DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 559–565

QUANTIZATION INDUCED BY GEOMETRY

GERALD FISCHER

1. Introduction

This contribution is guided by the idea of storing information about a geometric system in a suitable ring of objects of this system. This idea is mainly used in algebraic geometry [EH] where the information about algebro geometric objects, the varieties, is carried by schemes which are special ringed spaces $(\mathcal{G}, spec\,\mathcal{G})$ which consist of a structure sheaf $\mathcal{G}$ and a topological space given by the spectrum of the ring $\mathcal{G}$. Quite similar in quantum theory the information about a physical system is given by a ring of observables $\mathcal{R}$ represented as operators $\mathcal{O}$ on a Hilbert space $\mathcal{H}$ [BLT]. The eigenvalue equations correspond to the physical measurements which yield elements in the ground field $\mathbb{C}$ as measurement values. In slight contrast to usual ring theory here we define the spectrum $spec\,R$ of a ring R as the set of its maximal left ideals and a scheme as a ringed space which looks like $(R,\, spec\,R)$. In this context the pairs $(\mathcal{O}\,,\mathcal{H})$ in quantum theory and $(\mathcal{G}\,,\, spec\,\mathcal{G})$ in algebraic geometry represent the same idea only in different domains of research. We apply the idea of a scheme to differential geometry what forces us to introduce the term of a geometric observable and structure sheaves formed by these observables.

2. Quantum Theory

The algebraic formulation of quantum theory is based on *-algebras (algebras with involution) [BLT][E]. An abstract observable algebra $\mathcal{R}$ is a *-algebra which is irreducibly represented as an operator algebra $\mathcal{O}$ on a Hilbert space $\mathcal{H}$ of states. By taking expectation values

$$F_\psi(A) = (\psi\,,\, A\psi)\ ,\ A \in \mathcal{O},\ \psi \in \mathcal{H} \tag{1}$$

the states can be interpreted as positive functionals F_ψ on the observable algebra $\mathcal{R}$, i.e. elements in the dual space $\mathcal{R}^*$. Positive functionals formed by the vectors in a representation space in such a way are called vector functionals. A positive functional F on a *-algebra $\mathcal{R}$ induces a representation of $\mathcal{R}$ on a Hilbert space $\mathcal{H}_F$ by the GNS-construction [E]. This construction creates a Hilbert space $\mathcal{H}_F$

1991 *Mathematics Subject Classification.* 53B50, 81Q99.
Key words and phrases. Geometric objects, quantization, geometric observables.
This paper is in final form and no version of it will be submitted for publication elsewhere.

by completion of the pre-Hilbert space $\mathcal{R}/N_F$ where N_F is a left ideal in $\mathcal{R}$ defined by

$$N_F = \{F_\psi(A^*A) = 0 \mid A \in \mathcal{R}\}\,. \tag{2}$$

The scalar product in $\mathcal{H}_F$ is given by

$$(\psi_A, \psi_B) := F(A^*B)\,,\ \psi_A, \psi_B \in \mathcal{H}_F\,,\ A, B \in \mathcal{R}. \tag{3}$$

The set of positive functionals is a convex set. Its boundary points, the so called extremal positive functionals, are of particular interest for the GNS-construction, because they yield irreducible representations [S].
Let $\mathcal{A}$ be a *-algebra and $\mathcal{N}$ the set of its proper left ideals. Then we have the theorem

Theorem 1. The maximal ideals N_m in $\mathcal{N}$ are of the form N_F for some positive functionals F and the representations of $\mathcal{A}$ constructed by the GNS-method with these functionals are irreducible.

The proof is based on Zorns lemma and the partial orderings in the space of positive functionals

$$F_1 \leq F_2 \text{ iff } F_1(A) \leq F_2(A)\,, \text{ for all} A \in \mathcal{A} \tag{4}$$

and in the set of left ideals in $\mathcal{A}$

$$N_1 \geq N_2 \text{ iff } N_2 \subset N_1, \tag{5}$$

which are related by $F_1 \leq F_2 \Rightarrow N_2 \subset N_1$. The fact that extremal positive functionals give irreducible representations by the GNS-construction completes the proof.
Let $\mathcal{R}$ be an abstract ring of quantum observables and let $\rho : \mathcal{R} \to \mathcal{O}(\mathcal{H})$ be a representation of it on a Hilbert space $\mathcal{H}$. The vectors ψ_p in $\mathcal{H}$ which form extremal positive vector functionals are called pure vectors. To every pure vector ψ_p corresponds a maximal ideal N_p in $\mathcal{R}$. The set of these ideals is a subspace $\widetilde{spec_{\mathcal{H}}}\mathcal{R}$ of $spec\,\mathcal{R}$. We use this for the definition

Definition 1. If $\mathcal{R}$ is an algebra of quantum observables (*-algebra) and $(\rho\,,\ \mathcal{O}(\mathcal{H}))$ a representation of it (irreducibility not required) then we call the ringed space $(\mathcal{O}\,,\ spec\,\mathcal{O}) := (\mathcal{O}\,,\ \widetilde{spec}_{\mathcal{H}}\mathcal{R})$ the quantum scheme of $\mathcal{R}$ associated to ρ.

So we see that the whole physical information can be stored in a scheme like in algebraic geometry.

3. Differential geometry

Differential geometry is the theory of geometric objects living on smooth manifolds. Geometric objects are tupels of numbers, like tensors, attached to the points of a C^∞-manifold and subjected to a special behaviour under coordinate transformations. More precisely one defines geometric objects as the points in a natural bundle [N].

Definition 2. A natural bundle is a quadruple $\mathcal{E}(X) = (E, \pi, B, X)$ where (E, π, X) is a fibre bundle and B is the so called lifting functor from the category $C(X)$ of local diffeomorphisms of X to the category $C(E)$ of local diffeomorphisms of E, subject to the conditions:

(1) If U is an object of $C(X)$, then $B(U) = \pi^{-1}(U)$.
(2) If $V \subset U$ is open and $f : U \to X$ is a morphism of $C(X)$, then $Bf\mid_{B(V)} = B(f \mid V)$.

The sections in natural bundles are geometric object fields. In this contribution we are more interested in the fields as in the objects, so we are forced to deal with natural sheaves [Ec] formed from the sections of natural bundles. Differentiable maps between natural bundles which are fibre and base point preserving and which are natural transformations between the lifting functors are called natural maps [N]. The points of the image of a natural map are again geometric objects.
In order to represent differential geometry in the light of schemes, we form a ring of distinct geometric object fields, the geometric observables. For the motivation of their definition we first have a look at the simplest geometric objects, the scalar ones, and their fields, the smooth functions. The ring of smooth functions $C^\infty(M, I\!R)$ on a smooth manifold M is a *-algebra by complex conjugation $f^* = \bar{f}$. A special set of positive functionals on this algebra is given by the points in M

$$F_p(f) = ev_p(f) := f(p)\,, \quad f \in C^\infty(M, I\!R)\,, \, p \in M. \tag{6}$$

The ideals

$$N_{F_p} = \{f \in C^\infty(M, I\!R) | F_p(f) = f(p) = 0\} \tag{7}$$

are exactly the maximal ideals in the ring of smooth functions. So, by theorem 1, the irreducible representations of $C^\infty(M, I\!R)$ are parametrized by the points in M and we have the equalities

$$Hom\,(C^\infty(M, I\!R), I\!R) = spec\, C^\infty(M, I\!R) = M. \tag{8}$$

This result is often referred to as Milnors exercise [KMS] or as Gelfand transformation [R].
So the scheme $(C^\infty(M, I\!R), spec\, C^\infty(M, I\!R)) = (C^\infty(M, I\!R), M)$ contains the whole information about the smooth manifold M, which is twofold. The first part is the set of points of M i.e. the space M itself, the second part is the C^∞-structure. The sheaf of germs of smooth functions on a smooth manifold, here denoted by $\mathcal{S}$, is the structure sheaf of a smooth manifold. It is a natural sheaf,

and the ring of its sections is equivalent to the ring of smooth functions $C^\infty(M, I\!R)$ by taking representatives. So the information about smooth manifolds, the carrier-spaces of differential geometry, is stored in a set of distinct geometric object fields. This example shows that the evaluation map ev_p attaches a number in the ground field to every geometric object of these fields i.e. a result of a measurement. Thus ev_p in the notation of quantum theory is the measurement by the state or positive functional parametrized by p. So M is dual to $C^\infty(M, I\!R)$ and the measurement corresponds to the assignment of unique duals to the geometric objects. Therefore we are forced to introduce the term of a geometric observable.

Definition 3. A **geometric observable** is a geometric object field which assigns to every base point a value in the ground field. That means there exists a unique dual to every geometric object of these fields and the contraction assigns a number to the object. Such an object is called measurable.

On a C^∞-manifold without any further structure there is no possibility to measure other objects than scalars, e.g. the length of a vector is not defined. So the only geometric observables on such a manifold are the sections in $\mathcal{S}$, or equivalently, the smooth functions, and the whole information about M is carried by these geometric observables in the form of the scheme

$$(\mathcal{S}\,,\, spec\,\mathcal{S})\,. \tag{9}$$

We call the elements in $\mathcal{S}$ the classical observables.

Quantization means introducing quantum observables created from classical ones. In physical terms the introduction of new observables corresponds to the introduction of new measurements i.e. of interaction. Hence it is of great interest how new geometric observables can be created from the classical ones. To this end we propose a two-step method. The first step generates new geometric object fields from the classical observables. The second step assigns a unique dual to the geometric objects of this new fields:

(1) The structure sheaf $\mathcal{S}$ is a fibre space $(\mathcal{S}, \pi, M)$ over M, its sections, the classical observables, are maps $f : M \to \mathcal{S}$. By jet-prolongation of maps one gets a new fibre space $(J^k\mathcal{S}, \pi^k, M)$ [KMS][Sa]

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{j^k} & J^k\mathcal{S} \\ f\Big\uparrow & & \Big\uparrow j^k f \\ M & \longrightarrow & M \end{array} \tag{10}$$

Jet-prolongation is a natural map [KMS] for the lifting functors of natural bundles, consequently the new fibre spaces are also natural. The jet spaces

$J^k\mathcal{S}$ are sheaves of modules over $\mathcal{S}$, and the dual spaces are given thus by the dualizing sheaves

$$J^k\mathcal{S}^* = Hom_{\mathcal{S}}(J^k\mathcal{S}, \mathcal{S}). \tag{11}$$

Because dualizing natural vector bundles is also a natural map [N] we have two kinds of new geometric object fields, the sections

$$\Gamma J^k\mathcal{S} \text{ and } \Gamma J^k\mathcal{S}^*. \tag{12}$$

(2) Assigning an unique dual to the geometric objects in $J^k\mathcal{S}$ and $J^k\mathcal{S}^*$ requires the identification of these spaces i.e. an isomorphism

$$Q : J^k\mathcal{S} \xrightarrow{\simeq} J^k\mathcal{S}^* \tag{13}$$

$$\Rightarrow J^k\mathcal{S} \times J^k\mathcal{S} \xrightarrow{Q} J^k\mathcal{S} \times J^k\mathcal{S}^* \xrightarrow{contraction} \mathcal{S} \tag{14}$$

what is a further structure on M which has to be specified. Thus the enlarging of structure (information) on M induces an enlarging of the set of observable geometric objects. How this structure looks like we show with one of the simplest but also most important examples, the 1.jet-prolongation.

Example
The 1.jet-prolongation of $(\mathcal{S}, \pi, M)$ leads from the classical observables to the covector fields or differentials [KMS]

$$\Gamma\mathcal{S} \xrightarrow{j^1} \Gamma J^1\mathcal{S} = \Omega^1(M) \tag{15}$$

$$f \mapsto j^1 f. \tag{16}$$

The dual identification

$$Q \quad : \quad J^1\mathcal{S} = \Omega^1(M) \longrightarrow J^1\mathcal{S}^* = \mathcal{X}(M) = Der(\mathcal{S}) \tag{17}$$

$$j^1 f \mapsto Der_f := Q(j^1 f) \tag{18}$$

attaches an unique vector field Der_f to the covector fields $j^1 f$ what designs the new structure Q to be an isomorphism between vectors and covectors i.e. a geometry on M. Thus Q makes vectors on M observable and the vector fields, the derivations of the structure sheaf $\mathcal{S}$, become geometric observables. Now we have arrived at the content of the title, a geometry on M induces new geometric observables resulting in a new structure sheaf $\mathcal{S}^Q$.

4. Quantization induced by geometry

Let (M, Q) be a smooth manifold with a geometry Q on it. Q attaches in a smooth way to every point p in M a non degenerate bilinear form on the tangent spaces

$$Q_p \quad : \quad T_pM \times T_pM \longrightarrow I\!R \tag{19}$$

$$Q_p \quad : \quad T_pM \xrightarrow{\simeq} T_p^*M. \tag{20}$$

This structure allows the measurement of geometric objects different from the scalars, and thus geometric observables different from the classical ones are induced. A further, nonlocal, part of information carried by (M, Q) is, that also extended objects can be measured e.g. length of curves for a Riemannian geometry or the content of surfaces for a symplectic geometry. The whole information on (M, Q) should be stored in a new scheme with a new structure sheaf $\mathcal{S}^Q$ which is constructed from the geometric observables on (M, Q)

$$\mathcal{S} \longrightarrow \mathcal{S}^Q \tag{21}$$

$$f \mapsto f^Q := f + Der_f. \tag{22}$$

Introducing a quantum physical parameter $\hbar$ the algebra of sections in $\mathcal{S}^Q$ shows itself as a deformation [G] of the algebra $\Gamma\mathcal{S}$

$$(\mathcal{S}, \cdot) \longrightarrow (\mathcal{S}, *_\hbar) \simeq \mathcal{S}^Q \tag{23}$$

with

$$\begin{aligned} f *_\hbar g : \quad &= \quad f^Q g^Q = (f + \hbar Der_f)(g + \hbar Der_g) \\ &= \quad f \cdot g + \hbar(Der_f g + g \cdot Der_f + f \cdot Der_g) \\ &\qquad + \hbar^2(Der_f \circ Der_g) \end{aligned} \tag{24}$$

and the star-product commutator [G]

$$[f, g] := \tfrac{1}{2\hbar}(f *_\hbar g - g *_\hbar f) \stackrel{\hbar \to 0}{\longrightarrow} Der_f g =: \{f, g\} \tag{25}$$

is a quantization of the Poisson structure $\{\,,\,\}$.
Like in the algebraic formulation of quantum theory we introduce the abstract algebra of geometric observables on (M, Q) denoted by $\hat{\mathcal{S}}^Q$. The maximal ideal space $spec\,\hat{\mathcal{S}}^Q$ is again the space of irreducible representations of $\hat{\mathcal{S}}^Q$ given by $Hom(T_S, \mathbb{C})$ where T_S is the maximal toral subalgebra or the complete set of commuting observables in $\hat{\mathcal{S}}^Q$. But this algebra is defined by its representation $\mathcal{S}^Q$ acting on $\Gamma\mathcal{S}$, and it is a *-algebra because $\mathcal{S}^Q$ is one by the involution

$$f^{Q*} = f^* + \hbar Der_f^* = \bar{f} + \hbar \overleftarrow{Der_f}\ . \tag{26}$$

Furthermore the classical observables $\Gamma\mathcal{S}$ form a Hilbert space using the L^2-norm and we can restrict $spec\,\hat{\mathcal{S}}^Q$ to $\widetilde{spec}_{\mathcal{S}}\hat{\mathcal{S}}^Q$, the space of pure vectors in $\Gamma\mathcal{S}$. Therefore $(\mathcal{S}^Q\ , spec\,\mathcal{S}^Q)$ is a quantum scheme. It is easy to see that locally the pure vectors in $\Gamma\mathcal{S}$ are determined by the maximal isotropic subspaces for Q_p of the complexified tangent spaces $T_pM^{\mathbb{C}}$, and so by the polarizations of M. This fact relates the space $spec\,\mathcal{S}^Q$ to the space of sections in the twistor space $J(M)$, the bundle of Q-isometric almost complex structures on M [F].

In a diagram the whole idea is represented as

$$\begin{array}{rcl} M : (\mathcal{S}, spec\,\mathcal{S}) & = & (\mathcal{S}, M) \\ & & \Big\downarrow \; geometry\,Q \\ (M,Q) : (\mathcal{S}^Q, spec\,\mathcal{S}^Q) & = & (\mathcal{S}^Q, \Gamma J(M)) \end{array}$$

We see that the points of M loose their meaning after quantizing M by a geometry, they are replaced by points in an infinite dimensional space.
Examples, where geometry induced quantization arise are the geometric quantization of phase spaces (symplectic manifolds) or the quantization of Riemannian manifolds [F]. In the second case the symmetry of the Riemannian metric results in graded commutators, and so it is natural to consider the Riemannian manifolds as supermanifolds, i.e. consider graded structure sheaves.

References

[BLT] N.N. Bogolubov, A.A. Logunov, I.T. Todorov, Introduction to Axiomatic Quantum Field Theory, The Benjamin/Cummings Publishing Company, London 1975
[E] Gerard G. Emch, Algebraic Methods in Statistical Mechanics and Quantum Field Theory, John Wiley & Sons,London 1972
[EH] David Eisenbud, Joe Harris, Schemes: The Language of Modern Algebraic Geometry, Wadsworth & Brooks/Cole Mathematics Series, California 1992
[Ec] David J. Eck, Natural Sheaves, Illionois Journal of Mathematics Vol. 31, Number 2, 1987
[F] Gerald Fischer, work in progress
[G] S. Gutt, Some Aspects of Deformation Theory and Quantization, in: Quantum Theories and Geometry, Mathematical Physics Studies 10, Kluwer Academic Publishers, London 1988
[KMS] Ivan Kolář, Peter W. Michor, Jan Slovák, Natural Operations in Differential Geometry, Springer, Berlin 1993
[N] Albert Nijenhuis, Natural Bundles and Their General Properties in Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo 1972
[R] Walter Rudin, Functional Analysis, Tata McGraw-Hill Publishing Company LTD., New Delhi 1974
[Sa] D.J. Saunders, The Geometry of Jet Bundles, LMS 142, Cambridge University Press, Cambridge 1989
[S] Konrad Schmüdgen, Unbounded Operator Algebras and Representation Theory, OT 37, Birkhäuser, Berlin 1990

Institut für Theoretische Physik
Universität Regensburg
Regensburg, GERMANY