DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 567–571

PROJECTIVE SYMMETRY AND HOLONOMY

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Abstract. Some results are presented which concern the relationship between the existence of projective symmetries and the reducibility of the holonomy group for Riemannian and Pseudo-Riemannian manifolds. A few general remarks are made on the existence of projective vector fields.

1. Introduction

Projective symmetries of a manifold M admitting a connection Γ are, roughly speaking, local diffeomorphisms of M which preserve the geodesics of Γ. If they also preserve the affine parameters of these geodesics they are called affine symmetries. If further Γ is the Levi-Civita connection of some metric g on M (of any signature) then an *affine* symmetry of M may also preserve g, either exactly or up to a constant conformal factor, or it may not. If it does not then it is referred to as a *proper affine symmetry*. Such a proper affine symmetry of Γ leads to an alternative metric g' for M, not related to g by a constant conformal factor, but whose Levi-Civita connection is also Γ. Hence the holonomy bundle of Γ is a proper subset of the (pseudo-) orthogonal bundle of g and so the holonomy group of Γ is reduced. The exact nature of the holonomy decomposition together with the de Rham-Wu theorem [1,2] can be used to study the nature of the existing affine symmetries (see e.g. [3,4] for the application to Einstein's general relativity).

In this paper a relation between the holonomy group and the existence of projective symmetries of Γ is explored and, in particular, a connection between holonomy reducibility and the existence of special projective symmetries is found.

In order to be more precise let M be a smooth manifold of dimension n with a smooth metric g (of any signature) and corresponding Levi-Civita connection Γ. A global smooth vector field X on M is called *projective* if its associated local (flow) diffeomorphisms ψ_t for appropriate $t \in \mathbf{R}$ each map a geodesic of Γ into another such geodesic. If each of the maps ψ_t always preserve affine parametrisation X is called an *affine vector field*. If the ψ_t also preserve the metric, X is called *Killing*

1991 *Mathematics Subject Classification*. 53B05.
Key words and phrases. Projective symmetry, connections, holonomy.
This paper is in final form and no version of it will be submitted for publication elsewhere.

whilst if the ψ_t preserve the metric only up to a constant conformal factor, X is called *homothetic*. It is remarked that it is only necessary to assume X is C^3 on M. The smoothness of X then follows from the smoothness of Γ [5].

Using a semi-colon to denote a covariant derivative with respect to Γ and $\mathcal{L}$ to denote a Lie derivative one may decompose the covariant derivative of X in any coordinate system as

$$X_{a;b} = \frac{1}{2}h_{ab} + F_{ab} \qquad (h_{ab} = h_{ba}, \quad F_{ab} = -F_{ba}) \tag{1}$$

where $h = \mathcal{L}_X g$. The condition that X is *projective* is then

$$h_{ab;c} = 2g_{ab}\phi_c + g_{ac}\phi_b + g_{bc}\phi_a \tag{2}$$

for some smooth function $\phi : M \to \mathbf{R}$ and where $\phi_a \equiv \phi_{,a} = \partial\phi/\partial x^a$. The equation (2) implies that $3h_{ab;c} - h_{bc;a} - h_{ac;b} = 4g_{ab}\phi_c$ and so the conditions that ϕ is constant on M and that h_{ab} is covariantly constant on M are equivalent and, if they hold, X is *affine*. If h is a constant multiple of g then X is *homothetic* and if $h \equiv 0$, X is *Killing*. If X is projective but not affine it is called *proper projective* and if X is proper projective but ϕ is restricted by the condition $\phi_{a;b} = 0$ then X is called *special projective*. For any projective vector field X one necessarily has, with the usual abuse of notation,

$$\mathcal{L}_X\, R^a{}_{bcd} = \delta^a_d\phi_{b;c} - \delta^a_c\phi_{b;d} \tag{3}$$

$$\mathcal{L}_X\, R_{ab} = (1-n)\phi_{a;b} \tag{4}$$

for the curvature tensor of Γ with components $R^a{}_{bcd}$ and the associated Ricci tensor with components $R_{ab} \equiv R^c{}_{acb}$. A vector field Y on M satisfying $\mathcal{L}_Y\, R^a{}_{bcd} = 0$ is called a *curvature collineation*. Thus a projective vector field on M is a curvature collineation if and only if $\phi_{a;b} = 0$, that is, Y is special projective.

2. The Main Results

One of the difficulties encountered in the study of projective symmetries is obtaining information on the function ϕ associated with a projective vector field X on M. In this respect the following theorem is helpful (see [5]).

Theorem 1. *Let M be a smooth manifold of dimension $n \geq 3$ with smooth metric g and let X be a projective vector field on M satisfying equations (1)-(4).*

(i) If for $m \in M$ there is a non-zero decomposable 2-form at m with components $G^{ab} = p^a q^b - q^a p^b$, where p and q are vectors at m, satisfying $R^a{}_{bcd}G^{cd} = 0$ then p and q are eigenvectors of the symmetric tensor $\phi_{a;b}$ (with respect to g_{ab}) at m with the same eigenvalues.

(ii) If for each $m \in M$ there is a coordinate neighbourhood U of m and a smooth vector field k on U satisfying $R^a{}_{bcd}k^d = 0$ on U then $\phi_{a;b} = 0$ on M.

(iii) If for each $m \in M$ there is a coordinate neighbourhood U of m and a smooth decomposable 2-form G on U satisfying the condition (i) at each point of U and

it is also given that $\phi_{a;b} = \alpha g_{ab}$ *for some function* $\alpha : M \to \mathbf{R}$ *then* $\alpha \equiv 0$ *on* M *and so* $\phi_{a;b} = 0$ *on* M.

This theorem can be used to establish the following results (c.f. [5]) concerning holonomy reducibility. It says, roughly speaking, that if g is positive definite and if Γ admits a special projective vector field then *all* proper projective vector fields admitted by Γ are special and the holonomy group of Γ is reducible. In this sense, special and non-special proper projective vector fields "cannot exist together". One also notes here that special projective vector fields satisfy a "uniqueness" theorem in the sense that if X is special projective and Y is proper projective ($\Rightarrow Y$ is special projective) on M then Y is a linear combination of X and an *affine* vector field on M.

Theorem 2. *Let* M *be a smooth connected manifold of dimension* $n \geq 3$ *with smooth positive definite metric* g *and associated Levi-Civita connection* Γ.

(i) If X *is a proper projective vector field on* M *then* X *is special projective if and only if the holonomy group of* Γ *is reducible.*

(ii) If Γ *is not flat and* M *admits a special projective vector field then* Γ *is not geodesically complete.*

In the case when M is a space-time, so that $n = 4$ and g has Lorentz signature, there is an analogue of Theorem 2. However one must now distinguish between the cases when the holonomy group of Γ is reducible and when it is *non-degenerately* reducible ([2], see also [6]) the latter meaning that at some (any) $m \in M$ a non-trivial, proper *non-null* subspace of the tangent space to M at m is holonomy invariant. The problems in this case arise from the more complicated nature of the subgroup structure of the Lorentz group (and hence the holonomy group of Γ) as compared with the corresponding orthogonal group [6].

Theorem 3. *Let* M *be a smooth space-time with smooth Lorentz metric* g *and associated Levi-Civita connection* Γ *and let* X *be a proper projective vector field on* M.

(i) If X *if special projective the holonomy group of* Γ *is reducible.*

(ii) If the holonomy group of Γ *is non-degenerately reducible,* X *is necessarily special projective.*

The converse of (ii) is false as the following example shows.

Choose global coordinates $u(= x^0), v(= x^1), x^2, x^3$ on $\mathbf{R}^4$ and let M be the open submanifold given by $u > 0$. Define a metric on M by [5,7]

$$ds^2 = 2du\,dv + u^2 P_{\alpha\beta} dx^\alpha dx^\beta \tag{5}$$

where α, β range over the indices 2 and 3 and the $P_{\alpha\beta}$ depend only on x^2 and x^3 with $P_{\alpha\beta}$ positive definite. Then the vector field with components $(uv, u^2, 0, 0)$ is a special projective vector field on M. However, the holonomy group of the Levi-Civita connection of the metric (5), although reducible, is not non-degenerately reducible. (It is a subgroup of the null rotation group about the null vector field $\partial/\partial v$ and is of the type referred to as R_{11} [3,6].)

Under certain extra assumptions (whose significance is not fully understood and which may not be necessary) a much more detailed version of Theorem 3 is available together with some remarks on the completeness of Γ [5]. It is also remarked here that special and non-special proper projective vector fields cannot exist on the same space-time and that special projective vector fields satisfy the same "uniqueness" theorem as that described at the beginning of this section in the positive definite case.

3. Further Remarks on Projective Vector Fields

In this section two further developments regarding projective vector fields are briefly discussed. They are, as yet, incomplete and further details will be published elsewhere.

The first remark is essentially a generalisation of Theorem 1(i). Let M be a general manifold with arbitrary metric g as discussed in section 1 and let X be a projective vector field on M. Suppose that at $p \in M$ the curvature tensor admits an *eigenbivector* G (not necessarily decomposable) with eigenvalue α (not necessarily zero) so that

$$R_{abcd}G^{cd} = \alpha G_{ab} \tag{6}$$

By taking the Lie derivative with respect to X of the relation $g_{ae}R^{e}{}_{bcd}+g_{be}R^{e}{}_{acd} = 0$ and using (6) and (3) one can show that, at p

$$P_{ac}G^{c}{}_{b} + P_{bc}G^{c}{}_{a} = 0 \qquad (P_{ab} = \alpha h_{ab} + 2\phi_{a;b}) \tag{7}$$

Equation (7) is formally a skew-self adjointness relation between $G^{a}{}_{b}$ and P_{ab} at p and reflects the close connection between h_{ab}, $\phi_{a;b}$ and the algebraic structure of the curvature tensor (and which is described by Theorem 1 in the case when G is decomposable and $\alpha = 0$). In particular if there are "sufficiently many" eigenbivectors of the curvature tensor at p *with the same eigenvalue* α then (7) gives at p

$$\alpha h_{ab} + 2\phi_{a;b} = \beta g_{ab} \qquad (\beta \in \mathbf{R}) \tag{8}$$

In the case when M has constant curvature (and is hence an Einstein space) there are $n(n-1)/2$ bivectors at each $p \in M$ with the same eigenvalue and so (8) holds (with $\beta = 0$ as follows from (4)).

The second remark is mainly concerned with the case of general relativity (so that $n = 4$ and g is a Lorentz metric). Let the components of the *Weyl projective curvature tensor* be $W^{a}{}_{bcd}$. Then

$$W^{a}{}_{bcd} = R^{a}{}_{bcd} - \frac{1}{3}(\delta^{a}_{c}R_{bd} - \delta^{a}_{d}R_{bc}) \tag{9}$$

This tensor has the following properties.

i. $W^{a}{}_{bcd} = C^{a}{}_{bcd} \Leftrightarrow M$ is an Einstein space (-time) (where $C^{a}{}_{bcd}$ are the components of the *Weyl conformal curvature tensor*).
ii. $W^{a}{}_{bcd} = R^{a}{}_{bcd} \Leftrightarrow M$ is a vacuum space-time.

iii. $W^a{}_{bcd} = 0 \Leftrightarrow M$ has constant curvature.
iv. If X is a projective vector field on M, $\mathcal{L}_X W^a{}_{bcd} = 0$.

The first three of these properties follow from (9) and the fourth (whose converse is easily seen to be false by considering curvature collineations in vacuum space-times [8] and comparing with [5]) is a well-known consequence of X being a projective vector field. In the study of projective vector fields in general relativity given in [5] it was found convenient to consider four types A, B, C and D of space-times determined by the algebraic structure of the curvature tensor. An alternative approach (which may be of more general applicability) is to show that for space–time types A, B and C the *projective tensor W on M uniquely determines the curvature tensor on M*. A consideration of the local flows of a projective vector field X on M for these types, together with iv above, then shows that these flows are local diffeomorphisms of M which preserve the projective tensor W. Hence they preserve the curvature tensor and so are necesssarily curvature collineations. It then follows from (3) that, for X, $\phi_{a;b} = 0$ and so one achieves the result that the projective vector field X is *special projective*.

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