthogonal projections associated with the observer o, we obtain the following splitting.

Proposition 2.5. *We have*

$$д \circ js = (д \circ js)^{\|} + (д \circ js)^{\perp} = \delta\,(д \circ o|s + \bar{\beta})\,,$$

where

$$\delta = (д \circ js) \lrcorner (\tau \circ o|s) = \frac{g(д \circ o|s, д \circ js)}{c^2} = \frac{c}{\sqrt{c^2 - \|\bar{\beta}\|^2}} : s \to I\!R$$

and

$$\bar{\beta} = \frac{c^2((д \circ js) \lrcorner (\vartheta \circ o|s))}{g(д \circ o, д \circ js)} : s \to \mathbb{T}^* \otimes T^{\perp} M\,. \quad \square$$

The vector $\bar{\beta} \in \mathbb{T}^* \otimes L_o^{\perp}$ can be interpreted as the *velocity of the motion s observed by the observer o*. We have

$$\delta > 1\,, \qquad \|\bar{\beta}\| = \frac{c\sqrt{\delta^2 - 1}}{\delta} < c\,,$$

which express, respectively, the time dilatation and the fact that the observed velocity of a motion is smaller than c.

2.4. Vertical bundle of the phase space

We have noticed that in the Einstein case the fibres of the phase space are just Riemannian manifolds; in spite of this fact, the Lorentz metric allows us to recover important technical results, which in the Galilei case follow from the affine structure of the phase space.

We shall be involved with the vertical tangent bundle VUM of $\pi^1_0 : UM \to M$.

Lemma 2.6. *The vertical prolongation of* д

$$V\text{д} : VUM \to V(\mathbb{T}^*{\otimes}TM) \simeq (\mathbb{T}^*{\otimes}TM) \underset{M}{\times} (\mathbb{T}^*{\otimes}TM)$$

factorises through a fibred isomorphism over UM

$$v^\perp : VUM \to \mathbb{T}^*{\otimes}T^\perp M\,,$$

according to the commutative diagram

$$\begin{tikzcd} VUM \arrow[rr, "V\text{д}"] \arrow[dr, "v^\perp"'] & & (\mathbb{T}^*{\otimes}TM) \underset{M}{\times} (\mathbb{T}^*{\otimes}TM) \\ & \mathbb{T}^*{\otimes}T^\perp M \arrow[ur] & \end{tikzcd}$$

We have the coordinate expressions

$$v^\perp = c\,\alpha\, d^i_0 \otimes b_i\,, \qquad v^{\perp\,-1} = \frac{1}{c\,\alpha}\vartheta^i \otimes \partial^0_i\,. \tag{2.8}$$

Proof. Let $\phi \in UM$ and $X \in V_\phi UM$. Then X can be expressed as $X = d\sigma(0)$, where $\sigma : I\!R \to U_{1\underline{\phi}}M$, is a vertical curve such that $\sigma(0) = \phi$. In virtue of the definition of д, we have $c^2 = g\circ(\text{д}\circ\sigma,\, \text{д}\circ\sigma)$; hence, by taking into account that σ is vertical, we can write

$$0 = g\big(T(\text{д}\circ\sigma)(0),\, (\text{д}\circ\sigma)(0)\big) = g\big((V\text{д}\circ d\sigma)(0),\, (\text{д}\circ\sigma)(0)\big) = g\big(V\text{д}(X),\, \text{д}(\phi)\big)\,.$$

Eventually, the coordinate expression (2.8) follows from (2.1). $\boxed{QED}$

The isomorphism $VUM \simeq \mathbb{T}^*{\otimes}T^\perp M$ is Einstein the analogue of $VJE \simeq JE \times_E \mathbb{T}^*{\otimes}VE$.

The map $v^{\perp\,-1}$ can be regarded as a scaled soldering 1-form

$$\Phi := v^{\perp\,-1} : UM \to \mathbb{T}{\otimes}T^*M \underset{UM}{\otimes} VUM\,.$$

This form is the Einstein analogue of $\vartheta : JE \to \mathbb{T}{\otimes}T^*E{\otimes}_{JE}VJE$.

2.5. Connections

In the Einstein case we can recover the mapping from space–time connections to connections on the phase space. However, we do not know the analogue of Corollary 1.1. In the Galilei case the connection on the phase space is affine; on the contrary, in the Einstein case, the connection on the phase space is highly non linear. However, our formalism allows us to handle also this case without problems.

For a connection K on TM we can repeat the considerations of the Galilei case.

A connection Γ on UM can be represented, equivalently, by a tangent valued form, or by a vector valued form

$$\Gamma : UM \to T^*M \underset{UM}{\otimes} TUM\,, \qquad v^{\perp}\circ\nu_{\Gamma} : UM \to T^*UM \underset{UM}{\otimes} (\mathbb{T}^*\otimes T^{\perp}M)\,,$$

with coordinate expressions

$$\Gamma = d^{\varphi}\otimes(\partial_{\varphi} + \Gamma_{\varphi}{}^{i}_{0}\partial_i^0)\,, \quad v^{\perp}\circ\nu_{\Gamma} = c\alpha(d_0^i - \Gamma_{\varphi}{}^{i}_{0}d^{\varphi})\otimes b_i\,, \tag{2.9}$$

$\Gamma_{\varphi}{}^{i}_{0} \in C^{\infty}(UM)$.

We can define the *torsion* of Γ as the scaled vertical valued 2-form (see Introduction)

$$[\Gamma, \Phi] : UM \to \mathbb{T}\otimes\wedge^2 T^*M \underset{UM}{\otimes} VUM\,.$$

Theorem 2.1. *For any linear connection K on TM the map*

$$v^{\perp}\circ\nu_{\Gamma} = \vartheta\circ\nu_K\circ T\text{д} \tag{2.10}$$

given by the following diagram

$$\begin{tikzcd} TUM \arrow[r, "\nu_{\Gamma}"] \arrow[d, "(\pi_{UM}, T\text{д})"'] & VUM \arrow[r, "v^{\perp}"] & \mathbb{T}^*\otimes T^{\perp}M \\ UM \underset{M}{\times} T(\mathbb{T}^*\otimes TM) \arrow[rr, "(\mathrm{id}_{UM}\times\nu_K)"] & & UM \underset{E}{\times} (\mathbb{T}^*\otimes TM) \arrow[u, "\vartheta"'] \end{tikzcd}$$

turns out to be a connection on the bundle $UM \to M$ with coordinate expression

$$\Gamma_{\varphi}{}^{i}_{0} = K_{\varphi}{}^{i}{}_{j}x_0^j + K_{\varphi}{}^{i}{}_{0} - x_0^i(K_{\varphi}{}^{0}{}_{j}x_0^j + K_{\varphi}{}^{0}{}_{0})\,. \tag{2.11}$$

Thus, we have obtained a map

$$\chi : K \mapsto \Gamma\,.$$

Proof. It can be proved in coordinates by using (1.6), (2.1) and (2.9). $\boxed{QED}$

Corollary 2.2. *The metric g yields the* gravitational *connections on TM and UM*

$$K^{\natural} := \varkappa\,, \qquad v^{\perp}\circ\nu_{\Gamma^{\natural}} := \vartheta\circ\nu_{K^{\natural}}\circ T\text{д}\,,$$

where $\varkappa$ is the Levi–Civita connection with the Christoffel symbols

$$\varkappa_{\varphi\psi}^{\sigma} = -\frac{g^{\sigma\tau}}{2}(\partial_{\varphi}g_{\tau\psi} + \partial_{\psi}g_{\tau\varphi} - \partial_{\tau}g_{\varphi\psi})\,. \quad \square \tag{2.12}$$

2.6. Curvature

In the Einstein case the curvatures fulfill relations analogous to the Galilei case.

The curvatures (see Introduction) of a connection K on TM and of a connection Γ on UM are, respectively, the 2–forms

$$R_K = \frac{1}{2}[K,K] : TM \to \wedge^2 T^*M \underset{M}{\otimes} TM, \;\; R_{\Gamma} = \frac{1}{2}[\Gamma,\Gamma] : UM \to \wedge^2 T^*M \underset{UM}{\otimes} VUM,$$

whose coordinate expressions can be easily computed by (0.1).

Moreover, we obtain

$$v^{\perp}\circ R_{\Gamma} : UM \to \wedge^2 T^*M \underset{UM}{\otimes} (\mathbb{T}^*\otimes T^{\perp}M)\,,$$

with coordinate expression

$$v^{\perp}\circ R_{\Gamma} = c\,\alpha\,(R_{\Gamma})_{\lambda\mu}{}^{i}_{0}\, d^{\lambda}\wedge d^{\mu}\otimes b_i\,.$$

Theorem 2.2. *If Γ is the connection on UM induced by a linear connection K on TM, then the following diagram commutes*

$$\begin{array}{ccc}
UM \underset{M}{\times} \mathbb{T}^*\otimes TM & \xrightarrow{\mathrm{id}_{UM}\times R_K} & UM \underset{M}{\times} (\wedge^2 T^*M \underset{M}{\otimes} \mathbb{T}^*\otimes TM) \\
\Big\uparrow{\scriptstyle (\mathrm{id}_{UM},\,\text{д})} & & \Big\downarrow{\scriptstyle \mathrm{id}_{\wedge^2 T^*M}\otimes\vartheta} \\
UM & \xrightarrow{v^{\perp}\circ R_{\Gamma}} & \wedge^2 T^*M \underset{UM}{\otimes} (\mathbb{T}^*\otimes T^{\perp}M)
\end{array}$$

i.e.,

$$R_{\Gamma} = \vartheta\circ R_K\circ \text{д}\,, \tag{2.13}$$

i.e. in coordinates

$$(R_{\Gamma})_{\lambda\mu}{}^{i}_{0} = (R_K)_{\lambda\mu}{}^{i}{}_{j}\, x_0^j + (R_K)_{\lambda\mu}{}^{i}{}_{0} - x_0^i\big((R_K)_{\lambda\mu}{}^{0}{}_{j}\, x_0^j + (R_K)_{\lambda\mu}{}^{0}{}_{0}\big)\,.$$

Proof. It can be proved in coordinates by using (0.1), (2.1), (2.7) and (2.11).

2.7. Second order connection

In the Einstein case we can also recover the analogues of the Galilei second order jet–contact structure and connection. This result is interesting because the second order connection provides the equation of inertial motion of particles.

Let us denote by $U_2M \subset J_2(M,1)$ the subspace of all 2–jets of motions.

Lemma 2.7. *For each motion $\iota_s : s \hookrightarrow M$, there is a unique linear map over $U_2M \to UM$*

$$\text{д}_{s2} : U_2M \underset{s}{\times} Ts \to TUM$$

such that the following diagram commutes

$$\begin{tikzcd} U_2M \underset{s}{\times} Ts \arrow[rr, "\text{д}_{s2}"] & & TUM \\ & Ts \arrow[ul, "(j_2 s \circ \pi_s, \mathrm{id}_{Ts})"] \arrow[ur, "Tjs"'] & \end{tikzcd} \qquad \square$$

Proposition 2.6. *The maps u_s and д_{s2} yield the fibred inclusion over UM*

$$\text{д}_2 : U_2M \to \mathbb{T}^* \otimes TUM\,,$$

with coordinate expression

$$\text{д}_2 = c\,\alpha\,(\partial_0 + x^i_0 \partial_i + x^i_{00} \partial^0_i)\,. \quad \square$$

Corollary 2.3. *Thus $U_2M \subset \mathbb{T}^* \otimes TUM$ turns out to be exactly the fibred submanifold over UM, which makes the following diagram commutative*

$$\begin{tikzcd} U_2M \arrow[r, "\text{д}_2"] \arrow[d, "\pi^2_1"'] & \mathbb{T}^* \otimes TUM \arrow[d, "(\pi_{UM}, \mathrm{id} \otimes T\pi^1_0)"] \\ UM \arrow[r, "(\mathrm{id}, \text{д})"] \arrow[d, "\pi^1"'] & UM \underset{M}{\times} (\mathbb{T}^* \otimes TM) \arrow[d, "\mathrm{id} \otimes \tau"] \\ M \arrow[r, "1_{\mathbb{T}}"] & \mathbb{T}^* \otimes \mathbb{T} \end{tikzcd} \qquad \square$$

A *second order connection* on M is defined to be a section

$$\gamma : UM \to U_2M\,.$$

Corollary 2.4. *By considering the inclusion д_2, every second order connections γ can be characterised as a scaled vector field*

$$\gamma : UM \to \mathbb{T}^* \otimes TUM\,,$$

which makes the following diagram commutative

$$\begin{tikzcd}
UM \arrow[r, "\gamma"] \arrow[d, "\mathrm{id}"'] & \mathbb{T}^*\otimes TUM \arrow[d, "(\pi_{UM}, \mathrm{id}\otimes T\pi^1_0)"] \\
UM \arrow[r, "(\mathrm{id}, \text{д})"] \arrow[d, "\pi^1"'] & UM \underset{M}{\times} (\mathbb{T}^*\otimes TM) \arrow[d, "\mathrm{id}\otimes\tau"] \\
M \arrow[r, "1_{\mathbb{T}}"] & \mathbb{T}^*\otimes\mathbb{T}
\end{tikzcd} \qquad \square$$

The above result is fully analogous to the Galilei case; however, now we have no fibring of UM over time, hence we cannot properly say that γ is a connection on UM over time.

The coordinate expression of a second order connection is of the type

$$\gamma = c\,\alpha\,(\partial_0 + x^i_0\partial_i + \gamma_0{}^i_0\partial^0_i)\,, \qquad \gamma_0{}^i_0 \in C^\infty(UM)\,. \tag{2.14}$$

Theorem 2.3. *If* Γ *is a connection on* UM*, then*

$$\gamma := \text{д} \lrcorner \Gamma : UM \to \mathbb{T}^*\otimes TUM$$

is a second order connection with coordinate expression (2.14), *where*

$$\gamma_0{}^i_0 = \Gamma_0{}^i_0 + \Gamma_j{}^i_0 x^j_0\,.$$

Proof. It follows from (2.1) and (2.9). $\boxed{QED}$

Corollary 2.5. *The metric* g *yields the second order* gravitational *connection*

$$\gamma^\natural := \text{д} \lrcorner \Gamma^\natural\,. \quad \square$$

2.8. Distinguished 2–forms

Next, we study the 2-forms induced on the phase space by the connections. Again, in the Einstein case the Lorentzian metric allows us to recover objects and relations which in the Galilei case are provided by the fibring over time.

Let us consider a linear connection K and recall the canonical form σ (see (0.3)). Then, we define the 2–form

$$\Omega(g, K) := \nu_K \bar{\wedge} \sigma : \mathbb{T}^*\otimes TM \to \mathbb{T}^*\otimes\mathbb{L}^2\otimes\wedge^2 T^*(\mathbb{T}^*\otimes TM)$$

with coordinate expression (see (1.5), (0.3))

$$\Omega(g, K) = \nu_K \bar{\wedge} \sigma = g_{\lambda\mu} u^0 \otimes (\dot{d}^\lambda_0 - K_\varphi{}^\lambda{}_\psi \dot{x}^\psi_0 d^\varphi) \wedge d^\mu\,.$$

Remark 2.5. Let us consider the maps

$$\pi_1 : UM \underset{M}{\otimes} (\mathbb{T}^*\otimes TM) \to UM\,, \qquad \pi_2 : UM \underset{M}{\otimes} (\mathbb{T}^*\otimes TM) \to \mathbb{T}^*\otimes TM\,.$$

Then, we can write on $UM \underset{M}{\otimes} (\mathbb{T}^*\otimes TM)$

$$\sigma \simeq \sigma\circ\pi_2 = (\lambda + \vartheta)\circ\pi_1 \simeq \lambda + \vartheta\,.$$

Therefore, we obtain the splitting of $\Omega(g,K)$ over $UM \underset{M}{\otimes} (\mathbb{T}^*\otimes TM)$ into parallel and orthogonal components

$$\Omega(g,K) \equiv \Omega^{||}(g,K) + \Omega^{\perp}(g,K) := \nu_K \bar{\wedge} \lambda + \nu_K \bar{\wedge} \vartheta\,,$$

where

$$\Omega^{||}(g,K)\,,\Omega^{\perp}(g,K) : UM \underset{M}{\times} (\mathbb{T}^*\otimes TM) \to \mathbb{T}^*\otimes\mathbb{L}^2\otimes\wedge^2 T^*(\mathbb{T}^*\otimes TM)$$

have the coordinate expressions

$$\Omega^{||}(g,K) = c^2 \tau_\lambda \tau_\mu u^0 \otimes (\dot{d}^\lambda_0 - K_\varphi{}^\lambda{}_\psi \dot{x}^\psi_0 d^\varphi) \wedge d^\mu\,,$$
$$\Omega^{\perp}(g,K) = (g_{\lambda\mu} - c^2 \tau_\lambda \tau_\mu) u^0 \otimes (\dot{d}^\lambda_0 - K_\varphi{}^\lambda{}_\psi \dot{x}^\psi_0 d^\varphi) \wedge d^\mu\,. \quad \square$$

The above forms on the tangent space of space–time induce via pullback analogous forms on the phase space.

Remark 2.6. We obtain on UM the 2-form

$$\omega(g,K) = \omega^{||}(g,K) + \omega^{\perp}(g,K) := д^*\Omega(g,K) : UM \to \mathbb{T}^*\otimes\mathbb{L}^2\otimes\wedge^2 T^*UM$$

where

$$\omega^{||}(g,K) := (id_{UM}, д)^*\Omega^{||}(g,K)\,, \qquad \omega^{\perp}(g,K) := (id_{UM}, д)^*\Omega^{\perp}(g,K)\,.$$

In coordinates

(2.15) $$\omega(g,K) = $$
$$= c\,\alpha\,\big((g_{i\mu} - c^2\,\tau_i\tau_\mu) d^i_0 - (\frac{1}{2}\,c\,\alpha\,\partial_\varphi(\alpha^{-2})\tau_\mu + g_{\lambda\mu}(K_\varphi{}^\lambda{}_0 + K_\varphi{}^\lambda{}_j x^j_0)) d^\varphi\big) \wedge d^\mu\,.$$

and

(2.16) $$\omega^{||}(g,K) \;=\; -c\,\alpha\,\big(\frac{1}{2}\,c\,\alpha\,\partial_\varphi(\alpha^{-2})\tau_\mu + c^2 \tau_\lambda \tau_\mu (K_\varphi{}^\lambda{}_0 + K_\varphi{}^\lambda{}_j x^j_0)\big) d^\varphi \wedge d^\mu$$

(2.17) $$\omega^{\perp}(g,K) = $$
$$= c\,\alpha\,(g_{i\mu} - c^2 \tau_i \tau_\mu)\big(d^i_0 - (K_\varphi{}^i{}_0 + K_\varphi{}^i{}_j x^j_0 - x^i_0(K_\varphi{}^0{}_0 + K_\varphi{}^0{}_j x^j_0)) d^\varphi\big) \wedge d^\mu\,. \quad \square$$

On the other hand a connection Γ on UM and the metric g yield the scaled 2–form on UM

$$\Omega(g,\Gamma) := (v^\perp \circ \nu_\Gamma) \bar{\wedge} \vartheta : UM \to \mathbb{T}^*\otimes\mathbb{L}^2\otimes\wedge^2 T^*UM\,,$$

with coordinate expression

(2.18) $$\Omega(g,\Gamma) = c\,\alpha\,(g_{i\mu} - c^2 \tau_i \tau_\mu)\,(d^i_0 - \Gamma_\varphi{}^i_0 d^\varphi) \wedge d^\mu$$

(2.19) $$= c\,\alpha\,(g_{ij} - c^2 \tau_i \tau_j)\,(d^i_0 - \Gamma_\varphi{}^i_0 d^\varphi) \wedge \vartheta^j$$

(2.20) $$= c\,\alpha\,(g_{ij} - c^2 \tau_i \tau_j)\,(d^i_0 - \gamma_0{}^i_0 d^0 - \Gamma_h{}^i_0 \vartheta^h) \wedge \vartheta^j\,,$$

where $\gamma := д \lrcorner \Gamma$.

Theorem 2.4. *If Γ is the connection on UM induced by a linear connection K on TM then*

$$\Omega(g, \Gamma) = \omega^{\perp}(g, K)\,. \tag{2.21}$$

PROOF. It can be proved by using (2.11), (2.17) and (2.19). $\boxed{QED}$

Theorem 2.5. *There is a unique second order connection γ such that*

$$\gamma \lrcorner \Omega(g, \Gamma) = 0\,.$$

Namely,

$$\gamma = д \lrcorner \Gamma : UM \to \mathbb{T}^*\otimes TUM\,.$$

PROOF. It follows from (2.1), (2.14) and (2.20). $\boxed{QED}$

2.9. Distinguished metric 2–forms

Eventually, we specialise the results of the above section by referring to the gravitational connections $K^{\natural} \equiv \varkappa$ and $\Gamma^{\natural}$ induced by g. We analyse the contact structure on UM via the form $c^2\tau$. This structure is expected to be important for our covariant quantisation programme in the Einstein background.

We start by studying the gravitational 2–form $\Omega(g, \varkappa)$ on $\mathbb{T}^*\otimes TM$.

Remark 2.7. The metric g yields the scaled Liouville 1–form

$$\theta : \mathbb{T}^*\otimes TM \to \mathbb{T}^*\otimes\mathbb{L}^2\otimes T^*(\mathbb{T}^*\otimes TM)$$

defined by

$$\theta(X) := g(T\pi_M(X)\,, \pi_{\mathbb{T}^*\otimes TM}(X))\,, \qquad \forall X \in T(\mathbb{T}^*\otimes TM)\,.$$

In coordinates

$$\theta = g_{\mu\lambda}\dot{x}_0^{\lambda} u^0 \otimes d^{\mu}\,. \;\square$$

Proposition 2.7. *The 2–form*

$$\Omega(g) := d\theta : \mathbb{T}^*\otimes TM \to \mathbb{T}^*\otimes\mathbb{L}^2\otimes\wedge^2 T^*(\mathbb{T}^*\otimes TM)\,, \tag{2.22}$$

with coordinate expression

$$\Omega(g) = u^0\otimes(\partial_{\varphi} g_{\mu\lambda}\dot{x}_0^{\lambda} d^{\varphi} + g_{\mu\lambda}\dot{d}_0^{\lambda}) \wedge d^{\mu}\,,$$

is a symplectic form on $\mathbb{T}^*\otimes TM$ and

$$\Omega(g) = \Omega(g, \varkappa)\,. \;\square \tag{2.23}$$

Next, we study the gravitational 2–form on UM

$$\Omega^{\natural} := \Omega(g, \Gamma^{\natural})\,.$$

Lemma 2.8. *We have*

$$\omega^{||}(g, \varkappa) = 0\,. \tag{2.24}$$

PROOF. It can be proved in coordinates, by using (2.16) and (2.12). QED

Proposition 2.8. *We obtain*

$$\omega(g) := д^* \Omega(g) = \omega(g, \varkappa)\,.$$

PROOF. It follows from (2.21), (2.24), (2.22). □

Lemma 2.9. *The form*

$$\eta := c^2 \tau \wedge \omega(g) \wedge \omega(g) \wedge \omega(g) : UM \to \mathbb{T}^{*4} \otimes \mathbb{L}^8 \otimes \wedge^7 T^* UM\,,$$

with coordinate expression

$$c^2 \tau \wedge \omega(g) \wedge \omega(g) \wedge \omega(g) = 6\, c^4 \alpha^4 \, |g| \, d^0 \wedge d^1 \wedge d^2 \wedge d^3 \wedge d^1_0 \wedge d^2_0 \wedge d^3_0\,,$$

is a scaled volume form on UM. □

Lemma 2.10. *We obtain* $\quad д^* \theta = c^2 \tau\,.$ □

Theorem 2.6. *The form of UM*

$$\Omega^\natural := \Omega(g, \Gamma^\natural) = \omega(g, \varkappa) = \omega(g) = c^2 \, d\tau\,.$$

is the scaled contact 2–form generated by the contact 1–form $c^2 \tau$. □

Indeed, the 2–form $\Omega^\natural$ is our candidate for the quantisation programme in the Einstein case along the lines of [4], [5]. The fact that this form is closed is interesting for our aim. We stress that the form $\Omega^\natural$ has a natural global potential; an analogous result is not true in the Galilei case.

2.10. Electromagnetic field and geometric structure

So far, we have assumed on space–time the gravitational structure associated with the Lorentzian metric g. Next, we introduce the electromagnetic field.

Assupption E.3. We assume the *electromagnetic field* to be a closed scaled 2-form on M

$$F : M \to (\mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes \wedge^2 T^* M\,. \ \square$$

We denote the local potential of F by $A : M \to (\mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes T^* M$. Thus, by definition, we set

$$2dA = F\,.$$

We define the *'universal' electric* and *'universal' magnetic fields* to be the scaled forms on UM

$$E := -д \lrcorner F : UM \to (\mathbb{T}^{*2} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes T^*_\perp M\,,$$

$$B := F + 2\tau \wedge E : UM \to (\mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes \wedge^2 T^*_\perp M\,.$$

Thus, we can write

$$F = -2\tau \wedge E + B.$$

Hence, the *electric* and *magnetic fields* associated with an observer $o : M \to UM$ are defined as the pullback forms

$$o^* E : M \to (\mathbb{T}^{*2} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes T^* M \,, \quad o^* B : M \to (\mathbb{T}^{*} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* M \,.$$

We stress that the universal electric field carries the full information of the electromagnetic field.

Remark 2.8. We have the coordinate expressions

$$F = 2 F_{0j} d^0 \wedge d^j + F_{ij} d^i \wedge d^j = -\frac{2}{c\alpha} E_j \lambda^0 \wedge \vartheta^j + B_{ij} \vartheta^i \wedge \vartheta^j \,,$$

and

$$E = E_i \vartheta^i = E_j d^j + E_0 d^0 \,, \qquad B = B_{ij} \vartheta^i \wedge \vartheta^j = B_{ij} d^i \wedge d^j + 2 B_{0j} d^0 \wedge d^j \,,$$

where

$$E_j = -c\alpha (F_{0j} + F_{hj} x^h_0) \qquad E_0 = -c\alpha F_{h0} x^h_0 \,,$$
$$B_{ij} = F_{ij} + \tau_i E_j - \tau_j E_i \qquad B_{0j} = -B_{hj} x^h_0 \,.$$

Therefore, in a chart adapted to an observer o, the coordinate expressions of the observed electric and magnetic fields are

$$o^* E = -\frac{c}{\sqrt{g_{00}}} F_{0j} d^j$$
$$o^* B = (F_{ij} - \frac{1}{g_{00}} (g_{i0} F_{0j} - g_{j0} F_{0i})) d^i \wedge d^j \,. \quad \square$$

Next, we show that the electromagnetic field can be naturally incorporated into the gravitational structures of the phase space.

Namely, let us consider the following gravitational objects on the phase space UM:

$$\Gamma^\natural : UM \to T^* M \underset{UM}{\otimes} TUM \,, \qquad \gamma^\natural : UM \to \mathbb{T}^* \otimes TUM \,,$$
$$c^2 \tau : UM \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes T^* UM \,, \qquad \Omega^\natural : UM \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \Lambda^2 T^* UM \,,$$

which fulfill the following structural relations

$$\gamma^\natural = д \lrcorner \Gamma^\natural \,, \qquad \Omega^\natural = \nu_{\Gamma^\natural} \bar{\wedge} \vartheta \,, \qquad \gamma^\natural \lrcorner \Omega^\natural = 0 \,, \qquad c^2 d\tau^\natural = \Omega^\natural \,.$$

We are looking for *total* objects obtained correcting the gravitational objects by an electromagnetic term, in such a way to preserve the above relations.

For this purpose we need a suitable coupling constant. So, we consider a particle with a given mass and charge

$$m \in \mathbb{M}, \qquad q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \,,$$

and refer to the coupling constant

$$\frac{q}{m} \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{*1/2}$$

We start from the obvious coupling of the electromagnetic field with the gravitational contact 2-form on UM.

Accordingly, we define the *total 2-form* to be

$$\Omega := \Omega^{\natural} + \frac{q}{2mc} F : UM \to \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \wedge^2 T^* UM \,.$$

Here, the factor $\frac{1}{2}$ is chosen in such a way to recover the standard normalization in practical formulas.

Of course, we obtain

$$d\Omega = 0 \,.$$

However, while $\Omega^{\natural}$ has a natural and global potential, the potential of Ω is defined up to a gauge and in general exists only locally. Indeed, we obtain locally

$$d(c^2 \tau) \equiv d(c^2 \tau^{\natural} + \frac{q}{mc} A) = \Omega \,.$$

We can write locally

$$(c^2 \tau^{\natural} + \frac{q}{mc} A) \wedge \Omega \wedge \Omega \wedge \Omega = (c^2 \tau^{\natural} + \frac{q}{mc} A) \wedge \Omega^{\natural} \wedge \Omega^{\natural} \wedge \Omega^{\natural} \,.$$

This 7–form needs not to be a local volume form; for instance, if $A = -g_{0\mu} d^{\mu}$, then this 7–form vanishes. Hence, Ω needs not to be locally a contact 2–form.

Next, we look for the total second order connection on UM.

Lemma 2.11. *A section $\gamma : UM \to \mathbb{T}^* \otimes TUM$ is a second order connection if and only if $\gamma - \gamma^{\natural}$ is a section $\gamma - \gamma^{\natural} : UM \to \mathbb{T}^* \otimes VUM$.* □

Theorem 2.7. *There is a unique second order connection γ such that*

$$\gamma \lrcorner \Omega = 0 \,.$$

Namely, γ is given by $\gamma = \gamma^{\natural} + \gamma^{e}$, where

$$\gamma^{e} = \frac{q}{mc} v^{\perp\,-1} \circ g^{\sharp} \circ E \,,$$

with coordinate expression

$$\gamma^{e} = -\frac{q}{mc} (g^{i\mu} - x_0^i g^{0\mu})(F_{0\mu} + F_{j\mu} x_0^j) \partial_i^0 = \frac{q}{mc^2 \alpha} g_{\perp}^{ij} E_j \partial_i^0 \,.$$

PROOF. In virtue of the expressions of $\Omega^{\natural}, \gamma^{\natural}$ and F, we have

$$\gamma^{\natural} \lrcorner F = д \lrcorner F \,, \qquad \gamma^{e} \lrcorner \Omega^{\natural} = g^{\flat} \circ v^{\perp} \circ \gamma^{e} \,.$$

Then, we can write

$$\begin{aligned} 0 = \gamma \lrcorner \Omega &= \gamma^{\natural} \lrcorner \Omega^{\natural} + \frac{q}{2mc} \gamma^{\natural} \lrcorner F + \gamma^{e} \lrcorner \Omega^{\natural} + \frac{q}{2mc} \gamma^{e} \lrcorner F \\ &= 0 + \frac{q}{2mc} д \lrcorner F + \frac{1}{2} g^{\flat} \circ v^{\perp} \circ \gamma^{e} + 0 \,. \quad \boxed{QED} \end{aligned}$$

Remark 2.9. Thus, we have recovered the Lorentz force

$$f := -\frac{q}{mc} \circ g^\sharp \circ E : UM \to \mathbb{T}^* \otimes T^\perp M \subset UM \underset{M}{\times} (\mathbb{T}^{*2} \otimes TM)\,,$$

with coordinate expression

$$f = -\frac{q\alpha}{m}(g^{i\mu} - x^i_0 g^{0\mu})(F_{0\mu} + F_{j\mu} x^j_0) b_i = \frac{q}{mc} g^{ij}_\perp E_j b_i\,,$$

by a natural procedure as a byproduct of the coupling between gravitational and electromagnetic field. □

Remark 2.10. The law of motion

$$\nabla_\gamma js := j_2 s - \gamma \circ js = 0$$

for the unknown motion $s \subset M$ reads

$$v^\perp \circ \nabla_{\gamma^\natural} js = f \circ js\,. \quad \square$$

Eventually, we look for the total connection on UM.

Let us set

$$g^{\sharp 2} : T^*M \underset{M}{\otimes} T^*M \to \mathbb{L}^{*2} \otimes T^*M \underset{M}{\otimes} TM : \alpha \otimes \beta \mapsto \alpha \otimes g^\sharp(\beta)\,.$$

Lemma 2.12. *We have the following sections*

$$\Gamma^e_E := -\frac{q}{2mc} v^{\perp\,-1} \circ \vartheta \circ g^{\sharp\,2} \circ (2\tau \wedge E) : UM \to T^*_{||}M \underset{M}{\otimes} VUM\,,$$
$$\Gamma^e_B := -\frac{q}{2mc} v^{\perp\,-1} \circ g^{\sharp\,2} \circ B : UM \to T^*_\perp M \underset{M}{\otimes} VUM\,,$$

with coordinate expressions

$$\Gamma^e_E = \frac{q}{2mc^2\alpha} g^{ij}_\perp E_j \tau \otimes \partial^0_i\,, \qquad \Gamma^e_B = -\frac{q}{2mc^2\alpha} g^{ij}_\perp B_{hj} \vartheta^h \otimes \partial^0_i\,.$$

Moreover, we obtain

$$д \lrcorner \Gamma^e_E = \gamma^e\,, \qquad -(v^\perp \circ \Gamma^e_E) \bar\wedge \vartheta = -\frac{q}{2mc} 2\tau \wedge E\,,$$
$$д \lrcorner \Gamma^e_B = 0\,, \qquad -(v^\perp \circ \Gamma^e_B) \bar\wedge \vartheta = \frac{q}{2mc} B\,.$$

Proof. It follows by a computation in coordinates, by using the base (λ^0, ϑ^i), Lemma (2.3), and Lemma (2.4). $\boxed{QED}$

Let us consider a section

$$H : UM \to T^*M \underset{UM}{\otimes} VUM\,.$$

We define the map

$$A(v^\perp \circ H) : UM \to \mathbb{T}^* \otimes \Lambda^2 T^*_\perp M$$

by means of the composition

$$UM \xrightarrow{H} T^*M \underset{UM}{\otimes} VUM \xrightarrow{\vartheta^*\otimes(g^\flat\circ v^\perp)} \mathbb{T}^*\otimes T^*_\perp M \underset{UM}{\otimes} T^*_\perp M \xrightarrow{\mathrm{Alt}} \mathbb{T}^*\otimes\Lambda^2 T^*_\perp M.$$

Lemma 2.13. *The following conditions are equivalent*

$$д \lrcorner H = 0\,, \qquad (v^\perp \lrcorner H)\bar{\wedge}\vartheta = 0\,;$$
$$H : UM \to T^*_\perp M \underset{UM}{\otimes} VUM \subset T^*M \underset{UM}{\otimes} VUM\,, \qquad A(v^\perp \circ H) = 0\,;$$
$$H = H_j{}^i \vartheta^j \otimes \partial^0_i\,, \qquad g^\perp_{ih} H_j{}^h = g^\perp_{ih} H_j{}^h\,.$$

Proof. It follows by a computation in coordinates, by using the base (λ^0, ϑ^i). $\boxed{QED}$

Theorem 2.8. *There is a unique connection* $\Gamma : UM \to T^*M \underset{UM}{\otimes} TUM$*, such that*

$$д \lrcorner \Gamma = \gamma\,, \qquad (v^\perp \circ \nu_\Gamma)\bar{\wedge}\vartheta = \Omega\,, \qquad A(v^\perp \circ (\Gamma - \Gamma^\natural)) = 0\,.$$

Namely, $\Gamma = \Gamma^\natural + \Gamma^e$, *where*

$$\Gamma^e = \Gamma^e_E + \Gamma^e_B = -\frac{q}{2mc} v^{\perp\,-1} \circ g^{\sharp 2} \circ (F + 4\tau \wedge E)\,,$$

i.e. , in coordinates

$$\Gamma^e = \frac{q}{2mc^2\alpha} g^{ij}_\perp (E_j \tau - B_{hj}\vartheta^h) \otimes \partial^0_i\,.$$

Proof. It follows from the above Lemmas and from the expressions of γ and Ω. $\boxed{QED}$

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