DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 479–485

CONTACT EQUIVALENCE OF MONGE-AMPÈRE EQUATIONS WITH TRANSITIVE SYMMETRIES

DIMITRY V. TUNITSKY

ABSTRACT. This paper deals with Monge-Ampère equations on 5-dimensional contact manifolds, i.e., M-A equations with two independent variables. If a M-A equation is in general position, then the unique affine connection can be put into correspondence with this equation in natural manner. This correspondence allows to formulate and prove a number of results on contact equivalence of M-A equations with transitive symmetry group using suitable properties of affine connections.

All objects are supposed to be smooth. All manifolds are real Hausdorff paracompact 5-dimensional connected and simply connected. For any fiber bundle G denote by $\Gamma(G)$ the set of all sections of this bundle.

1. Monge-Ampère equations

Consider a contact manifold M with contact structure generated by a linear differential contact form ω_0,

$$d\omega_0 \bigwedge d\omega_0 \bigwedge \omega_0 \neq 0.$$

The contact ideal

$$I_0 = (\omega_0, d\omega_0) \tag{1}$$

is generated by the contact form ω_0 and it's differential $d\omega_0$. A differential 2-form ω is called *effective* if

$$\omega \bigwedge d\omega_0 \bigwedge \omega_0 = 0$$

1991 *Mathematics Subject Classification.* 58G03, 58G16, 53C07, 53C30.
Key words and phrases. Monge-Ampère equation, characteristic bundle, contact equivalence, characteristic affine connection, invariance with respect to parallelism, contact symmetry, homogeneous M-A equation.
Research was partially supported by International Science Foundation and Russian Government grant N RO4300 and by Russian Fundamental Researches Fund grant N 93-01-00454.
This paper is in final form and no version of it will be submitted for publication elsewhere.

(cf. [1]). Let

$$I = (\omega_0, d\omega_0, \omega) \tag{2}$$

be the ideal of the ring $\Gamma(\Lambda^*(M))$ that is generated by the differential forms ω_0, $d\omega_0$ and ω. An ideal of the type (2) is called a *Monge-Ampère ideal.* A system of interior differential equations

$$I = 0 \tag{3}$$

is called a *Monge-Ampère equation.*

2. Characteristic bundles

Consider the complex tangent bundle

$$T^C M = C \otimes_R TM$$

of the manifold M. A complex vector $\xi \in T^C_m M$ is called a *characteristic vector* of the equation (3) at a point $m \in M$ if $\xi \lrcorner \omega_0 = 0$ and $\xi \lrcorner I(m) = \xi \lrcorner I_0(m)$ for the ideals (1) and (2). Put

$$H = \bigcup_{m \in M} H(m),$$

where

$$H(m) = \left\{ \xi \in T^C_m M : \xi \lrcorner \omega_0 = 0, \xi \lrcorner I(m) = \xi \lrcorner I_0(m) \right\}. \tag{4}$$

Lemma 1. *For any equation (3) there exist the two-dimensional vector bundles H_1 and H_2 of the complex tangent bundle $T^C M$ such that the set (4) can be represented in the form*

$$H(m) = H_1(m) \bigcup H_2(m)$$

for all points $m \in M$.

Proof. See [2]. $\square$

The fiber bundles H_1 and H_2 are called *characteristic bundles* of the equation (3). This equation is called *elliptic* at a point $m \in M$ if

$$H_1(m) \neq H_2(m), \quad \overline{H}_1(m) = H_2(m),$$

parabolic if

$$H_1(m) = H_2(m),$$

and *hyperbolic* if

$$H_1(m) \neq H_2(m), \quad \overline{H}_j(m) = H_j(m)$$

for $j = 1, 2$.

Remark 1. It can be shown that this definition of equation's (3) type coincides with traditional one (cf. [3, ch. 5, app. 1, par. 2]).

Fix a point $m \in M$ and consider two sections ξ_{j1} and ξ_{j2} of the equation's (3) bundle H_j that are linearly independent at m.

Definition 1. Suppose the equation (3) is elliptic or hyperbolic at the point m. This equation is in *general position* at the point m if the vectors

$$\xi_{j1}, \xi_{j2}, [\xi_{j1}, \xi_{j2}], [\xi_{j1}, [\xi_{j1}, \xi_{j2}]], [\xi_{j2}, [\xi_{j1}, \xi_{j2}]]$$

are linearly independent at m for $j = 1, 2$.

3. Contact equivalence

Consider another contact manifold $\widetilde{M}$ with contact form $\widetilde{\omega}_0$, Monge-Ampère ideal

$$\widetilde{I} = (\widetilde{\omega}_0, d\widetilde{\omega}_0, \widetilde{\omega}) \tag{5}$$

and Monge-Ampère equation

$$\widetilde{I} = 0, \tag{6}$$

where $\widetilde{\omega}$ is an effective differential 2-form.

Monge-Ampère equations (3) and (6) are called *contactly equivalent* if there exists a diffeomorphism

$$f : \widetilde{M} \to M \tag{7}$$

that takes the ideal (2) to the ideal (5):

$$f^* I = \widetilde{I}. \tag{8}$$

Remark 2. It follows from (8) that (7) is a contact diffeomorphism, i.e.,

$$f^* I_0 = \widetilde{I}_0.$$

4. Characteristic connections

Below all Monge-Ampère equations are in general position.

Consider a Monge-Ampère equation (3) and it's characteristic bundles H_1 and H_2, which are described by lemma 1. The first derivatives $H_1^{(1)}$ and $H_2^{(1)}$ of these bundles are 3-dimensional complex vector subbundles of the tangent bundle $T^C M$. That is why we have the representation

$$T^C M = H_0 \oplus H_1 \oplus H_2, \tag{9}$$

where

$$H_0 = H_1^{(1)} \bigcap H_2^{(1)}$$

is a 1-dimensional bundle. This representation determines projections

$$\chi^l : T^C M \to H_l, \tag{10}$$

$l = 0, 1, 2$, onto the corresponding summands at the right side of the representation (9).

Theorem 1. *Any Monge-Ampère equation (3) on the contact manifold M uniquely defines the affine connection Σ on M such that the following propositions hold for this connection.*

(a) For two arbitrary points $m, \widetilde{m} \in M$ and for any continuous curve that connects these points the parallel displacement

$$F : T_{\widetilde{m}} M \to T_m M \tag{11}$$

along this curve takes the equation's (3) set $H_1 \bigcup H_2$ of all characteristic vectors at $\widetilde{m}$ to the set of all characteristic vectors at m:

$$F(H_1(\widetilde{m}) \bigcup H_2(\widetilde{m})) = H_1(m) \bigcup H_2(m). \tag{12}$$

(b) Let (6) be a Monge-Ampère equation on the contact manifold $\widetilde{M}$, $\widetilde{\Sigma}$ its affine connection, and (7) a diffeomorphism. This diffeomorphism satisfies equality (8) if and only if it is affine, i.e.,

$$f_*(\widetilde{\Sigma}) = \Sigma \tag{13}$$

and for some points $m \in M$ and $\widetilde{m} \in \widetilde{M}$ such that

$$m = f(\widetilde{m}) \tag{14}$$

it takes the equation's (6) set $\widetilde{H}_1 \bigcup \widetilde{H}_2$ of all characteristic vectors at $\widetilde{m}$ to the equation's (3) set $H_1 \bigcup H_2$ of all characteristic vectors at m:

$$f_*(\widetilde{H}_1(\widetilde{m}) \bigcup \widetilde{H}_2(\widetilde{m})) = H_1(m) \bigcup H_2(m). \tag{15}$$

Proof. A construction of the stated affine connection Σ is based on the representation (9). This connection is the direct sum

$$\Sigma = \Sigma_0 \oplus \Sigma_1 \oplus \Sigma_2 \tag{16}$$

of linear connections Σ_0, Σ_1 and Σ_2 on the bundles H_0, H_1 and H_2 respectively. The definition of these connections is the following.

The covariant differentiation operator ∇^j of the linear connection Σ_j, $j = 1, 2$, is uniquely defined by the properties:

$$\nabla^j_X Y = \chi^j([X, Y])$$

for $X \in \Gamma(H_0 \oplus H_{3-j})$, $Y \in \Gamma(H_j)$ and

$$R^j(X, Y) = 0$$

for $X \in \Gamma(H_0)$, $Y \in \Gamma(H_{3-j})$, where R^j is the curvature of Σ_j and χ^j is the projection operator (10).

The linear connection Σ_0 on the bundle H_0 is a mean of linear connections $\Sigma_{0,1}$ and $\Sigma_{0,2}$:

$$\Sigma_0 = \frac{1}{2}(\Sigma_{0,1} + \Sigma_{0,2}). \tag{17}$$

The covariant differentiation operators $\nabla^{0,1}$ and $\nabla^{0,2}$ of the linear connections $\Sigma_{0,1}$ and $\Sigma_{0,2}$ are uniquely defined by the properties:

$$\nabla^{0,j}_X Y = \chi^0([X, Y]) \tag{18}$$

for $X \in \Gamma(H_1 \oplus H_2)$, $Y \in \Gamma(H_0)$ and

$$R^{0,j}(X, Y) = 0 \tag{19}$$

for $X, Y \in \Gamma(H_j)$, where $R^{0,j}$ is the curvature of $\Sigma_{0,j}$, $j = 1, 2$.

The proof of propositions (a) and (b) follows immediately from the described construction of the affine connection Σ. See [4] for technique details. □

Remark 3. The construction of connections (17) and (16) implies reality of these connections.

Definition 2. The affine connection that is uniquely put into correspondence with any Monge-Ampère equation in general position according to theorem 1 is called a *characteristic affine connection* of this equation.

Remark 4. It can be seen from the theorem's 2 proof that the construction of characteristic connections is based on the property of the equations under consideration to be in general position. Nevertheless the definition of the linear connection Σ_0 on H_0 by formulas (17), (18), and (19) is well-posed for all hyperbolic and elliptic Monge-Ampère equations. In particular, the curvature of this connection is a differential 2-form on M.

5. Equivalence theorems

Definition 3. A Monge-Ampère equation (3) in general position is *invariant with respect to parallelism* if the following property holds for the characteristic affine connection Σ of this equation and two arbitrary points $m, \widetilde{m} \in M$. For any continuous curve that connects these points and the parallel displacement (11) along this curve there exist neighborhoods U of m, $\widetilde{U}$ of $\widetilde{m}$ and a diffeomorphism

$$f : \widetilde{U} \to U \tag{20}$$

that satisfies equalities (14),

$$f_*(m) = F,$$

and

$$f^*(I \mid U) = I \mid \widetilde{U}. \tag{21}$$

Theorem 2. *Consider a Monge-Ampère equation (3) in general position and the characteristic affine connection Σ of this equation. By ∇, R and T denote the covariant differentiation, tensors of curvature and torsion of the connection Σ. The equation (3) is invariant with respect to parallelism if and only if the equalities*

$$\nabla R = 0, \quad \nabla T = 0 \tag{22}$$

hold at all points of manifold M.

Proof. Follows from theorem 1 and the well known properties of affine connections [5, ch. 6, par. 7]. □

Theorem 3. *Consider Monge-Ampère equations (3) and (6) in general position. Denote by Σ and $\widetilde{\Sigma}$ the characteristic affine connections of these equations. If the considered equations are invariant with respect to parallelism, then the following propositions are equivalent.*

(a) For any points $m \in M$ and $\widetilde{m} \in \widetilde{M}$ there exist neighborhoods U of m, $\widetilde{U}$ of $\widetilde{m}$ and a diffeomorphism (20) such that equalities (14) and (21) hold.

(b) There exists at least one point $m \in M$ and one point $\widetilde{m} \in \widetilde{M}$ such that there is a linear isomorphism (11) that satisfies condition (12) and takes the connection's $\widetilde{\Sigma}$ tensors of curvature $\widetilde{R}(\widetilde{m})$ and torsion $\widetilde{T}(\widetilde{m})$ into the connection's Σ tensors of curvature $R(m)$ and torsion $T(m)$ respectively.

Proof. Follows from theorem 2 and the well known properties of affine connections [5, ch. 6, par. 7]. □

Remind that a diffeomorphism

$$f : M \to M$$

is a *contact symmetry* of a Monge-Ampère equation (3) on the contact manifold M if

$$f^*(I) = I.$$

An equation (3) is called *homogeneous* if this equation's group of contact symmetries is a Lie group that acts transitively on M. Thus M is homogeneous. If the homogeneous manifold M is reductive, then an equation (3) is called a *homogeneous reductive* Monge-Ampère equation.

Theorem 4. *Consider a Monge-Ampère equation (3) in general position on the contact manifold M and the characteristic affine connection Σ of this equation. Denote by ∇, R and T the covariant differentiation, tensors of curvature and torsion of the connection Σ. The following propositions are equivalent.*

(a) The equation (3) is a homogeneous reductive Monge-Ampère equation and Σ is a canonical connection of the manifold M.

(b) The connection Σ is complete and equalities (22) hold at all points of M.

Proof. Follows from theorem 1 and the well known properties of invariant connections on homogeneous reductive manifolds [5, ch. 10, par. 2]. □

Theorem 5. *Consider Monge-Ampère equations (3) and (6) in general position and the characteristic affine connections Σ and $\widetilde{\Sigma}$ of these equations. Suppose these equations satisfy proposition (a) of theorem 4. Then the following propositions are equivalent.*

(a) For any points $m \in M$ and $\widetilde{m} \in \widetilde{M}$ there exists a diffeomorphism (7) such that equalities (14) and (8) hold.

(b) Proposition (b) of theorem 3.

Proof. Follows from theorem 4 and the well known properties of affine connections [5, ch. 6, par. 7]. □

References

1. V.V. Lychagin. Contact geometry and nonlinear second-order differential equations. *Russian Math. Surv.* V. 34(1) (1979), pp. 149-180.
2. T. Morimoto. La géométrie des équations de Monge-Ampère. *C. R. Acad. Sc. Paris.* T. 289 (1979), no 1, pp. 25-28.
3. R. Courant. Partial differential equations, Interscience, New York - London, 1962.
4. D.V. Tunitsky. Equivalence and characteristic connections of Monge-Ampère equations. *To be published in Matematicheskii Sbornik* (Russian).
5. S. Kobayashi, K. Nomizu. Foundations of differential geometry, V. 1, 2, Interscience, New York - London, 1963, 1969.

International Sophus Lie Center
9-3-19 Pluscheva St.
Moscow 111524, RUSSIA

E–mail: tunitsky@glas.apc.org